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SCRIPTORES LOGARITHMICI;

OR

A COLLECTION

OF

SEVERAL CURIOUS TRACTS

ON THE

NATURE AND CONSTRUCTION

OF

LOGARITHMS,

MENTIONED IN DR. HUTTON'S HISTORICAL INTRODUCTION TO HIS NEW
EDITION OF SHERWIN'S MATHEMATICAL TABLES:

TOGETHER WITH

SOME TRACTS ON THE BINOMIAL THEOREM AND OTHER SUBJECTS
CONNECTED WITH THE DOCTRINE OF LOGARITHMS.

VOLUME IV.

LONDON:

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THE

P R E F A C E.

I NOW present my Readers with a Fourth Volume of the *Scriptores Logarithmici*, of the contents of which I shall here give a short account.

The first Tract in this Volume is intitled "*An Appendix to the Tract of Dr. Edmund Halley, intitled 'An Easy Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line, or Sum of the Secants; with various Methods of computing the same to the utmost exactness;'*" which has been re-printed in the second Volume of this Collection of Tracts, intitled, *Scriptores Logarithmici*, in pages 76, 77, 78, &c. - - 84: Containing the Solution of a curious Problem relating to Navigation, proposed by Dr. Halley in the latter part of the said Tract."

This Problem, relating to Navigation, was proposed by Dr. Halley towards the end of the aforesaid Tract, intitled, "*An easy Demonstration of the Analogy, &c*" in the year 1696, in a passage expressed in these words; "*I need not shew how, by regressive work, to find the latitudes from the meridional parts; the method being sufficiently obvious: I shall only conclude with the proposal of a Problem, which remains to make this doctrine compleat; and that is this.*"

A P R O B L E M.

A Ship sails from a given latitude; and, having run a certain number of leagues, has altered her longitude by a given angle. It is required to find the course steered.

The Solution hereof would be very acceptable, if not to the Publick, at least to the Author of this Treatise; being likely to open some further light into the mysteries of Geometry." See Vol. II of this present Collection of Treatises, called Scriptores Logarithmici, page 83.

In this Problem the ship is supposed to have sailed always in the same course during the whole voyage, or in such a line as to have always cut the several meridian-lines (or circumferences of great circles of the earth passing through it's two poles,) over which it has passed, in the same angle: and the earth is supposed to be of a spherical, and not of a spheroidal, or other oval, figure; and consequently all the meridian-lines (or circumferences of great circles of the earth that pass through the two poles,) will be equal to the circumference of the equator: And by the difference of the ship's longitude at the beginning and at the end of the voyage, (which difference is here supposed to be given, or known,) we are to understand the arch of the circumference of the equator that is intercepted between two meridian-lines (or the circumferences of two great circles of the earth passing through the two poles,) that are drawn through the ship's two positions at the beginning and at the end of the voyage. These things must be thoroughly understood, and carefully attended-to, in order to have a clear idea of the meaning of the Problem of which Dr. Halley wished to see a Solution.

This Problem has never yet received a compleat and accurate Solution from the time when Dr. Halley proposed it in the year 1696, to this day; though it has been solved since that time in a tentative way by several Mathematicians; and particularly by Mr. John Robertson, the late Librarian of the Royal Society (who had been formerly the Head Master of the Royal Military Academy at Portsmouth,) in his valuable Treatise on Navigation, Book VIII, Sect. IV, Problem X, page 560; and by Monsieur Pierre Bouguer, (a distinguished member of the Royal Academy of Sciences at Paris,) in his copious and learned

learned treatise on Navigation, (published at Paris in the year 1753, and intitled *Nouveau Traité de Navigation, contenant la Théorie et la Pratique du Pilotage*;) in the 5th book of the said treatise, Sect. I, Chapter II, pages 352, 353, under the title of *Sixième Problème général*; and likewise by the late Mr. *Israel Lyons*, in the Appendix to the Nautical Almanack for the year 1772. And, indeed, long before it was proposed by Dr. Halley in the year 1696, and even long before the birth of Dr. Halley (which was in the year 1656,) this Problem had been very ably solved in *this tentative way* by a very eminent Dutch Mathematician, named *Snellius*, or *Willebrord Snell*, in his excellent treatise on Navigation, written in Latin and intitled *Tiphys Batavus, sive Hiftridromice; de navium cursibus et re navali*, and published at Leyden in Holland in the year 1624; and it had been proposed 30 years before that time, to wit, in the year 1594, by Mr. *Robert Hues*, of the University of Oxford, in a very valuable treatise on the Use of the Globes, written in Latin and published in that year, as *Snellius* informs us in his *Tiphys Batavus*. But all this seems to have been unknown to Dr. Halley, who (in the passage above-cited from his Tract called *An easy Demonstration of the Analogy, &c.*) speaks of this Problem as having been then proposed for the first time to the consideration of learned men. This seems rather surprizing in a man of Dr. Halley's character, who was not only a very learned and skillfull Mathematician, but particularly conversant with Geography and Navigation, and had even performed a long sea-voyage to the Island of St. Helena, on the south side of the equator, in the capacity of a master, or captain, of a ship, and who therefore might be supposed to have been acquainted both with the *Tiphys Batavus* of *Snellius* and the more antient, but much-esteemed, treatise of *Hues* on the Use of the Globes. But so the fact is, however difficult it may be to account for it: since it is by no means probable that, if Dr. Halley had known these two works of *Hues* and *Snellius*, (of which the former had mentioned and proposed this Problem, and the latter had actually given a very good solution of it,) he would have proposed this Problem to the mathematical world as an absolutely new question worthy of their attention and investigation, and necessary to compleat the doctrine of Navigation.

It seems probable, however, that, if Dr. Halley had been acquainted with *Snellius's* tentative Solution of this Problem, he would not have been perfectly

satisfied with it, though it is a very good one, and sufficient for all practical purposes. This I collect from his intimating an opinion *that the Solution of this Problem was likely to open some further light into the mysteries of Geometry*; by which words I conceive he must have meant, that a *direct and accurate* Solution of it was likely to be attended with this desirable consequence, which could hardly be expected to follow from a mere tentative solution of it. And therefore I conjecture that the Solution of this Problem in the present Volume, (which is a direct and accurate solution of it, grounded on a Series invented by Mr. James Gregory for expressing the length of the logarithmick tangent of a circular arch that is greater than 45 degrees, but less than 90 degrees, in a circle of which the radius is known, or the length of the logarithm of the ratio of the natural tangent of such an arch to the radius of the circle, taken on the axis, or asymptote, of a logarithmick curve of which the subtangent is equal to the radius of the said circle,) is such a Solution as Dr. Halley wished to see, though I doubt whether it is likely to produce the good effect he had in view, *of opening some further light into the mysteries of Geometry*. The substance of this Solution was invented and communicated to me by the learned Mr. George Atwood, A. M. who was formerly a Fellow of Trinity-College, Cambridge. But, as I have drawn it up at great length and with much labour in it's present form, I have thought I might without impropriety call it mine, after making the present acknowledgement. The Problem is a very abstruse one; and the Solution here given of it is very subtle and difficult to understand, as well as very tedious and laborious in the application of it to particular examples, insomuch that, if any sea-faring person should ever have occasion to solve this Problem, and to find, in a particular case, the course which a ship had steered, from the knowledge of the latitude at the beginning of the voyage, the number of leagues run, and the difference of the longitudes at the beginning and the end of the voyage, I would advise him to solve it by the *tentative* method taught us by *Snellius*, or by some other method of the same kind, rather than to have recourse to the direct and accurate Solution that is given of it in this Tract. For this Solution requires the resolution of an infinite equation, in which x , or the unknown quantity, or root, is the co-sine of the angle sought, or the angle made by the ship in it's course with the several meridian-lines which it has crossed, and not only the left-hand side of this final equation, (which involves the several powers of x) is an infinite series, but the co-efficient of each of the powers of x contained in

in it is likewise an infinite series, or the result of an infinite series combined with some other known quantity; so that there is a great deal of laborious calculation required to obtain the said final equation, before we enter upon the labour of resolving it by Mr. Raphson's, or some other, Method of Approximation; which labour is likewise very great.

I have, however, done all that was in my power to lessen the difficulty both of understanding and of reducing to practice the Solution of this Problem. For I have divided the Problem into three separate cases, and have given a separate solution of each case, and an example to each of the three separate solutions. The first case is, when the place from which the ship sails and that at which it arrives are both on the same side of the equator, and the former place is more distant than the latter from the equator; or, in other words, when the ship sails, from any place not in the equator, in an oblique line, *towards* the equator, but does not pass it: the second case is, when the ship sails, from any place not in the equator, in an oblique line, *from* the equator; which is the reverse of the former case: and the third case is, when the ship sails from a place on one side of the equator, in an oblique line, to a place on the other side of the equator. The first 21 pages of this Tract are taken up with the demonstration of three Lemmas, or preliminary propositions to the Solution of the Problem itself; after which the solution of the first case of the Problem itself extends from page 21 to page 33, and is afterwards illustrated by an application to a particular example, which extends from page 33 to almost the end of page 45, and by a proof of the truth of the conclusion obtained by it, (to a considerable degree of exactness,) in pages 45, 46, and 47; so that this first case of the Problem takes up no less than 26 pages. The solution of the second case of the Problem begins in page 48, and extends to the end of page 59; after which it is applied to a particular example, which begins in page 60, and extends to page 75; and this example is followed by a proof, or confirmation, of the truth of the conclusion obtained by it, which reaches through pages 76, 77, and 78; so that this second case of the Problem takes up 30 pages, or still more pages than the first case. And the solution of this second case is followed by the statement and solution of a certain additional Lemma, (or Problem, preliminary to the solution of the third case of the Problem itself,) which begins in page 79, and extends to the bottom of page 80; after which the solution of the said

third case of the Problem itself begins in page 81, and extends to the end of page 94, and is then applied to a particular example, which begins in page 95, and extends to page 111; and this example is followed by a proof, or confirmation, of the truth of the conclusion obtained by it, which extends from page 111 through pages 112 and 113; so that this third case of the Problem takes up no less than 35 pages. And with this solution of the said third case of the Problem, and the illustration of the said solution by an example, and a proof of the truth of the value of x obtained by it, this first Tract concludes.

This Problem is no otherwise connected with the subject of Logarithms, and therefore proper to be admitted into this Collection of Tracts, (which relates to that subject,) than as it's solution is derived from Mr. James Gregory's infinite Series for expressing the length of what he calls *the logarithmick tangent* of a given circular arch, greater than 45 degrees but less than 90 degrees in a given circle, by which he means the logarithm of the ratio of the natural tangent of such a circular arch to the tangent of an arch of 45 degrees, or to the radius of the circle, taken upon the axis, or asymptote, of a logarithmick curve, of which the subtangent is equal to the radius of the given circle.

When I had completed the direct Solution of Dr. Halley's Nautical Problem at so great length and with so much pains, I thought it would be proper to accompany it with a description of the tentative Solution that had been given of the same Problem in the year 1624 by Willebrord Snell in his Treatise of Navigation called *Tiphys Batavus*, which appeared to me to be a very good solution, and, for practical purposes, even preferable to the direct Solution that makes the subject of the first Tract in this Volume. And this induced me to look into this treatise of Snellius with more care and attention than I had hitherto bestowed upon it. And the result was, that this treatise appeared to me to be the most regular, well-written, and scientific book that I had ever met with on the subject of Navigation. I also found that the Preface, or Introduction, to it was a most curious and interesting collection of every thing that could be found in the books published at that time, both antient and modern, relating to the History of Navigation. It contains, amongst other things, a very diligent and learned inquiry into the form of the vessels of the Antients, made to be rowed by three, or four, or five, or more benches of rowers placed one above the other; and called *triremes*, *quadrيرهmes*, *quinquيرهmes*,

remes, &c. and into the manner of managing the oars, and into the size or capacity of these vessels, and the number of men they were capable of holding, and into the figures of heathen gods, or of celebrated heroes, or of some brute animal, (such as a lion, or a bull, or a ram,) that were placed by the Ancients at the heads of their ships, and from which the ships took their names;—and it likewise contains an account of the discovery that an iron needle that had been touched by a load-stone, would, if moving freely upon a center, always point towards the North Pole; which remarkable and most useful property he supposes to have been discovered about four hundred years before the time in which he wrote, that is, about the year of Christ 1220;—and an account of the discovery of America by Christopher Columbus, a native of Genoa, who was induced by a remarkable adventure that had happened to another man, to undertake a voyage across the Atlantick Ocean in hopes of finding land on the other side of it; which undertaking terminated in the discovery of America. This adventure, as Snellius informs us, was reported to be as follows. The master of a vessel in the Spanish Province of Andalusia, (whose name, however, is not now known,) who had been accustomed to a sea-faring life from his early youth, had made many voyages which, in the humble state of Navigation in those times, (about the year 1470,) were considered as very long and *venturefome*; and in one of these voyages he had sailed west-ward beyond the Island of Madeira, and (whether by design, or driven thither by stress of weather, is not certain,) had arrived at some part of the opposite coast of America; from whence he soon after returned to Madeira, after losing a great part of his crew, and with his own health very much impaired and broken by the fatigues and hardships he had undergone; and soon after died at Madeira, at the house of the said Christopher Columbus, (who was then settled in that island, and got his living by making sea-charts for the use of trading-vessels,) to whom he communicated the Journal of his voyage, and the several courses he had followed in performing it, (as pointed-out by the needle and compass by which his voyage had been directed,) and the parallel of latitude in which the new land he had touched-at was situated. This information made Columbus resolve to make, if possible, another attempt to discover this new Continent; in which at last, by the assistance he received towards this important undertaking from *Ferdinand* and *Isabella*, king and queen of Arragon and Castile in Spain, he was successful. This Preface of Snellius afterwards gives an account of further improvements made in the Science of Navigation by
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the great sagacity, learning, and industry, of *Peter Novius*, or *Notes*, the celebrated Portuguese, in the year 1530, and by *Gerard Mercator* and *Stevinus*, in Flanders and Holland, and *Robert Ilues* and *Edward Wright* in England. To all these writers he gives just commendations; but yet intimates, in the last page of his Preface, that it appeared to him that there was still wanting to the completion of the Theory of Navigation a treatise that should point-out with a considerable degree of exactness, and to every minute of *longitude*, the variation of the longitude of a ship's position corresponding to a given number of leagues run by it on a given course, or the relation that subsists between the distances run by the ship in any given times on a given course and the changes of it's latitude and longitude in the same times: and such a treatise it was therefore his intention to compose. This (if I understand it right,) is the meaning of the following passage. *Sed, ut ingenuè dicam quod res est, coronis et operis perfecta delineatio desiderari videtur, [scilicet,] ut "quantum loxodromia singulae, et earum partes, in longitudine evariarent," ad minutum constaret: neque id rudi aut mechanicæ aliquâ faciliæ, sed accuratiore illâ quæ minimas quasque particulas subducit. Id igitur secutus sum, ut ab ipso fundamento omnia solidè demonstrata, et ordine constituta, nunc demùm exhiberem.* These words express the Author's intentions of treating this subject in a regular and scientific manner: and he appears to me to have executed this design with ability and success. And therefore, as this book of Snellius is now grown scarce, I thought it would be an acceptable thing to the Students of this usefull branch of the Mathematicks to have a new edition of it; and I consequently resolved to re-print it in this Collection, though it has (it must be confessed,) but little connection with Logarithms, which are the principal subject of these Volumes. I have printed this treatise of Snellius faithfully and carefully from the Leyden edition in the year 1624, with the Tables that belonged to it and were there printed with it: and I have added nothing of my own to it, but a note in page 172 of the length of about half a page, and another note at the end of the Tables, in pages 269, 270, and 271, concerning the rule of false position, which is adopted by the Author as a method of approximation to some of the quantities he has occasion to investigate. This treatise of Snellius extends from page 115 to page 271.

The third Tract in this Volume relates to the same subject as the first, or to the Solution of Dr. Halley's Nautical Problem above-mentioned, being a
new

new Solution of that Problem in an indirect, or tentative, way by Dr. Andrew Mackay, Doctor of Laws in the University of Aberdeen in Scotland. It was communicated to me by that learned gentleman (who is uncommonly well acquainted with the subject of Navigation,) in consequence of my having sent him a copy of the direct and accurate Solution of it printed in the first Tract of this Volume, and a copy of Snellius's *Tiphys Batavus*. It differs from the tentative Solutions of the same Problem given by former Mathematicians; and Dr. Mackay has, in this tract, instituted a comparison between it and the Solutions given of this Problem by Snellius in the *Tiphys Batavus*, by Monsieur Bouguer, Mr. Robertson, Mr. Emerson, Mr. Israel Lyons, and Monsieur Bezout; so that the reader, after perusing this tract, will be able to form his own judgement of the merits of these several different Solutions. This Tract of Dr. Mackay begins in page 275, and ends in page 299; and I consider it as a very valuable addition to the foregoing Tracts on the subject of Navigation.

The fourth Tract in this Volume is Dr. James Wilson's *Historical Dissertation on the Rise and Progress of the Modern Art of Navigation*, which had been printed before in the year 1772 at the beginning of the 3d edition of Robertson's *Elements of Navigation*. This Tract begins in page 303, and ends in page 332. It is full of curious historical matter, and has suggested to my mind a wish that some person of affluence, fond of the subject of Navigation, and who should have been indebted to it, perhaps, for his rank or fortune, would cause a collection of all the Authors on that subject whose works are mentioned in this Dissertation, to be made; and reprinted in a handsome manner in a set of quarto volumes of the size of these volumes of the *Scriptores Logarithmici*, under the title of *Scriptores Nautici*. Such collections of learned tracts on particular subjects, under various titles suited to the several subjects of which they treated, would be very convenient in the present state of science; which is extended to such a variety of subjects, and dispersed in such a number of different books, that it is very difficult and very expensive for a person, fond of any particular branch of science, to procure himself all the books that relate to it. Besides the collection called *Scriptores Nautici*, relating to Navigation, there might be a collection called *Scriptores Statici*, relating to the doctrine of *Statics*, or bodies at rest that form an equilibrium, or counter-poise, to each other; under which head all the books of merit that treat of the *Lever*, the *Inclined Plane*, and the other mecha-

nical powers, would be comprized, and those that treat of the catenary curve, and of the partial immersion and the positions of bodies floating in liquids of greater specifick gravity than themselves, and of many other curious subjects of the like nature. And there might be another collection called *Scriptores Phoronomici*, relating to the doctrine of bodies in motion; under which head would be comprised Galileo's Mechanical Dialogues, of which the 3d and 4th contain the doctrine of the fall of heavy bodies to the earth with the law of their acceleration, and of their motion on inclined planes, and of the motion of pendulums in circular arches, and of the motion of projectiles, which (abstracting from the resistance of the air,) would describe parabolas; and under the same head would be comprised Mr. Huygens's tract on the motions of perfectly elastic bodies striking against each other, and his admirable treatise *De Horologio Oscillatorio*, or on the motion of a pendulum-clock, and his tract on central forces, and all Sir Isaac Newton's most profound, but very difficult, work called the *Principia*, or *Mathematical Principles of Natural Philosophy*, with the several commentators on it, and Hérman's *Phoronomia*, and Euler's work *De Motu*. Another Collection might relate to the finding the centers of gravity of different bodies; which is, I believe, a more subtle and difficult subject than is generally supposed. This Collection might be called *Scriptores Centrobatici*. And another Collection might consist of all the writers on Opticks, under the title of *Scriptores Optici*. This Collection should comprize the work of Euclid, or that which has been ascribed to him, on this subject, and those of Alhazen, and Vitellio, and Roger Bacon, (the learned English monk,) and *Antonio De Dominis*, and Willebrord Snell, and Des Cartes, and Mr. Huygens's Dioptricks, and his treatise *De Lumine*, and other works of his on the subject of Opticks, and James Gregory's *Optica Promota*, and Dr. Barrow's *Lectiones Opticae*, and Sir Isaac Newton's *Lectiones Opticae*, and his Treatise of Opticks, or Experiments on Light and Colours, and Molineux's Dioptricks, and Dr. Smith's Compleat System of Opticks, and Harris's Opticks, and many papers in the Philosophical Transactions relating to the same subject. If such separate Collections of Authors were published, every person, who was devoted to any particular branch of these sciences, (and no man can attend to all of them, or even to many of them, with any great prospect of becoming master of them,) might buy the Collection which related to his particular branch, at a moderate expence.—But, to return from this digression to Dr. Wilson's Dissertation on Navigation;

Navigation ; I will add that the author of it was a doctor of Physick, and a man of great skill and knowledge in the Mathematicks, and was an intimate friend of Dr. Henry Pemberton, the learned Mathematician who assisted Sir Isaac Newton in the publication of the third and last edition of his *Principia*, and who distinguished himself by giving a demonstration of a famous Theorem of Mr. Cotes concerning a remarkable property of the circle, a description of which had been found amongst Mr. Cotes's papers after his death, but without a demonstration, and had been published in that manner in Mr. Cotes's *Harmonia Mensurarum* amongst his other works, by Dr. Robert Smith, who succeeded him as Plumian Professor of Astronomy and Experimental Philosophy, and who was afterwards Master of Trinity College, Cambridge. Dr. Pemberton's demonstration of this Theorem was published in a thin quarto pamphlet, under the title of *Epistola ad Amicum J. W. de Cotesii inventis*, and was addressed to this Dr. James Wilson, who was the person meant by the word *Amicum* with the two letters *J. W.*, which were the initial letters of his name. Dr. Wilson was also an intimate friend of the celebrated Mr. Benjamin Robins, who was one of the greatest Mathematicians of the middle of the present century, and was still more admired for the accuracy and perspicuity of his style in treating of the most difficult parts of those sciences than for his extensive knowledge of them. He was also a very able engineer, and was employed in that capacity by the East India Company, in whose service he died at Madras in the year 1751 at only 44 years of age. And some years after the death of this eminent man, this Dr. Wilson collected and published a new edition of his Mathematical and Philosophical works in two volumes octavo in the year 1761. It is reasonable to presume that this intimate friend and companion of two such eminent Mathematicians as Dr. Pemberton and Mr. Robins, and who published the works of the latter, must himself have been deeply learned in the same sciences : and the Dissertation here re-printed is worthy of a man of that character.

The fifth Treatise in this Volume contains, first, the construction, and, afterwards, the Algebraical Solution, of a curious Geometrical Problem that had been proposed by James Glenie, Esq., who was some years ago a Lieutenant in the Corps of Engineers, and who is known to be profoundly skilled in the Mathematicks. It was, first, proposed in the *Ladies' Diary* for the year 1794, page 48, in the words following :

In the Palace of one of the Persian kings, it is said, there was a triangular area, such, that the cubes of two of the sides were, together, equal to thrice the cube of the third side, (which was 200 feet in length,) and that the area itself contained 10,000 superficial feet. Supposing this to have been really the case, it is required to construct the triangle by common, or plane, Geometry. And in the Ladies' Diary for the year 1795 there were published Mr. Glenie's own construction of this Problem and two other constructions of it by other persons.

This Problem is very fully examined and discussed in the present Volume, of which it takes up no less than 78 pages, to wit, from page 334 to page 412, of the contents of which it will be proper to give some account.

The Problem is stated in Art. 1, page 335; and Mr. Glenie's construction of it (which is very simple and elegant,) is given in page 336, which is followed by a demonstration of that construction in pages 336, 337, and 338: and this demonstration gives rise to a Scholium that contains some curious matter relating to the extraction of the square-root of a binomial quantity, of which one of the members is a surd quantity, or the square-root of a non-square number. For in the course of that demonstration it became necessary to extract the square-root of the quantity $\frac{14 + \sqrt{96}}{3}$, which is equal to the binomial quantity $2 + \frac{\sqrt{2}}{\sqrt{3}}$, as will appear by squaring the said binomial quantity and observing that its square will be equal to the said quantity $\frac{14 + \sqrt{96}}{3}$. But how to discover that the square-root of the quantity $\frac{14 + \sqrt{96}}{3}$ (which involves in it the surd quantity $\sqrt{96}$;) will be the binomial quantity $2 + \frac{\sqrt{2}}{\sqrt{3}}$ is by no means obvious. The method of doing this is therefore explained in the said Scholium in pages 338, 339, 340, 341, and 342.

The first step taken in the investigation of the square-root of the quantity $\frac{14 + \sqrt{96}}{3}$ is to reduce this quantity to its lowest terms, or smallest numbers, by shewing that it is equal to $\frac{2}{3} \times \sqrt{7 + \sqrt{24}}$; whence it follows that its square-root will be $= \frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{7 + \sqrt{24}}$. And then it becomes necessary to find the

the square-root of the binomial quantity $7 + \sqrt{24}$, of which the first member is the whole number 7, and the second member is the surd number $\sqrt{24}$, or the square-root of the non-square number 24. And this is done in Art. 7 and 8, in which the square-root of $7 + \sqrt{24}$ is found to be equal to $\sqrt{6} + 1$, or $1 + \sqrt{6}$. And in Art. 9 it is shewn that this binomial quantity $1 + \sqrt{6}$ is truly equal to the square-root of the former binomial quantity $7 + \sqrt{24}$, by actually squaring it, or multiplying it into itself; by which we obtain a product that is $= 7 + \sqrt{24}$. And then, in Art. 10, it is shewn that the binomial quantity $2 + \frac{\sqrt{2}}{\sqrt{3}}$ will be equal to $\frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{1 + \sqrt{6}}$, and consequently to $\frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{7 + \sqrt{24}}$, and therefore to the square-root of the first quantity $\frac{14 + \sqrt{96}}{3}$. And thus the investigation of the binomial square-root of the quantity $\frac{14 + \sqrt{96}}{3}$ is completed.

After this, in Art. 11, 12, 13, and 14, there is a similar investigation of the binomial square-root of the quantity $\frac{14 - \sqrt{96}}{3}$, by, first, shewing that the said quantity is equal to $\frac{2}{3} \times \sqrt{7 - \sqrt{24}}$, and consequently that its square-root will be $= \frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{7 - \sqrt{24}}$, and then investigating the square-root of the binomial, or residual, quantity $7 - \sqrt{24}$, which is found to be $= \sqrt{6} - 1$, or $6^{\frac{1}{2}} - 1$; whence it follows that the square-root of the quantity $\frac{14 - \sqrt{96}}{3}$ will be equal to $\frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{6^{\frac{1}{2}} - 1}$ and consequently to $2 - \frac{\sqrt{2}}{\sqrt{3}}$.

These investigations of the square-roots of the quantities $\frac{14 + \sqrt{96}}{3}$ and $\frac{14 - \sqrt{96}}{3}$, and of the square-roots of the binomial quantities $7 + \sqrt{24}$ and $7 - \sqrt{24}$, are followed by some usefull observations on the limitations to which the methods used in those investigations for finding the square-roots of the said binomial quantities $7 + \sqrt{24}$ and $7 - \sqrt{24}$, or, in general terms, for finding the square-roots of the binomial quantities $A + \sqrt{B}$ and $A - \sqrt{B}$, (A and B being any whole numbers,) are subject; and it is shewn that these methods of finding such binomial square-roots of the binomial quantities $A + \sqrt{B}$ and $A -$

A -

$A - \sqrt{B}$, can be applied only when the whole number A is greater than the surd number $\sqrt{B}$, or when AA is greater than B . Thus, for example, it cannot be applied to the finding of a binomial square-root of the quantity $3 + \sqrt{24}$, or of the quantity $\sqrt{24} - 3$, in which the rational member 3 is less than the surd member $\sqrt{24}$; as is shewn in Art. 16, page 343. This limitation, I have observed, is not mentioned in some very celebrated books of Algebra which treat of this subject. But it is a circumstance very proper to be attended-to, in order to avoid attempting investigations which will lead to absurd conclusions.

Having gone through Mr. Glenie's construction of this Problem, and the demonstration of that construction, and the observations concerning the extraction of the binomial square-roots of such binomial quantities as $A + \sqrt{B}$ and $A - \sqrt{B}$, to which the said demonstration gave rise, I, in Art. 20, page 345, &c, undertake the Solution of Mr. Glenie's Problem *de novo* by means of Algebra, and without the help of his construction. And, for this purpose, I, first, state the Problem over again in Art. 20, and then, before I enter upon the Solution of it, I shew that the Problem is possible, or that it is possible for a triangle to exist which shall have the properties required by the conditions of the Problem, to wit, 1st, that it's height shall be equal to half it's base, and, 2ndly, that the sum of the cubes of the other two sides of it shall be triple of the cube of it's base. And then I shew further, that, if from the point C , or the right-hand extremity of the given base BC , in fig. 2, page 345, we draw the right line CE at right angles to the base BC and equal to BF , or half the base BC , (which is the height of the intended triangle ABC), and from the point E we draw the right line EB to the other extremity B of the base BC , the sum of the cubes of the two lines EB and EC will be less than three times the cube of the base BC , and consequently the line EB will be less than the line AC , or the greater side of the intended triangle ABC . And hence it follows that, if from A , or the vertex of the intended triangle, a right line were to be drawn at right angles to the base BC , or to the said base produced, if need be, and cutting the said base, or base produced, in the point H , the said point H would fall without the said triangle ABC , and the line BH would be longer than the base BC of the said triangle. And, after these preliminary observations, I put a for half the given base BC of the intended triangle,

triangle, and x for the, hitherto unknown, external line CH , the determination of which will enable us to solve the Problem. For, when the length of CH is discovered, we need only draw the line HA at right angles to the line BC , or BH , and equal to $\frac{BC}{2}$, and from the point A draw the lines AB and AC ; and the triangle ABC , so constituted, will be the triangle which was required to be found. And thus the Solution of the Problem is reduced to the investigation of the length of the external line CH , that extends from the right-hand extremity C of the said base BC to the point H , from which, if the right line HA be drawn at right angles to BC , or BH , and equal to $\frac{BC}{2}$, and from the point A be drawn the right lines AC and AB , the sum of the cubes of AC and AB will be equal to $3 \times BC^2$, or to $3 \times 2a^2$, or to $3 \times 8a^2$, or to $24a^2$.

After this preparation I enter upon the investigation of the magnitude of x , or CH , or of it's relation to $\frac{BC}{2}$, or a , and I find it to be expressed by the following long equation of the 10th order, or involving the 10th power of the unknown quantity x , to wit, $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$; by the resolution of which we may discover the value of x , or CH , and from that value may derive the value of xx , or CH^2 , and that of $aa + xx$, (or $AH^2 + CH^2$), or AC^2 , and that of $5aa + 4ax + xx$, (or of $aa + 4aa + 4ax + xx$, or of $aa + 2a + x$), or of $AH^2 + \overline{BC + CH}^2$, or of $AH^2 + BH^2$, or of AB^2 , and consequently those of AC and AB , or the sides of the triangle ABC , which we are required to describe.

This investigation of the value of x , or CH , or of the equation expressing it's relation to a , or $\frac{BC}{2}$, takes up pages 348, 349, 350, and 351, and contains many tedious and difficult Algebraick operations, in which it seemed reasonable to apprehend that some mistake might have been made, which would render the said final long equation inaccurate. I therefore thought it prudent and expedient to have recourse also to another method of obtaining an equation that should express the relation of x to a . And I found that this second

second method of proceeding produced the same long equation above-mentioned, which had been obtained by the first method. This second method of obtaining this equation takes up pages 352, 353, 354, and part of 355. And from the agreement of the results obtained by both these methods it seems reasonable to conclude with confidence that no arithmetical mistakes have been made in the operations by which the said final long equation has been obtained, and consequently that the said equation does truly express the relation between the given line a , or DF , or $\frac{BC}{a}$, and the unknown line x , or CH . And therefore all that remains to be done, in order to the Solution of the Problem, is to resolve the said long, final, equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^{10} = 12,625a^{10}$.

Now this equation is capable of being reduced to a quadraick equation by, first, subtracting it's absolute term, or known quantity, $12,625a^{10}$, from both sides of it, and then dividing the remainder of such subtraction, to wit, the compound quantity $-12,625a^{10} + 16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9$, consisting of eleven terms, (which will be equal to 0,) by the compound quantity $-2525a^8 + 360a^7x + 1404a^6x^2 + 1640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, consisting of nine terms, by which the former compound quantity is exactly divisible, or without leaving any remainder. For the quotient of this division will be the trinomial quantity $5aa - 6ax - 3xx$, from which we may derive the quadratick equation $5aa = 6ax + 3xx$, or $xx + 2ax = \frac{5aa}{3}$, by reasoning in the manner following. Since the dividend of the foregoing division, to wit, the above-mentioned compound quantity consisting of eleven terms, is equal to 0, it follows that either the divisor of it, or the quotient, or both the divisor and the quotient, must be equal to 0 likewise. For in every division the product of the multiplication of the divisor into the quotient is always equal to the dividend. And in this case the dividend is equal to 0. Therefore the said product of the multiplication of the divisor into the quotient will in this case be equal to 0. Now this would be impossible, if both the divisor and the quotient were finite quantities, but will be possible if either one, or both of them, is equal to 0. Therefore either the said divisor, consisting of nine terms, or the said quotient $5aa - 6ax - 3xx$,

or both the said divisor and the said quotient, will be equal to 0. We have reason therefore to suppose that the said quotient $5ax - 6ax - 3xx$ is equal to 0, and consequently that $xx + 2ax$ is $= \frac{5aa}{3}$; which equation may be constructed geometrically, or by drawing only right lines and circles, in the manner described by Mr. Glenie, and likewise in other ways, one of which is exhibited in page 360 of this Volume.

But this method of reducing the said long equation of the 10th power to the quadratick equation $xx + 2ax = \frac{5aa}{3}$ by the above-mentioned operation of division by the aforefaid divisor consisting of nine terms, is what, I must confess, I should never have thought of, if I had not been previously acquainted with Mr. Glenie's construction. For I know no method of discovering that the aforefaid compound quantity consisting of eleven terms, (which is equal to 0,) is exactly divisible by any other compound quantity consisting of nine terms and involving all the powers of x as far as x^9 ; and, if I had been told, or had of my own accord suspected, that it was divisible by such a compound quantity, I should not have known how to find the terms of such a compound quantity.

But the said long equation may also be resolved without being reduced to a lower equation, though not without a great deal of arithmetical calculation. And this way of resolving it appears to me to be the plainest and clearest, and therefore, upon the whole, the best, way of resolving it. And accordingly I have resolved it in this manner by Mr. Raphson's excellent method of approximation. This resolution is performed in Art. 45, 46, 47, &c. - - - 51, pages 367, 368, 369, &c. - - - 380, and the result of it is, that x , or CH, (in fig. 2, page 345,) is nearly equal to $0.632,993,15 \times a$, or $0.632,993,15 \times DF$, or $0.632,993,15 \times \frac{BC}{3}$. And in Art. 52, pages 380, 381, it is shewn that, if x , or CH, be taken $= 0.632,993,15 \times a$, we shall have AC (or $\sqrt{aa + xx}$) $= 1.183,503,41 \times a$, and AB (or $\sqrt{5aa + 4ax + x^2}$) $= 2.816,496,569 \times a$; which will answer the conditions of the Problem. For AC³ will be $(= 1.183,503,41)^3 \times a^3 = 1.657,709,893 \times a^3$, and AB³ will be $(= 2.816,496,569)^3 \times a^3 = 22.342,289,740 \times a^3$, and consequently

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$AC^3 + AB^3$ will be $(= 1.657,709,893 \times a^3 + 22.342,289,740 \times a^3) = 23,999,999,633 \times a^3$; which is exceedingly near to $24 \times a^3$, or $3 \times 8a^3$, or $3 \times 2a^3$, or $3 \times BC^3$, or three times the cube of the given base BC , agreeably to the conditions of the Problem.

But, though the Problem is solved by this determination of the length of the line CH , yet the long equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 9a^{10} = 12,625a^{10}$, deduced from the said Problem, is not yet completely resolved. For this equation has another root greater than CH , or 0.632 , $0.9315 \times a$, and nearly $= 0.791,626,8 \times a$, which relates to another Problem that much resembles the former Problem and produces the very same equation. This Problem may be expressed as follows: "To describe the triangle BaC , (in fig. 4, page 396,) of which the base BC is given, and is equal to 200 feet, and the height DF is also given and is equal to half the base BC , or to 100 feet, (as in the former Problem,) and of which the sides aC and aB are of such lengths that $aB^3 - aC^3$, or the difference of their cubes, (instead of their sum,) shall be equal to three times the cube of the base BC ." For, if the base BC be prolonged to the point b , so that the external line Cb shall be $= 0.791,626,8 \times a$, or $0.791,626,8 \times \frac{BC}{2}$, or $0.791,626,8 \times DF$, and from the point b the right line ba be drawn at right angles to the line Bb , and equal to DF , and from the point a the right lines aB and aC be drawn to the extremities of the given base BC ; the triangle aBC will be the triangle sought, or the excess of the cube of aB above the cube of aC will be equal to three times the cube of the base BC . All these things are fully investigated and proved in Art. 53, 54, 55, &c. . . . 64, pages 381, 382, 383, &c, ... 398, and are well worth the attention of the curious reader, as they afford a remarkable instance of an equation that is equally related to two different Problems, and fitted by one of it's roots to afford a Solution of one of the Problems, and by it's other root to afford a Solution of the other Problem.

As to this second Problem, (which relates to the difference of the cubes of the triangle, required to be found,) I very much doubt whether it can be constructed.

fructed in the same manner as Mr. Glenie has constructed the first Problem, or by drawing only right lines or circles.

The second, or greater, root of the long equation of the 10th power that results from both these Problems, to wit, $0.791,626,8 \times a$, is the root of the equation $360a^2x + 1,404a^3x^2 + 1,640a^4x^3 + 978a^5x^4 + 408a^6x^5 + 124a^7x^6 + 24ax^7 + 3x^8 = 2,525a^8$, which arises from supposing the above-mentioned long compound quantity, consisting of nine terms, to wit, the quantity $-2,525a^8 + 360a^2x + 1,404a^3x^2 + 1,640a^4x^3 + 978a^5x^4 + 408a^6x^5 + 124a^7x^6 + 24ax^7 + 3x^8$, (by means of which, as a divisor, we before reduced the long final equation of the 10th power, derived from Mr. Glenie's Problem, to a quadratick equation,) to be equal to 0. This is shewn in Art. 69, pages 403, 404, and 405.

In Art. 65, page 398, I resume the subject of the reduction of the long final equation of the 10th power, resulting from Mr. Glenie's Problem, to a quadratick equation by means of the operation of division, and shew how some subtleties and difficulties that attended that reduction, (as it was managed above in Art. 33, pages 355 and 356, and as the like reductions of high equations are managed in all the books of Algebra that I have seen,) may be avoided. In making that reduction in Art. 33, the terms of that equation of the 10th power were all brought to the same side of the equation, whereby they became equal to 0, and then this quantity, or, rather, *shadow* of a quantity, that was equal to nothing, was divided by a long compound quantity consisting of nine terms, and produced the trinomial quantity $5aa - 6ax - 3xx$ for the quotient, which is likewise equal to 0. All this is obscure and mysterious, and therefore disgraceful to the Science of Algebra, or Universal Arithmetick, which ought to be remarkable for it's perspicuity. And I therefore endeavoured to find some method of performing the same reduction in a clearer and plainer manner, and have found a method of doing it which appears to me perfectly satisfactory. This method is set forth in Art. 65, and is as follows.

I first add $90ax^9 + 9x^{10}$ to both sides of the final equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, and I thereby obtain the equation

$$16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 \\ - 1,848a^3x^7 - 501a^2x^8 = 12,625a^{10} + 90ax^9 + 9x^{10}.$$

I then subtract $12,625a^{10}$ from both sides, and thereby obtain the equation
 $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 \\ - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 = 90ax^9 + 9x^{10}.$

Now let the long divisor $-2,525a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 \\ + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$ be multiplied into $5aa$, and the product will be the compound quantity $-12,625a^{10} + 1,800a^9x \\ + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 \\ + 15a^2x^8$, in which the literal parts of the several terms are the same as the literal parts of the corresponding terms in the left-hand side of the last equation
 $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 \\ - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 = 90ax^9 + 9x^{10}.$

Let the long divisor $-2,525a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 \\ + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$ be, for the sake of brevity, denoted by the letter D. And the said product of the multiplication of the said long divisor into $5aa$ will be equal to $5aaD$.

Further, let the long compound quantity $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 \\ - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 \\ - 501a^2x^8$, (which forms the left-hand side of the last equation,) be, for the sake of brevity, denoted by the letter A. And we shall then have $A = 90ax^9 \\ + 9x^{10}.$

Let $5aaD$ be subtracted from both sides of this equation, and let the remainder be called R.

And we shall then have $A - 5aaD = -5aaD + 90ax^9 + 9x^{10}$, or $R = -5aaD + 90ax^9 + 9x^{10}.$

Now, since $A - 5aaD$ is R , we shall have $A = R + 5aaD.$

But A is $= 90ax^9 + 9x^{10}.$

Therefore

Therefore $R + 5aaD$ will be $= 90ax^9 + 9x^{10}$, and consequently $5aD$ will be $= -R + 90ax^9 + 9x^{10}$.

But R (which is $= A - 5aD$), is $= -12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,849a^3x^7 - 501a^2x^8 + 12,625a^{10} - 1,800a^9x - 7,020a^8x^2 - 8,200a^7x^3 - 4,890a^6x^4 - 2,040a^5x^5 - 620a^4x^6 - 120a^3x^7 - 15a^2x^8 = 15,150a^9x + 5,415a^8x^2 - 9,504a^7x^3 - 14,052a^6x^4 - 10,788a^5x^5 - 5,382a^4x^6 - 1,968a^3x^7 - 516a^2x^8$.

Therefore $-R + 90ax^9 + 9x^{10}$ will be $= -15,150a^9x - 5,415a^8x^2 + 9,504a^7x^3 + 14,052a^6x^4 + 10,788a^5x^5 + 5,382a^4x^6 + 1,968a^3x^7 + 516a^2x^8 + 90ax^9 + 9x^{10}$.

Therefore $5aD$ (which is equal to $-R + 90ax^9 + 9x^{10}$), will also be equal to the compound quantity $-15,150a^9x - 5,415a^8x^2 + 9,504a^7x^3 + 14,052a^6x^4 + 10,788a^5x^5 + 5,382a^4x^6 + 1,968a^3x^7 + 516a^2x^8 + 90ax^9 + 9x^{10}$.

Now let both sides of this equation be divided by D , or the compound quantity $-2,525a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$.

And we shall have $5a = 6ax + 3xx$, and consequently $xx + 2ax = \frac{5aa}{3}$. And thus the foregoing long equation of the 10th power, derived from Mr. Glenie's Problem, is reduced to the quadratick equation $xx + 2ax = \frac{5aa}{3}$ by intelligible operations performed with finite quantities, without any mention of quantities equal to nothing.

I have seen two other Algebræick Solutions of Mr. Glenie's Problem that produce equations of the fifth order instead of producing the above-mentioned equation of the tenth order; one of which Solutions was communicated to me by the Reverend Mr. George Walker, of Manchester, (the author of a new treatise on the Conick Sections, in quarto, published about three or four years ago,) and the other by Mr. John Playfair, Professor of Mathematicks in the University

University of Edinburgh. But I have not seen any Solution of it that produces immediately, (or without such a reduction, by means of the operation of division, as has been above-described,) a quadratick equation (such as the equation $xx + 2x = \frac{500}{3}$;) and consequently leads us to a geometrical construction of the Problem, to be performed by drawing only right lines and circles, in the manner required by Mr. Glenie.

This fifth Tract, concerning Mr. Glenie's Problem, has, it must be confessed, no connection with the subject of Logarithms, and no pretensions therefore to be admitted into a Collection of Tracts intitled *Scriptores Logarithmici*. But, as this Collection has run into a greater number of volumes than was originally intended, I have taken the liberty of inserting in it some Tracts upon other Mathematical subjects that contained matters that I thought might be interesting to the lovers of these sciences.

The sixth Tract in this Volume has a better right to a place in it, having a near relation to the subject of Logarithms. It is intitled, *An important Proposition concerning the Asymptotick Areas of an Equilateral Hyperbola; from which it follows that they are Logarithms, or measures, of the ratios of the ordinates that bound them. Demonstrated by the late Dr. William Ridlington, L. L. D. formerly Fellow of Trinity Hall in the University of Cambridge, and Professor of Civil Law in the same University.*

This Tract begins in page 415, and ends in page 420.

The seventh Tract in this Volume consists of two Problems concerning the variations of the ratio, or proportion, of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$, upon a supposition that a denotes a line of a given magnitude, and that x denotes a variable line that increases continually from 0 *ad infinitum*. This inquiry, concerning the variations of this proportion of $aa + 2ax + xx$ to $aa - ax + xx$ while x increases from 0 *ad infinitum*, was suggested to me by the Reverend Mr. John Hellins, B. D. Vicar of Potter's Pury in Northamptonshire, in consequence of an answer given to a question that had been proposed by him in the Ladies' Diary for the year 1794, in the words following:

“ XIII.

" XIII. Question 981, by the Reverend Mr. John Hellins.

" It is proposed to find the correct value of z from the equation $\dot{z} = \frac{z}{1+x^2}$,
 "(where x and z begin together,) in series which shall converge when x is
 " greater than 1 ; and to perform the whole by means of series."

Two answers were given to this question in the Diary for the year 1795 ; one by Mr. Hellins himself, and the other by another person under the signature of *Amicus*. - It is in the latter answer that the variable quantity $\frac{aa+2ax+xx}{aa-ax+xx}$, (which gave rise to the present Tract,) was made use of. I, however, had not seen it there, and had no knowledge of it till it was mentioned to me by Mr. Hellins. The question proposed by Mr. Hellins is not a mere idle speculation designed to try the acuteness and sagacity of Mathematicians, but occurred, as he informed me, in the Solution of a curious and useful Problem in Physical Astronomy. And the inquiry contained in the present Tract into the variations of the proportion of $aa+2ax+xx$ to $aa-ax+xx$ during the increase of x from 0 *ad infinitum*, has also this further use in it, that it seems very well fitted to illustrate the principles of the curious and amusing, but oftentimes subtle and difficult, doctrine of finding the *maximums* of variable quantities, and particularly of fractions, in which the variable quantity enters into both the numerator and the denominator. This Tract begins in page 423, and ends in page 434.

After this 7th Tract I have re-printed the whole of Monsieur Pascal's works relating to Arithmetick and Algebra, from the last edition of his works published at the Hague in the year 1779. These works are so full of Genius and Invention, and are written in so clear and pleasant a style, that I thought I should do a service to the Mathematicians of Great-Britain (where Mr. Pascal's works are not often met with,) by re-publishing them in this Collection. And some of them, and more especially his *Arithmetical Triangle*, have a considerable connection with Logarithms by affording a good demonstration of Sir Isaac Newton's Binomial Theorem in the case of integral and affirmative powers, which is of great use in the construction of Logarithms, as has been shewn in the two first Volumes of this Collection. These extracts from Pascal's works extend from page 437 to page 570.

In

In one of these Tracts of Monsieur Pascal there are some remarks on the Gravity of bodies near the surface of the earth, or their tendency to fall towards it in right lines passing through it's center, which deserve particular notice. They occur in pages 467, 468, 469, and 470, of this Volume, in a Letter written jointly by Mr. Pascal and Mr. Roberval (the learned Professor of Mathematicks at Paris,) in the year 1636, to Monsieur Fermat, (a Counsellor of the Parliament of Toulouse and one of the greatest Geometricians of that time,) concerning a certain principle, or maxim, relating to the fall of heavy bodies, which he was inclined to lay down as perfectly evident and incontestable, and to make the ground-work of the Solution of all questions that might be proposed on that subject. But this principle Mr. Pascal and Mr. Roberval refused to assent to, on account of their ignorance, not only of the cause, but even of the nature, of Gravity. Upon this subject they express themselves in the following manner, in page 467. *La diversité des opinions touchant l'origine de la pesanteur des corps, (des quelles aucune n'a été jusqu'ici, ni démontrée, ni convaincue de fausseté par démonstration,) est un ample témoignage de l'ignorance humaine en ce point. La commune opinion est, que la Pesanteur est une qualité qui réside dans le corps même qui tombe. D'autres sont d'avis que la descente des corps précède de l'attraction d'un autre corps qui attire celui qui descend, comme le globe de la terre paroît attirer une pierre qui tombe. Il y a une troisième opinion, qui n'est pas hors de vraisemblance; " que c'est une attraction mutuelle entre les corps, causée par un désir naturel que ces corps ont de s'unir ensemble; comme il est évident au fer et à l'aimant; les quels sont tels, que, si l'aimant est arrêté, le fer, ne l'étant pas, ira le trouver; et, si le fer est arrêté, l'aimant ira vers lui; et, si tous deux sont libres, ils s'approcheront réciproquement l'un de l'autre; en sorte toutefois que le plus fort des deux fera le moins de chemin."* That is, in English, "The difference of the opinions that are maintained by different persons concerning the origin, or cause, of the gravity of bodies (of which opinions no one has hitherto been shewn with certainty to be either true or false,) is an ample proof of the ignorance of mankind on this subject. The most common opinion is, that Gravity is a quality residing, or inherent, in the body itself that falls to the ground. Others are of opinion, that the descent of bodies is owing to their being attracted by other bodies towards which they fall, or that the latter bodies draw the former towards them, as the globe of the earth seems to draw a stone that falls towards it. And there is a third opinion entertained by some persons, which is not without some appearance of probability;

" which

" which is, that Gravity is owing to a mutual attraction between two bodies, caused by a natural desire in the two bodies to meet and unite together; which is evidently the case with a load-stone and a piece of iron: for in these it is constantly seen, that, if the load-stone is held fast and prevented from moving, and the piece of iron is at liberty to move, the iron will move towards the load stone; and, if the iron is held fast and prevented from moving, and the load-stone is loose and free to move, the load-stone will move towards the iron; and, if both the load-stone and the iron are loose and free to move, they will both of them move towards each other, but not through equal spaces, the stronger of the two going through a smaller space than the other."

These were the opinions that were entertained in France in the year 1636 concerning the Gravity of bodies. The third, or last, of these opinions, to wit, " that Gravity is owing to the mutual attraction of bodies," is that which Sir Isaac Newton has since adopted, and made the foundation of his manner of explaining the motions of the heavenly bodies; and most Philosophers of the present age accede to this opinion, and to the important consequences which Sir Isaac has deduced from it in Astronomy. But we see by this passage, that in the year 1636 this opinion concerning Gravity was less common than either of the two former, and particularly than the first of them.

There is another passage in this Letter of Monsieur Pascal and Monsieur Roberval, which likewise deserves particular attention. It is in page 470 of the present Volume, in these words. *Pour ces considérations, dans nos conférences de Méchanique, nous appellons des poids égaux, ou inégaux, ceux qui ont égale, ou inégale, puissance de se porter vers le centre commun des choses pesantes; et nous entendons un même corps avoir un même poids, quand il a toujours cette même puissance: que, si cette puissance augmente ou diminue, alors, quoique ce soit le même corps, nous ne le considérons plus comme le même poids. Or, que cela arrive, ou non, aux corps qui s'éloignent ou s'approchent du centre commun des choses pesantes, c'est chose que nous désirerions bien de savoir: mais, ne trouvant rien qui nous satisfasse sur ce sujet, nous laissons cette question indéciée, raisonnant seulement sur ce que les Anciens et nous avons pu découvrir de vrai jusqu'à maintenant.* That is, in English, " For these reasons, in our conferences with each other on the principles of Mechanicks, we consider those bodies as equally, or unequally, heavy, and call

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"them equal, or unequal, weights, which have an equal, or unequal, tendency,
 "or power, to move towards the common center of heavy bodies; and we say
 "that the same body has *the same weight* only so long as it has the same
 "tendency, or power, to move towards this common center: so that, if this
 "tendency, or power, in any body, to move towards this common center,
 "should, from any circumstance, be increased or diminished, we should, in
 "such case, though the body itself would still continue to be the very same
 "body that it was before, no longer call it *the same weight*. Now, "whether,
 "or no, any change in this tendency, or power of moving, towards such
 "common center, happens to bodies in consequence of their approaching
 "towards, or receding from, the said common center of heavy bodies," is
 "what we do not know, but should be very desirous of knowing, if we could
 "find any method of discovering it: but, as we have not hitherto been able
 "to find any such method that has appeared to us clear and satisfactory, we
 "leave this matter undecided, and, in all our inquiries concerning questions of
 "Mechanicks, we form our reasonings on such grounds and principles only as
 "the Antients and ourselves have been able to discover with certainty to be
 "agreeable to truth."

In this passage we have a plain confession that these two very learned men, Monsieur Pascal and Monsieur Roberval, though they seemed to suspect that the Gravity of bodies might be different at different distances from the common center of heavy bodies, that is, from the center of the Earth, and consequently that, if a body were to be removed to a considerable height above the surface of the Earth, (as, for example, to the height of three, or four, semidiameters of the Earth above it's surface,) the space through which it would fall in a line tending to the center of the Earth in a given portion of time, (as, for example, in a second of time,) might be greater, or less, than the space through which it would fall in the same time near the surface of the Earth, yet could discover no means of ascertaining in a satisfactory manner whether there was, or was not, any such variation in it. But this discovery, which those learned and ingenious Mathematicians so much wished to make, has been since made by the sagacity of Sir Isaac Newton, which led him to consider the Moon as being a body that is either attracted, or impelled, towards the center of the Earth by the same power (whatever that power be,) by which a stone near the surface of the Earth falls to the ground, and which is called *Gravity*, and to inquire, what would be the velocity with

with which the Moon would move in her orbit, or describe a circle round the center of the Earth, and consequently what would be the time in which she would describe such a circle, if the gravity of a body at that distance from the Earth was the same as it is at the surface of the Earth, and then to compare the said velocity and time with the velocity and time that really belong to the Moon's motion. Now these inquiries may be made, by means of the doctrine of the Motion of falling bodies laid down by *Galileo* in his 3d and 4th Mechanical Dialogues, and by *Mr. Huygens* in his excellent treatise on Pendulum-Clocks, intitled *De Horologio Oscillatorio*, and of the thirteen Theorems, in the last part of that excellent treatise in the first edition of it at Paris in the year 1673, concerning the centrifugal forces of bodies revolving in a circle, by reasoning in the manner following.

If a stone, or a cannon-ball, or any other body, were to be projected from any place near the surface of the Earth in a direction parallel to the horizon, or at right angles to a line drawn through the said body to the center of the Earth, with any such force as human art could employ for that purpose, and were to meet with no obstacles in it's passage, it would move in the arch of a curve that would be concave towards the Earth, till it came to the ground; and it's range (or the distance to which it would fly before it came to the ground, measured on the surface of the Earth,) would be greater, or less, according to the force with which it had been projected. And, if this force could be increased at pleasure, the said body might be projected with so great a velocity that, if there were no resistance of the air to it's motion, and it met with no obstacles in it's passage (as would be the case if it were projected horizontally from a building erected on the top of the highest mountain in the world, where-ever that mountain may be,) it would never fall to the ground at all, but would move round and round the Earth for ever and ever in a circle at the same distance from the surface of the Earth as that at which it was first projected. This velocity would be a prodigiously great one, and seems at first sight difficult to be discovered. But *Mr. Huygens*, in the 5th of those thirteen Theorems at the end of his *Horologium Oscillatorium* which we just now mentioned, relating to the centrifugal forces of bodies revolving in circular orbits, has taught us how to find it. For he there informs us that it is the velocity which a heavy body would acquire by falling perpendicularly downwards towards the surface of the Earth through a line equal to a fourth part of the diameter of the circle that is to be described. Now this circle (as, on the present occasion, we shall suppose

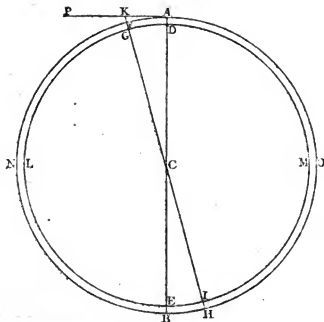
the Earth to be an exact sphere, though, in consequence of it's diurnal rotation round it's axis, the diameter of the equator is, in a small degree, greater than it's axis, and it's figure therefore deviates a little from that of a perfect sphere,) will be a little greater than the equator, or any other great circle of the Earth, because the radius of it is greater than the radius of the Earth by the height of the mountain from the top of which the body is supposed to be projected. But this height, even in the highest mountains on the Earth, is not more than 5 miles, which is but a small quantity in comparison to the radius of the Earth, which is near 4000 miles. And therefore we may, without deviating materially from the truth, suppose this circle to be equal to a great circle of the Earth, and it's diameter to be equal to the diameter of the Earth, or to 8000 miles. And then the fourth part of the diameter of this circle will be 2000 miles, (or 2000 times 5280 feet,) or 10,560,000 feet. And consequently the requisite velocity will be that which would be acquired by a heavy body in falling perpendicularly downwards towards the surface of the Earth through a line equal to 10,560,000 feet, upon a supposition that the force of gravity by which the said body was drawn downwards to the surface of the Earth through the said line of 10,560,000 feet was, throughout the whole time of the said fall, as great as at the end of it, when the body had arrived at the surface of the Earth. Now the force of gravity at the surface of the Earth is such, according to Mr. Huygens's estimation of it, as would make a body fall through 15 feet and one inch, or 181 inches, of Paris measure, in a second of time, if the body met with no resistance from the air; or (because the Paris foot is greater than the English foot in the proportion of 1068 to 1000,) the force of gravity is such as would make a body fall, if it met with no resistance from the air, through $\left(\frac{1068}{1000} \times 181\right)$ English inches, or $\frac{193,308}{1000}$ English inches, or 193.308 inches, or 194 inches + 1.308 inches, or 16 feet + 1.308 inches, or 16 feet + $\frac{7.308}{12}$ feet, or 16 feet + 0.109 of a foot, or) 16.109 English feet. Now the velocity acquired by falling through a line of 10,560,000 feet is to the velocity acquired by falling through a line of 16.109 feet in the proportion of the square-root of 10,560,000 to the square-root of 16.109, that is, in the proportion of 3249.6153 to 4.0136, (or of $\frac{3249.6153}{4.0136}$ to $\frac{4.0136}{4.0136}$), or of 809.65 to 1. But the velocity acquired by a heavy body in falling through a line of 16.109 feet is such as, if it continued uniform, would carry it through twice that

that length, or through ($2 \times 16,109$ feet, or) $32,218$ feet, in a second of time. Therefore the velocity acquired by falling through a line of $10,560,000$ feet, or 2000 miles, would carry it (through $809.65 \times 32,218$ feet, or) through $26,085,303.70$ feet, or through ($\frac{26,085,303.70}{5280}$ miles, or) $4,940,398$ miles, in a second of time. Therefore the whole circumference of the Earth, which is about $25,000$ English miles, would, with this velocity, be described in a portion of time that would be greater than 1 second of time in the same proportion as $25,000$ miles is greater than $4,940,398$ miles, or in a portion of time equal to $\frac{25,000,000,000}{4,940,398}$ seconds, or to 5060 seconds, (or to $\frac{5060}{60}$ minutes,) or to 84 minutes and 20 seconds, or to an hour, and 24 minutes, and 20 seconds. Therefore the body that would have been projected horizontally from the top of a high mountain with the velocity acquired by falling perpendicularly through a line of 2000 miles, or $10,560,000$ feet, or with the velocity of $4,940,398$ miles in a second of time, would move round the Earth for ever and ever in a circle, and would perform one revolution in the space of an hour, and 24 minutes, and 20 seconds.

Q. E. I.

And "that a body projected in a horizontal direction from the top of a high mountain with this velocity of $4,940,398$ miles in a second of time, would move thus in a circle round the Earth without ever coming to the ground," may likewise be proved synthetically in the following manner.

With C as a center, and CD as a radius, describe the circle $CDLEM$ representing a great circle of the Earth. And let the radius CD be produced to the point A ; so that the little line DA may represent the height of some very high mountain, from which a body is supposed to be projected in the direction of the line AP , which is at right angles to the line CA ; and let the velocity with which the said body is supposed to be projected in the said line AP be such as, if it continued uniform, would carry it through $4,940,398$ miles in a second of time. With C as a center, and CA as a radius, describe another circle $ANBO$. And in the line AP (which, it is evident, will be a tangent to this second, or outer, circle in the point A .) take the line AK equal to $4,940,398$ miles. And from the point K , through the center C , draw the right line KCH cutting the outer circle in the points F and H , and the inner circle in the points G and I .



DEMONSTRATION.

Since the body at A is projected in the direction AP with a velocity of 4.940,398 miles in a second of time, and the line AK is taken equal to 4.940,398 miles, it is evident that, if the force of gravity did not act upon the body, and draw it out of the right line AK, the said body would, in the course of a second of time, describe the right line AK, and arrive at the point K. Now the point K is more distant than the point A from C, the center of the two circles, or the center of the Earth, by the little line KF, by which CK, or the secant of the arch AF, exceeds the radius CF, or, it's equal, the radius CA. Therefore the said motion of the body projected from the point A along the line AK would have caused it to recede from the center C, so as to have increased it's distance therefrom by the little line KF, though the said receds would

would not have been made in a direct line from the center C, or in the line CA continued upwards beyond the point A, but in a lateral, or oblique, direction in consequence of it's motion along the right line, or tangent, AK. And this tendency of the projected body to recede from the center C in consequence of the velocity impressed in the direction AP, or AK, (and which, if the action of gravity upon the body was taken off, would make it actually recede from the said center, or increase it's distance from it, by the little line KF,) is what Mr. Huygens calls it's *centrifugal force*. And other writers since Mr. Huygens's time (who was, I believe, the first person that considered the motions of bodies revolving in circular orbits round a given point, or, at least, the first person who published any thing concerning them,) have adopted the same language, and given the same name of *centrifugal force* to this tendency of a projected body to recede from a given point, which necessarily results from the projectile force impressed upon it which urges it to move in a right line, if it is not restrained, or drawn aside, by some other force.

Now the quantity of this recess, or increase of distance, of the said body projected at A, from C the center of the Earth, in a second of time, or while the said body (when the action of gravity is taken off, or suspended,) describes the line AK, is the little line KF. But, (by Euclid's Elements, Book III, Prop. 36,) this little line KF is equal to $\frac{AK^2}{KH}$. But, because KF is extremely small in comparison to FH, we may consider KH, or KF + FH, as being only equal to FH, and consequently KF (which is equal to $\frac{AK^2}{KH}$,) as being equal to $\frac{AK^2}{FH}$. And, because FH (the diameter of the outer circle,) is very little greater than GI (the diameter of the inner circle, or the diameter of the Earth,) we may consider $\frac{AK^2}{FH}$ as being equal to $\frac{AK^2}{GI}$, and consequently KF as being also equal to $\frac{AK^2}{GI}$.

Now AK is = 4.940,398 miles (= 5280 × 4.940,398 feet) = 26,085,301,440 feet; and GI, or the diameter of the Earth, is = 8000 miles
(or

(or 8000×5280 feet,) or 42,240,000 feet. Therefore $\frac{AK^2}{GI}$ will be ($=$
 $\frac{26,085,301,440^2}{42,240,000}$ feet $=$ $\frac{680,442,951,215,666,273,600}{42,240,000}$ feet,) $= 16,1089$ feet, or, very

nearly, 16.109 feet. Therefore the little line KF will be $= 16.109$ feet; that is, the projectile velocity of 4 940,398 miles in a second of time, will, on this supposition of the suspension of the action of gravity upon the projected body, have caused it to recede from C, the center of the Earth, through the distance of 16.109 English feet in the course of a second of time. But the action of gravity would, if the body were at rest in the point A, instead of being impelled by a projectile force in the direction AP, cause it to move directly downwards in the line AD, or AC, towards C, or the center of the Earth, through the very same space of 16.109 feet in a second of time. Therefore, when the body is acted upon in the point A both by the force of gravity drawing it downwards towards the center C, and the projectile force impelling it in the direction AP, or AK, with a velocity of 4 940,398 miles in a second of time, it will, at the end of the said second of time, be situated in the point F, or at the extremity of the radius CF, and therefore will be at the same distance from the center C as it was at the beginning of it's motion, or when it was in the point A; that is, it will neither have approached towards the said center C, nor have receded from it, in the course of the said second of time. And therefore, with this velocity of 4.940,398 miles in a second of time, and the centrifugal force thence arising, (which tends to make it recede from the center C in such a degree as to increase it's distance therefrom by the space of 16.109 feet in a second of time, while the force of gravity tends to make it approach towards the said center through exactly the same space in the same second of time,) it must continue always at the same distance from the said center C, and must describe a circle round the said center C, of which the radius will be CA, and must, if not acted upon by any new force, continue it's revolutions in the said circle for ever and ever, in the same manner as the Moon does. Q. E. D.

The force of gravity by which the said body is continually drawn downwards towards C, the center of the Earth, and prevented from flying out of the circle in which it moves, in a right line, as AP, that is a tangent to the circle it describes, is often called it's *centripetal force*, as the tendency it has to recede from the center C, in consequence of it's projectile velocity in the line AP, is called

called it's *centrifugal force*. And therefore in the motion of the said body in the said circular orbit, of which CA is the radius, the *centripetal* and *centrifugal* forces are exactly equal to each other. And so they always must be when bodies revolve in circles round a point towards which they are attracted.

Now let us suppose a body to be carried to the distance of the Moon from the Earth, that is, to the distance of 60 semi-diameters of the Earth, or of 60 times 4000 miles, or of 240,000 miles, from the center of the Earth; and let us suppose the force of gravity, by which the said body would be drawn downwards towards the center of the Earth, to be the same as it is on the surface of the Earth, or such as to make a body fall from rest perpendicularly downwards through a line of 16.109 English feet in a second of time. And let us inquire, by means of the same 5th Theorem laid down by Mr. Huygens at the end of his *Horologium Oscillatorium*, what would be the velocity with which the body ought to be projected, in a line forming a right angle with a line drawn from the body to the center of the Earth, in order that the centrifugal force of the said body, arising from the said velocity, or generated by it, should be exactly equal to it's centripetal force, or the gravity which draws it downwards towards the center of the Earth, (and which, if the projectile force were destroyed, would make it fall through 16.109 feet in a second of time,) and consequently that the said body should be made to revolve round the center of the Earth for ever and ever in a circle of which the radius would be 60 semi-diameters of the Earth, or 240,000 miles.

Now, according to that 5th Theorem of Mr. Huygens, this velocity would be that which would be acquired by a body in falling perpendicularly downwards towards the Earth through a line equal to the fourth part of the diameter of the circle that was to be described, that is, to a fourth part of 480,000 miles, or to 120,000 miles, (or to 120,000 times 5280 feet,) or to 633,600,000 feet. But the velocity acquired in falling through 633,600,000 feet is to the velocity acquired in falling through 16.109 feet as the square-root of 633,600,000 is to the square-root of 16.109, that is, as 25,171.4123 is to 4.0136, (or as $\frac{25,171.4123}{4.0136}$ is to $\frac{4.0136}{4.0136}$) or as 6271.5 is to 1. And the velocity acquired by falling through 16.109 feet is a velocity that, if uniformly continued, would carry a body through (twice 16.109 feet, or) 32.218 feet in a second of time.

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Therefore

Therefore the velocity acquired by falling through 633,600,000 feet would carry a body through $(6271.5 \times 32.218 \text{ feet, or } 202,055.1870 \text{ feet in a second of time. Therefore, if a body at the distance of the Moon, or of 60 semi-diameters of the Earth from the center of the Earth, was to be projected, in a right line at right angles to a line drawn from the said body to the center of the Earth, with a velocity of } 202,055.1870 \text{ feet in a second of time, the centrifugal force of the said body would be equal to it's centripetal force, or gravity, and the said body would consequently revolve round the Earth for ever and ever, in a circle of which the radius would be it's first distance from the center of the Earth, or } 240,000 \text{ miles.}$

Q. E. I.

We will next inquire, "in what time the said body, so projected, would describe the said circle, or perform one revolution round the Earth." Now this time may be easily found in the following manner.

Since the diameter of this circle is 480,000 miles, (or 480,000 times 5280 feet,) or 2,534,400,000 feet, it's circumference will be $= (\frac{355}{113} \times 2,534,400,000 \text{ feet, or } \frac{809,712,000,000}{113} \text{ feet, or } 7,962,053,097 \text{ feet. But the velocity with which this body moves will carry it through } 202,055.1870 \text{ feet in a second of time. Therefore it will carry it through } 7,962,053,097 \text{ feet, or the whole circumference of the circle, in a portion of time that will be greater than 1 second in the same proportion as the whole circumference, consisting of } 7,962,053,097 \text{ feet, is greater than } 202,055.1870 \text{ feet, that is, (in } \frac{7,962,053,097}{202,055.1870} \text{ seconds, or) in } 39,405 \text{ seconds, (or in } \frac{39,405}{60} \text{ minutes, or in } 656 \text{ minutes and } 45 \text{ seconds, or in } \frac{656}{60} \text{ hours and } 45 \text{ seconds,) or in } 10 \text{ hours, } 56 \text{ minutes, and } 45 \text{ seconds. Therefore the said body would, with this velocity of } 202,055.1870 \text{ feet in a second of time, describe the whole circumference of the circle, or perform one revolution round the Earth, in the short period of } 10 \text{ hours, } 56 \text{ minutes, and } 45 \text{ seconds.}$

Q. E. I.

Here again we may demonstrate synthetically (as we did before in the case of a body projected horizontally from the top of a high mountain on the Earth,)

Earth,) "that a body projected horizontally with this velocity of 202,055.1870 feet in a second of time, from a place at the distance of 60 semi-diameters of the Earth, or 240,000 miles, from the center of the Earth, would have it's centrifugal force exactly equal to it's gravity, or centripetal force, and consequently would move for ever and ever in a circle round the Earth, without ever falling upon it or approaching nearer to it." This demonstration may be as follows.

DEMONSTRATION.

Since the circumference of every circle contains 360 degrees, and every degree contains 60 minutes, and every minute, or first minute, contains 60 seconds, or second minutes, it follows that the circumference of every circle will contain $360 \times 60 \times 60$ second minutes, (or $21,600 \times 60$ second minutes,) or 1,296,000 second minutes. Now the whole circumference of the circle described by the body projected in the manner above-mentioned, is run-over by the said body in 39,405 seconds of time. Therefore the number of second minutes of a degree in the circumference of the said circle that will be gone-over by the said body in one second of time will be ($\frac{1,296,000}{39,405}$, or) 32.889 second minutes of a degree, which is but little more than half a first minute of a degree, (or a 120th part of one degree, or of the 360th part of the whole circumference,) or than the 43,200th part of the whole circumference. This is a very small portion of the whole circumference: and in these very small circular arches the tangents of the arches may be considered as equal to the arches, (the differences between them being extremely small in comparison of either the arches or the tangents,) and the excess of the secant of any such small arch will be very nearly equal to the square of either the tangent, or the arch, divided by the diameter of the circle. Therefore the excess of the secant of this little arch of 32.889 second minutes of a degree, which is run-over by the said body in a second of time, will be equal to the square of the said arch, or of it's tangent, divided by the diameter of the circle, that is, to the square of 202,055.1870 feet divided by 2,534,400,000 feet, (or to $\frac{202,055.1870^2}{2,534,400,000}$ feet, or

to $\frac{49,816,298,593.604,769,00}{2,534,400,000}$ feet,) or to 16,1038 feet, or, very nearly, 16,109 feet. Therefore the centrifugal force of the said body, arising from it's projectile velocity of 202,055,1870 feet in a second, is such as, if the action of gravity had been suspended, and the body had consequently moved on uniformly during a second of time in a tight line that would have been a tangent to it's orbit, would have caused the said body, in the said second of time, to recede from the center of the earth, or increase it's distance from it, by the space of 16,109 feet; which is the very distance through which, on the other hand, it would have been made to move directly downwards by the force of gravity, or it's centripetal force, if it had not received any projectile impulse. Therefore, when it is acted upon both by it's gravity and it's projectile impulse, it's centrifugal force will be equal to it's centripetal force, or gravity, and it consequently will continue always at the same distance from the center of the Earth, and revolve round the Earth in a circle for ever and ever. Q. E. D.

We may further observe, that the motion of this body, and the time of it's performing a revolution in a circle round the Earth, will continue the same, whether the said body be a small body or a great one. For, if the body were to be made double, or triple, or quadruple, or any other multiple, of what it was before, so as to weigh twice, or three times, or four times, or any number of times denoted by the letter *m*, as much as it did before, it's centrifugal force would, at the same time, become double, or triple, or quadruple, or *m* times as great as it was before; and consequently the equality between the centrifugal force of the body and it's gravity, or centripetal force, would still be preserved. And therefore, if the said body were to be made equal to the Moon, and it were to be projected horizontally with the aforesaid velocity of 202,055,1870 feet in a second of time; it would perform a revolution in the same period as before, to wit, in the course of 10 hours, 56 minutes, and 45 seconds. And, if the velocity with which it was projected was less than that of 202,055,1870 feet in a second of time, it's centrifugal force (which arises from it's velocity,) would be less than it's gravity, or centripetal force, and it would fall towards the Earth.

This period of 10 hours, 56 minutes, and 45 seconds would be the time of one revolution of the aforesaid body (which was supposed to be carried to the distance

distance of the Moon, or of 60 semi-diameters of the Earth from the center of the Earth, and there projected in a line at right angles to the line drawn from the said body to the center of the Earth,) if the force of gravity towards the Earth continues the same at that distance from the Earth as it is at the Earth's surface. But, if the force of gravity decreases when the distance of the body from the center of the Earth increases, the centrifugal force, requisite to balance the gravity, or centripetal force, of the body, will be less than in the former case; and consequently the velocity of the said body in it's orbit (from which the said centrifugal force arises,) will also be less, and the time of performing a revolution will be proportionally longer, than in the former case. And, if it were possible to human art to carry a body to that distance from the Earth, and there to project it, in the manner above-described, with a sufficient velocity to cause it to move in a circle round the Earth, and this experiment had been tried, and it had been found that a much less velocity than that of 202,055.1870 feet in a second of time had been sufficient for this purpose, and consequently that the time in which the said body, moving thus in a circle round the Earth, would perform a single revolution, would be much longer than 10 hours, 56 minutes, and 45 seconds, it might be justly concluded, *à converſo*, from these circumstances, "that the force of gravity at that distance from the Earth was not the same as at the surface of the Earth, but very much less." Now Sir Isaac Newton thought that the Moon was a body whose relation to the Earth and whose velocity in her orbit (which is not very different from a circular orbit,) and time of performing her revolutions, might answer the same purpose as such an experiment, which it is beyond the reach of human power to make. For he thought it reasonable to suppose that the force of gravity, by which all bodies here on the surface of the Earth fall to it's surface in lines tending to it's center, (whatever may be the cause of it, which is hitherto perfectly unknown,) might extend to the distance of the Moon, and be the cause of the Moon's constantly moving round the Earth, instead of going off in a right line that was a tangent to her orbit, according to the natural tendency of all bodies in motion, to continue to move in the same direction, or right line, with the same degree of velocity for ever and ever, if they are not restrained from doing so by some force that compels them to change their direction. Here then, if we suppose with Sir Isaac Newton that the Moon is acted-upon by gravity towards the Earth, and is thereby retained in her orbit, and, if we consider her orbit as being circular, (which does not differ greatly from the truth,) we may apply the

foregoing reasonings to the Moon, and consider her as a body which has been projected in the manner above-described, by the Creator of the world, at the aforesaid distance of 60 semi-diameters of the Earth from the Earth's center. And then we must observe that her time of performing one revolution round the Earth, instead of being only the short period of 10 hours, 56 minutes, and 45 seconds, is no less than 27 days, 7 hours, and 43 minutes, and consequently that her velocity is less than the former velocity of 202,055.1870 feet in a second of time, in the same proportion as the former short period of 10 hours, 56 minutes, and 45 seconds, is less than the Moon's period of 27 days, 7 hours, and 43 minutes, and therefore that her centrifugal force (which arises from her velocity in her orbit,) must be very much less than the centrifugal force of the former body, which was sufficient to make it recede from, or increase it's distance from, the center of the Earth in a second of time by a line of 16.109 feet, or the space through which a body falls perpendicularly downwards in the same time by the force of gravity on the surface of the Earth. Yet, since the Moon revolves in a circle round the Earth, her centripetal force, or her gravity towards the Earth, must be equal to, and not greater than, her centrifugal force: for, if it were greater than her centrifugal force, it would cause the Moon to fall towards the Earth. It follows therefore that the centripetal force of the Moon, or her gravity towards the Earth, must be much less than the centripetal force of the former body, or than the force of gravity on the surface of the Earth, by which bodies are made to fall perpendicularly downwards through 16.109 feet in a second of time. Therefore, if gravity towards the Earth is the cause that retains the Moon in her orbit, we may conclude that the force of gravity is much less at the distance of the Moon from the center of the Earth, or at the distance of 60 semi-diameters of the Earth from the said center, than it is at the surface of the Earth, or at the distance of one semi-diameter of it from it's center.

Q₂ E. I.

Having thus discovered that the gravity of bodies towards the Earth at the distance of the Moon, or of 60 semi-diameters of the Earth from the Earth's center, is very much less than at the surface of the Earth, it is natural to inquire in the next place, "how much less it is" or what proportion these different degrees of gravity bear to each other. Now this may be easily determined in the following manner.

The

The Moon performs one revolution round the Earth in the space of 27 days, 7 hours, and 43 minutes. Now 27 days, 7 hours, and 43 minutes, are equal to (27 times 24 hours together with 7 hours and 43 minutes, that is, to 648 hours and 7 hours and 43 minutes, or to 655 hours and 43 minutes, or to 655 times 60 minutes together with 43 minutes, or to 39,300 minutes together with 43 minutes, or to) 39,343 minutes, (or to 39,343 times 60 seconds,) or to 2,360,580 seconds. Therefore in one second of time the Moon performs the 2,360,580th part of a revolution, or goes over the 2,360,580th part of her orbit. But her orbit is = 7,962,053,097 feet. Therefore in a second of time the Moon moves through the 2,360,580th part of 7,962,053,097 feet, that is, through 3372.9 feet. Therefore the excess of the secant of the little arch of the Moon's orbit which is described in a second of time, (and which we have found to be equal to 3372.9 feet,) above the radius of her orbit, will be equal to the square of the said little arch divided by the diameter of the Moon's orbit, that is, it will be equal to

$$\frac{3372.9 \times 3372.9}{2,534,400,000} (= \frac{11,376,454.41}{2,534,400,000} = 0.004,488,8) \text{ or } \frac{44,888}{10,000,000} \text{ parts of a foot.}$$

Therefore the centrifugal force of the Moon is such as would, if the action of gravity were to be suspended, cause her to recede from the center of the Earth, or to increase her distance from the said center by only 44,888 ten-millionth parts of a foot in a second of time, instead of receding (as in the former case,) through 16.109 feet in the same time. Therefore the centripetal force of the Moon, or her gravity towards the center of the Earth, (which counter-balances her centrifugal force, and causes her to move round the Earth in a circle,) must be only such as would, if she had no projectile motion, cause her to fall towards the center of the Earth through 44,888 ten-millionth parts of a foot in a second of time. Therefore the force of gravity at the distance of the Moon, or of 60 semi-diameters of the Earth from the Earth's center, is to the force of gravity on the surface of the Earth, or at the distance of only one semi-diameter from it's center, as $\frac{44,888}{10,000,000}$ of a foot is to 16.109 feet, or to 16.109,000,0,

$$\text{or } \frac{161,090,000}{10,000,000} \text{ feet, or as } 44,888 \text{ is to } 161,090,000, \text{ (or as } \frac{44,888}{44,888} \text{ is to}$$

$\frac{161,090,000}{44,888}$), or as 1 is to 3588. It appears therefore that the force of gravity at the distance of the Moon, or of 60 semi-diameters of the Earth from the Earth's center, is equal to only the 3588th part of the force of gravity on the surface

surface of the Earth; or at the distance of one semi-diameter of the Earth from the Earth's center.

Q. E. 1.

This number 3588 is pretty nearly equal to 3600, or the square of 60. And therefore it appears that the force of gravity at the distance of the Moon, or of 60 semi-diameters of the Earth from the Earth's center, is to the force of gravity on the surface of the Earth; or at the distance of one semi-diameter of the Earth from the Earth's center; in, nearly, the same proportion as 1×1 , or 1, the square of the latter distance, to wit, 1 semi-diameter of the Earth, is to 3600, the square of the former distance, or 60 semi-diameters of the Earth. And hence it seems reasonable to conclude that the force of gravity, by which bodies are made to fall towards the center of the Earth, decreases, when their distances from the said center increase, in the same proportion in which the squares of those distances increase.

Q. E. 1.

It was, as I conceive, by the foregoing train of reasoning, or some other equivalent to it, that Sir Isaac Newton derived from the knowledge of the Moon's distance from the Earth, and the period of her revolutions round it, and the consequent knowledge of the velocity of her motion in her orbit, and the centrifugal force arising from such velocity, his conclusions concerning the decrease of the force of gravity upon bodies as they recede from the center of the Earth, and concerning the law of such decrease: which are the points that Mr. Pascal and Mr. Roberval wished so much to be able to discover.

This Letter of Mr. Pascal and Mr. Roberval is dated in the year 1636, when *Mr. Blaise Pascal*, the great Genius and Author of the *Arithmetical Triangles* and other curious tracts here re-printed, was only thirteen years old. And therefore, if the date of this Letter is not mis-printed, I think it is probable that this Letter was not written by him, but by his father *Mr. Stephen Pascal* (who was also a very ingenious and learned Mathematician and Philosopher, though not equal in these respects to his son,) in conjunction with Mr. Roberval. But in either case the contents of the Letter are equally curious and deserving of attention. And, amongst other things contained in this Letter, there are some remarks on the Centers of Gravity of solid figures that seem very proper to be considered by such persons as undertake to write on the subtle and difficult

difficult subject of rational Mechanicks, or the Theory of the Motion of bodies, and desire to preserve the ideas of all the terms they make use of on these occasions, (such as *Center of Gravity, Center of Oscillation, Center of Percussion, Force of Gravity, Centripetal Force, Centrifugal Force, Resistance*, and the like,) perfectly clear and free from ambiguity. The learned writers of this Letter even declare "that they do not conceive that any solid body, except a sphere, can have a center of gravity, according to the definition of that center given by Pappus Alexandrinus and other antient authors." See page 468, lines 10th and 9th from the bottom of the page. If persons of such eminent talents as Mr. Pascal and Mr. Roberval express doubts of this kind on the subject of rational Mechanicks, or the Theory of Motion, it is surely incumbent on the writers who treat of Physical Astronomy and other such abstruse branches of this theory, to endeavour to remove these doubts and explain clearly the grounds and principles of the science they undertake to teach.

Other parts of the works of Monsieur Pascal here reprinted relate to the fundamental principles of the Doctrine of Chances, which began about the year 1654 to be cultivated in France by Monsieur Pascal, Monsieur Roberval, and Monsieur Fermat, and some other learned Mathematicians, and of which Mr. Huygens afterwards wrote the first regular treatise in the year 1656 under the title of *De Ratiociniis in Ludo Aleæ*. And other parts of these works of Mr. Pascal contain very curious properties of numbers.

The 9th Tract in this Volume relates to the Summation of infinite Serieses, and is a proper supplement to a Tract contained in the Third Volume of this Collection of Mathematical Tracts called *Scriptores Logarithmici*, which extends from page 255 of that Volume to page 312, and is intitled *The Investigation of the foregoing Differential Series*. The Differential Series here alluded to is as follows,

$$a - \frac{bx}{1+x} - \frac{D^1 x^2}{1+x^2} - \frac{D^{11} x^3}{1+x^3} - \frac{D^{111} x^4}{1+x^4} - \frac{D^{1111} x^5}{1+x^5} - \&c, \text{ and con-}$$

verges with a considerable degree of swiftness even when x is very little less than 1, as, for example, when it is equal to $\frac{9}{10}$, or $\frac{99}{100}$, and even when it is equal to $\frac{1}{2}$; and this differential series is in all cases equal to the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + ix^8 - kx^9 + \&c$, when x is of any magnitude not greater than 1, and $b, c, d, e, f, g, h, i, k, \&c$, the

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co-efficients

co-efficients of the successive powers of x , form a decreasing progression of terms; and the differences of these co-efficients, to wit, $b - c$, $c - d$, $d - e$, $e - f$, $f - g$, $g - h$, $h - i$, $i - k$, &c, also form a decreasing progression of terms; and the differences of these differences, to wit, $b - c - \sqrt{c - d}$, $c - d - \sqrt{d - e}$, $d - e - \sqrt{e - f}$, $e - f - \sqrt{f - g}$, $f - g - \sqrt{g - h}$, $g - h - \sqrt{h - i}$, $h - i - \sqrt{i - k}$, &c, or $b - 2c + d$, $c - 2d + e$, $d - 2e + f$, $e - 2f + g$, $f - 2g + h$, $g - 2h + i$, $h - 2i + k$, &c, or the second differences of the said co-efficients, also form a decreasing progression of terms; and the third differences of the said co-efficients, or the differences of the said second differences, also form a decreasing progression of terms; and the fourth differences, and the fifth differences, and the sixth differences, and the seventh differences, of the said co-efficients, and their differences of every following order, also form decreasing progressions. With these conditions it will always be true that, however slowly the powers of x may decrease, or even, if x is equal to 1, and consequently the said powers do not decrease at all,—and however slowly the co-efficients b , c , d , e , f , g , h , i , &c may decrease,—and, consequently, however slowly the terms of the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + ix^8 - kx^9 + \&c$ may decrease,—the

said series will be equal to the differential series $a - \frac{bx}{1+x} - \frac{D^1bx^2}{(1+x)^2} - \frac{D^2bx^3}{(1+x)^3} - \frac{D^3bx^4}{(1+x)^4} - \frac{D^4bx^5}{(1+x)^5} - \frac{D^5bx^6}{(1+x)^6} - \frac{D^6bx^7}{(1+x)^7} - \frac{D^7bx^8}{(1+x)^8} - \frac{D^8bx^9}{(1+x)^9} - \&c$ *ad infinitum*,

in which D^1 is put for $b - c$, or the first difference of the said co-efficients b , c , d , e , f , g , h , i , k , &c of the first order; and D^2 is put for $b - c - \sqrt{c - d}$, or $b - 2c + d$, or the first difference of the second order of differences of the said co-efficients; and D^3 is put for $b - 2c + d - \sqrt{c - 2d + e}$, or $b - 3c + 3d - e$, or the first difference of the third order of differences of the said co-efficients; and D^4 , D^5 , D^6 , D^7 , D^8 , D^9 , &c are put for the first differences of the fourth, fifth, sixth, seventh, eighth, and ninth, and other following orders of differences of the said co-efficients, respectively. And consequently, however slowly the terms of the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + ix^8 - kx^9 + \&c$ may converge; their sum may always be obtained by computing a moderate number of the terms of the differential

series $a - \frac{bx}{1+x} - \frac{D^1bx^2}{(1+x)^2} - \frac{D^2bx^3}{(1+x)^3} - \frac{D^3bx^4}{(1+x)^4} - \frac{D^4bx^5}{(1+x)^5} - \frac{D^5bx^6}{(1+x)^6} - \frac{D^6bx^7}{(1+x)^7} - \frac{D^7bx^8}{(1+x)^8} - \frac{D^8bx^9}{(1+x)^9} - \&c$

$\frac{D^{11}x^3}{1+x} - \frac{D^{11}x^3}{1+x^2}$ &c. *ad infinitum*, which always converges with considerable swiftness. This Differential Series is described and set forth, and illustrated by several apposite examples, in a Tract in the 3d Volume of this Collection, which is intitled *A Method of finding the Value of a slowly-converging infinite Series of decreasing Quantities of a certain Form, when it converges too slowly to be summed in the common way by the mere computation, and addition or subtraction, of some of it's initial terms*, and which begins in page 219 and ends in page 252. And the manner of obtaining it is also briefly described in the said Tract in Art. 5, pages 221, 222. But afterwards I gave a much fuller investigation of it in another Tract which is also printed in the same 3d Volume of this Collection of Tracts immediately after the former Tract, and is intitled *The Investigation of the foregoing Differential Series*, and which extends from page 255 to page 312. Yet even in this second Tract there was one thing wanting to make the investigation of this Differential Series completely satisfactory, and which I had not been able to discover; and that was a proof that the remaining terms of the said Differential Series, after those that shall have been actually investigated, will be such as arise from the law of continuation that is observed to take place among those which have been actually investigated. In that second Tract I had actually investigated the first

five terms of the said Differential Series, and had shewn them to be $a - \frac{bx}{1+x} - [b - c \times \frac{xx}{1+x}] - [b - 2c + d \times \frac{x^2}{1+x}] - [b - 3c + 3d - e \times \frac{x^3}{1+x}]$,

or $a - \frac{bx}{1+x} - \frac{D^1x^3}{1+x} - \frac{D^2x^3}{1+x} - \frac{D^3x^3}{1+x}$; and I had, for my own satisfaction, investigated also the three following terms, and found them to be equal to

$[b - 4c + 6d - 4e + f] \times \frac{x^4}{1+x}$, and $[b - 5c + 10d - 10e + 5f - g] \times \frac{x^5}{1+x}$, and $[b - 6c + 15d - 20e + 15f - 6g + h] \times \frac{x^6}{1+x}$; or to $\frac{D^4x^3}{1+x}$,

$\frac{D^5x^3}{1+x}$, and $\frac{D^6x^3}{1+x}$. And, from the analogy of these terms so investigated, I thought it reasonable to conjecture that the ninth, and tenth, and eleventh, and

other following terms of this series would be $\frac{D^8x^3}{1+x}$, $\frac{D^9x^3}{1+x}$, and $\frac{D^{10}x^3}{1+x}$, &c.

But this was only a conjecture, and I was unable to convert it into a demonstration.

stration. This defect, however, is now supplied in the ninth Tract of the present Volume, in which I have set forth, in as full and clear a manner as I could; another manner of investigating the several terms of this Differential Series which has been invented and communicated to me by the learned and ingenious Mr. John Hellins, B. D. Vicar of Potter's Pury in Northamptonshire, and which is of such a nature as to enable us to perceive that the law of continuation, or generation, of the terms of the series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \frac{Wx^6}{1+x}, \frac{Xx^7}{1+x},$ &c *ad infinitum*, (which, in both Mr. Hellins's investigation and mine, is assumed to be equal to the original series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$, and from which the differential series $a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x} - \frac{D^2x^3}{1+x} - \frac{D^3x^4}{1+x} - \frac{D^4x^5}{1+x} - \frac{D^5x^6}{1+x} - \frac{D^6x^7}{1+x} - \&c$ was derived by investigating successively the values of P, Q, R, S, T, V, W, X, &c, and the signs that are to be prefixed to the second and other following terms of the said series,) one from another, which is found to take place in the terms that are actually investigated, must likewise take place in all the following terms which have not been investigated, to whatever number of terms the said series may be continued; so that we may now conclude with certainty (instead of only conjecturing, as we did before,) that the ninth, tenth, eleventh, twelfth, and other following terms of this series will be $-\frac{D^{10}x^9}{1+x} - \frac{D^{11}x^{10}}{1+x} - \frac{D^{12}x^{11}}{1+x} - \&c$, agreeably to the analogy that has taken place with respect to the former eight terms of it, to wit, $a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x} - \frac{D^2x^3}{1+x} - \frac{D^3x^4}{1+x} - \frac{D^4x^5}{1+x} - \frac{D^5x^6}{1+x} - \frac{D^6x^7}{1+x}$, which had been actually computed. This Tract begins in page 571, and ends in page 614.

The 10th and last Tract of this Volume is likewise an invention of the same ingenious and learned Mathematician, the Reverend Mr. John Hellins, from whom I had the improvement of the investigation of the differential series

$$a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x} - \frac{D^2x^3}{1+x} - \frac{D^3x^4}{1+x} - \frac{D^4x^5}{1+x} - \&c \text{ that is the subject of}$$

the

the 9th Tract. And it relates to the same subject as the foregoing, or 9th, Tract, to wit, the summation of infinite series, when they converge too slowly to be summed in the common way, or by computing a moderate number of their initial terms and taking the values of the sums of those terms for pretty near values of the whole series. For it is remarkable that serieses of the form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$, in which all the terms after the first term are marked with the sign +, or are added to the first term, are more difficult to be summed, when they converge with excessive slowness, than serieses of the form $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$, in which the signs are marked alternately with the sign - and the sign +: though one would naturally be inclined to suppose that the former series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ should (from the greater simplicity of it's composition, which arises from addition only, instead of arising, as that of the other series does, from both addition and subtraction,) be more easily managed than the latter for the purpose of summation as well as for every other purpose whatsoever. It is, however, found that serieses of the form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$, when their terms decrease very slowly, are often exceedingly difficult to sum; and no general series has yet been found by the eminent Mathematicians who have written upon this subject, that has been equal to the said series, and yet has converged with so considerable a degree of swiftness that eight, or ten, or a dozen, of it's first terms have been equal to three, or four, hundred terms of the former series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$, of which we are required to find the sum, as happens in the case of the differential series $a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x^2} - \frac{D^2x^3}{1+x^3} - \frac{D^3x^4}{1+x^4} - \frac{D^4x^5}{1+x^5} - \&c$ with respect to the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$. The finding therefore such a swiftly-converging series as will be equal to the slowly-converging series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$, and as will consequently enable us to find, with a moderate degree of labour, the sum of such a slowly-converging series to a considerable degree of exactness, is still a great desideratum among Mathematicians. But Mr. Hellins has, in some degree,

overcome this difficulty, by shewing us that the very differential series $a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x^2} - \frac{D^2x^3}{1+x^3} - \frac{D^3x^4}{1+x^4} - \frac{D^4x^5}{1+x^5} - \frac{D^5x^6}{1+x^6} - \&c$ itself may be successfully

successfully employed for this purpose, or so as to enable us to find the value of the series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ to any degree of exactness that we may desire, though its terms should decrease with an excessive degree of slowness, if we will have the patience to make four, or more, successive applications of the said differential series to certain serieses of the form of the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$, which are derived from the original series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ of which we are endeavouring to find the sum, And the explanation of Mr. Hellins's method of making the said successive applications of the said differential series to this purpose, together with a very full illustration of this method in the case of a particular example, namely, in finding the sum of the slowly-converging series $\frac{1}{33} + \frac{0.9}{34} + \frac{0.9^2}{35} + \frac{0.9^3}{36} + \frac{0.9^4}{37} + \frac{0.9^5}{38} + \frac{0.9^6}{39} + \frac{0.9^7}{40} + \&c$, make up the chief contents of this 10th Tract.

But it must be confessed that these repeated applications of the said Differential Series to this purpose are so exceedingly tedious and laborious, that it is probable few persons will ever have recourse to them for the purpose of finding the value of a slowly-converging series of the said form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$. And therefore it is still to be wished that ingenious Mathematicians would exert their talents and sagacity to find-out some easier way of finding the value of such a series.

The description and explanation of this method of applying the said differential series to the summation of the slowly-converging series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ begin in page 615 and extend to the end of Art. 15 in page 628; and the illustration of the method by applying it to the summation of the series $\frac{1}{33} + \frac{0.9}{34} + \frac{0.9^2}{35} + \frac{0.9^3}{36} + \frac{0.9^4}{37} + \frac{0.9^5}{38} + \frac{0.9^6}{39} + \frac{0.9^7}{40} + \&c$, begins in page 628, and ends in page 680. The degree of exactness obtained by these long and laborious calculations is such as gives us the number 2.302,585,092,573,344,422 for Napier's logarithm of 10, or of the ratio of 10 to 1; of which number the first ten figures 2.302,585,092

are

are exact, the more accurate value of that logarithm being 2.302,585,092,994, 045,684,017, &c. This is a considerable degree of exactness; but it is too dearly purchased by such an immense quantity of calculation as has been employed to obtain it.

The last 15 pages of this Volume, from page 680 to the end of page 695, contain an account of some conjectural methods of finding the sums of slowly-converging serieses of the foregoing form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$, that have occurred to me, and by which we may obtain the value of the series exact to about three places of decimal figures.

FRANCIS MASERES.

INNER TEMPLE, *January 20, 1801.*

A
T A B L E
OF THE
C O N T E N T S
OF THE
F O U R T H V O L U M E.

NUMBER I.

AN Appendix to the Tract of Dr. Edmund Halley, intitled "*An Easy Demonstration of the Analogy of the Logarithmick Tangents to the Meridian-Line, or Sum of the Secants, with various Methods of computing the same to the utmost Exactness*"; which has been reprinted in the Second Volume of this Collection of Mathematical Tracts, intitled, *Scriptores Logarithmici*, in pages 76, 77, 78, &c, - - - 84 of the said Second Volume:

Containing

The Solution of a curious and very difficult Problem, relating to Navigation, proposed by Dr. Halley in the latter part of the said Tract.

By Francis Maferes, Esq. F. R. S. Curator Baron of the Court of Exchequer.

In pages 3, 4, 5, &c, - - - 114.

NUMBER II.

A new Edition of an excellent Treatise on Navigation, written in Latin by Willebrord Snell, an eminent Dutch Mathematician, and published at Leyden in Holland in the year 1624, under the following title, to wit,

Willebrordi Snellii, à Royen, R. F. [id est, Rodolphi Filii,] Tiphys Batavus, sive Hiftiodromice, De Navium Curvibus et Rc Navali.

In pages 117, 118, 119, &c, - - - 271.

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Ouvrages de Mathématique, du célèbre Mr. Blaise Pascal; extraits de l'édition des Œuvres du dit Mr. Pascal faite à la Haye en l'année 1779, chez Detune, Libraire.

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An Appendix to the Tract *On the Summation of Infinite Series*es contained in the Third Volume of this Collection of Tracts called *Scriptores Logarithmici*, pages 255, 256, 257, &c., - - - 312, in which an Investigation is given

$$\text{of the Differential Series } a = \frac{bx}{1+x} - \frac{D^1 x^2}{(1+x)^2} - \frac{D^2 x^3}{(1+x)^3} - \frac{D^3 x^4}{(1+x)^4} - \frac{D^4 x^5}{(1+x)^5} \\ - \frac{D^5 x^6}{(1+x)^6} - \frac{D^6 x^7}{(1+x)^7} - \text{\&c ad infinitum};$$

Containing

An Improvement on the said Investigation, that has lately been made by the Reverend Mr. John Hellins, B. D. Vicar of Potter's Pury in Northamptonshire.

In pages 571, 572, 573, &c., - - - 614.

NUMBER X.

A Method of summing an Infinite Series of Decreasing Quantities of the following form, to wit, $a + bx + cxx + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \text{\&c ad infinitum}$, with exactness, when the several terms of the said Series decrease so slowly as to make the Summation of it in the common way, or by the mere computation and addition of a moderate number of it's initial terms, be very insufficient for that purpose.

Invented by the Reverend Mr. John Hellins, Vicar of Potter's Pury in Northamptonshire.

In pages 615, 616, 617, &c., - - - 695.

NUMBER XI.

A Table of Errata in this Volume, with their Corrections.

In pages 697, 698, 699, 700.

AN APPENDIX
TO THE TRACT OF
DR. EDMUND HALLEY,

INTITLED,

*"An easy Demonstration of the Analogy of the Logarithmick Tangents to the
Meridian Line, or Sum of the Secants; with various Methods
of Computing the same to the utmost Exactness;"*

Which has been Re-printed in the Second Volume of this Collection of
Mathematical Tracts, intituled, *Scriptores Logarithmici*, in pages 76, 77,
78, &c. - - - 84: Containing the Solution of a Curious Problem,
relating to Navigation, propos'd by Dr. HALLEY in the latter part of
the said Tract.

BY FRANCIS MASERES, Esq. F. R. S.
CURSOR BARON OF THE COURT OF EXCHEQUER*.

* N. B. I have put my name to the following Solution of this curious and difficult Problem, because I have drawn it out in its present form. But the substance of it was communicated to me by my learned friend Mr. George Atwood, M. A. and F. R. S. late Fellow of Trinity College, Cambridge.

A N
A P P E N D I X
TO THE TRACT OF
DR. EDMUND HALLEY,

INTITLED,

“ *An easy Demonstration of the Analogy of the Logarithmic Tangents to the Meridian
“ Line, or Sum of the Secants; with various Methods of Computing the same to
“ the utmost Exactness, &c.*”

Article 1. **T**HE above-mentioned Tract of Dr. HALLEY was first published in the Philosophical Transactions of the Royal Society of London, in the year 1695-6, in Number 219, Vol. XIX of the said Transactions; and was published a second time in the year 1708, in the second volume of the Collection of Mathematical and Philosophical Tracts, in three volumes, octavo, called *Miscellanea Curiosa*, pages 20, 21, 22, &c., - - 36; and, lastly, has been published a third time in the beginning of the year 1792, in the second volume of the present Collection of Mathematical Tracts, called *Scriptores Logarithmici*, pages 76, 77, 78, &c., - - 84. Towards the end of this Tract Dr. HALLEY has proposed a certain Problem relating to Navigation, which, I believe, has never received a compleat and accurate solution to this day; though it has been resolved in a tentative way by Mr. JOHN ROBERTSON, the late Librarian of the Royal Society (who had formerly been the head Master of the Royal Military Academy at Portsmouth,) in his valuable Treatise on Navigation, Book VIII, Sect. IV, Problem X, page 560, and likewise by *Monsieur Bouguer*, a distinguished Member of the Royal Academy of Sciences at Paris, in his copious and learned Treatise on Navigation, published at Paris, in the year 1753, and intitled, *Nouveau Traité de Navigation, contenant la Théorie et la Pratique du Pilotage*, in the 5th Book of the said Treatise, Section I, Chapter II, pages 352, 353, under the title of *Sixième Problème Général*; and likewise by the late Mr. ISRAEL LYONS, in the Appendix to

t's Nautical Almanack for the year 1772. The passage of Dr. HALLEY's said discourse, in which he makes mention of this Problem, is in the words following. "I need not shew how, by regressive work, to find the latitudes from the meridional parts; the method being sufficiently obvious: I shall only conclude with the proposal of a Problem, which remains to make this doctrine compleat; and that is this:

"A ship sails from a given latitude, and, having run a certain number of leagues, has altered her longitude by a given angle. It is required to find the course steered.

"The Solution hereof would be very acceptable, if not to the Publick, at least to the Author of this Treat; being likely to open some further light into the mysteries of Geometry." See Vol. II. of this present Collection of Tracts, called, *Scriptores Logarithmici*, page 83.

Art. 2. In this Problem the ship is supposed to have sailed always in the same course during the whole voyage, or in such a line as to have always cut the several meridian lines, or circumferences of great circles of the Earth passing through its two Poles,) over which it has passed, in the same angle: and

the Earth is supposed to be of a spherical, and not a spheroidal, or other oval, figure; and consequently all the meridian lines, (or circumferences of great circles of the Earth that pass through the two Poles,) will be equal to the circumference of the Equator: And by the difference of the ship's longitude at the beginning and at the end of the voyage, (which difference is here supposed to be given, or known,) we are to understand the arch of the circumference of the Equator that is intercepted between two meridian lines, (or the circumferences of two great circles of the Earth passing through the two Poles,) that are drawn through the ship's two positions at the beginning and at the end of the voyage. These things must be thoroughly understood, and carefully attended to, in order to have a clear idea of the meaning of the Problem of which we are going to attempt the Solution.

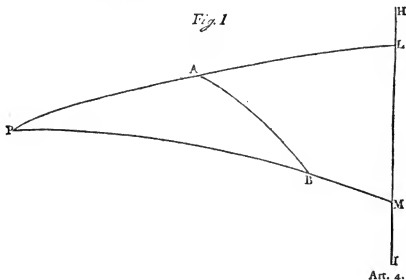
Art. 3. Let the point P, in Fig. 1, denote one of the Poles of the Earth, and the line HI denote a part of the circumference of the Equator, or great circle of the Earth which is drawn through its center at right angles to its axis. Let A and B be two points on the surface of the Earth, both situated on the same side of the Equator, but at unequal distances from it; and let the distance of A from the Equator be greater than the distance of B from it, or, in other words, let the latitude of A be greater than the latitude of B. Also let A and B be situated in different meridian lines P A L and P B M, which cut the line HI, or the circumference of the equator, in the points L and M. Then, it is evident, the arch L M of the circumference of the equator will be the difference of the longitudes of the points A and B, from whatever original meridian line the said longitudes may be reckoned.

Further, let a ship be supposed to sail from the point A to the point B in a line that cuts all the meridian lines, over which it passes, in the same angle in

in which its first direction, when it sets out from A, cuts the meridian line P A L which passes through its first position A, or in an angle equal to the angle L A B; by which angle we are to understand the rectilineal angle contained between a tangent to the meridian circle P A L in the point A, and a tangent to the curve line A B (described by the ship in her passage from A to B,) in the same point A.

This curve line A B, described by the ship in her passage from A to B, is called a *loxodromick line*, or *loxodromick curve*, from the two Greek words λοξός, *oblique*, and δρομή, *a course*, because it is oblique in its course, or direction, or cuts the several meridian lines, over which it passes, in an oblique angle. And it is also called a *rhumb-line*; and the angle L A B, in which it cuts the several meridian lines over which it passes, is called *the angle of the rhumb*, or *the rhumb-angle*.

Now in the foregoing Problem, proposed by Dr. HALLEY, the arch A L, or the latitude of the point A (from which the ship sets out upon her voyage,) is supposed to be known; but not the arch B M, or the latitude of the point B, at which the ship arrives after passing along the loxodromick-line, or rhumb-line, A B. And the arch L M, or the difference of the longitudes of the points A and B, is also supposed to be known; and likewise the length of the loxodromick-line, or rhumb-line, A B, which the ship has described in her passage from A to B. And from these *data* we are required to find the rhumb-angle L A B, in which the rhumb-line A B cuts all the meridian lines over which the ship has passed.



Art. 4. Before we enter upon the Solution of this Problem, it will be necessary to shew the relation of the rhumb-line, or loxodromick line, AB , or distance run by the ship, to the difference of the latitudes of the points A and B , and consequently how BM , or the latitude of the point B , may be derived from AL , or the latitude of the point A , and the length of the rhumb-line AB , together with the magnitude of the rhumb-angle LAB , when AL , AB , and the angle LAB are previously known. Now this relation of the difference of the latitudes of the points A and B to the length of the rhumb-line AB and to the rhumb-angle LAB , may be demonstrated in the following manner.

LEMMA I. A THEOREM.

If there be two points A and B (as in Fig. 2,) on the surface of the Earth, both on the same side of the Equator HI , but at different distances from it; and the distance of the point A from the Equator is greater than the distance of the point B from it; and the said points are situated in different meridian circles PAL and PBM , which cut the Equator in the points L and M respectively; and AB be a loxodromick line, or rhumb-line, extending from the point A to the point B ; the length of the said line AB will be to the difference of the two circular arches AL and BM , (which are the latitudes of the two points A and B ,) as the radius of a circle is to the co-sine of the rhumb-angle LAB .

In the meridian circle PAL , which passes through the point A , take the arch PQ equal to the arch PB of the meridian circle PBM , which passes through the point B . Then will the arch QL be equal to the arch BM , or the latitude of the point B ; and consequently the arch AQ (which is the excess of the arch AL above the arch QL ,) will be equal to the excess of the arch AL above the arch BM , or will be the difference of the latitudes of the two points A and B . We are therefore to shew that the length of the rhumb-line AB will be to the length of the circular arch AQ as the radius of a circle is to the co-sine of an angle in the said circle equal to the rhumb-angle LAB , or QAB .

DEMONSTRATION.

Let the line QB represent the arch of a circle of which the pole P is the center, and either of the lines PQ and PB is the radius, or the arch of a parallel

parallel of latitude passing through the point B. And let this arch QB be supposed to be divided into a very great number of very small and equal parts, such as B*c*, *c*g, g*k*, *k*n, &c; as, for example, into ten thousand million of such parts. And through each of the points of division, *c*, g, *k*, n, &c, let the meridian circles P*c*, P*g*, P*k*, P*n*, &c, be supposed to be drawn, cutting the loxodromick, or rhumb, line BA in the points C, D, E, F, &c.

Further, through the point C draw a parallel of latitude, or circle parallel to the Equator, cutting the meridian-line P*g* in the point G, and the meridian line PL, or PQ, in the point R. And, in like manner, through the point D draw another parallel of latitude, or circle parallel to the Equator, cutting the meridian-line P*k* in K, and the meridian-line PQ in S; and through the point E draw another parallel of latitude, or circle parallel to the Equator, cutting the meridian-line P*n* in N, and the meridian-line PQ in T; and, in like manner, conceive other parallels of latitude, or circles parallel to the Equator, to be drawn through all the other points of division in the loxodromick line AB, however great their number may be.

Art. 5. Then it is evident that, since the circles BQ and CR are parallel to each other, they will intercept equal portions of the two meridian circles P*c* and PQ; and consequently the little arch RQ of the meridian circle PQ will be equal to the little arch C*c* of the meridian circle P*c*. And, for a like reason, the little arch SR of the meridian circle PQ will be equal to the little arch DG of the meridian circle P*g*; and the little arch TS of the meridian circle PQ will be equal to the little arch EK of the meridian circle P*k*; and the little arch OT of the meridian circle PQ will be equal to the little arch FN of the meridian circle P*n*. And the like equality will take place between all the following little arches of the meridian circle PQ, lying between the points A and O, and the corresponding little arches of the several following meridian circles drawn through the following points of division in the loxodromick line AB lying between the points A and F. Therefore the sum of all the little arches RQ, SR, TS, OT, &c, of the meridian circle PQ, continued to the point A, that is, the whole arch AQ, will be equal to the sum of all the little arches C*c*, DG, EK, FN, &c, of the several meridian circles P*c*, P*g*, P*k*, P*n*, &c, lying between the points A and B.

And the whole loxodromick arch AB is equal to the sum of all its parts, or the small arches BC, CD, DE, EF, &c, continued to the point A.

Therefore the whole loxodromick line AB will be to the whole circular arch AQ, or to the difference of the latitudes of the points A and B, in the same proportion as the sum of all the little loxodromick arches BC, CD, DE, EF, &c, continued to the point A, is to the sum of all the little circular arches C*c*, DG, EK, FN, &c, of the different meridian circles P*c*, P*g*, P*k*, P*n*, &c, corresponding to them. We must therefore endeavour to shew that this latter proportion, to wit, the proportion of the sum of all the little loxodromick arches BC, CD, DE, EF, &c, continued to the point A, to the sum of all
the

the little circular arches Cc , DG , EK , FN , &c, corresponding to them, is equal to that of the radius of any circle to the co-sine of an angle equal to the rhumb-angle LAB , in the same circle.

Art. 6. Now, because the circular arches Bc , cg , gk , kn , &c, continued to the point Q , are supposed to be all equal to each other, and extremely numerous, as, for example, not fewer than ten thousand millions, it follows that the said arches must all be extremely small, and consequently that the corresponding loxodromick arches BC , CD , DE , EF , &c, continued to the point A , (into which the whole loxodromick arch AB is divided by the several meridian circles Pc , Pg , Pk , Pn , &c, continued to the point Q ;) will also be extremely small, and consequently that they will be very nearly equal to their several chords respectively, or will exceed their several and respective chords in very small ratios of majority. And it follows likewise that the little circular arches Cc , DG , EK , FN , &c, of the several meridian circles Pc , Pg , Pk , Pn , &c, and likewise the little circular arches CG , DK , EN , &c, of the several parallels of latitude CR , DS , ET , &c, will be all extremely small, and consequently that they will be very nearly equal to their several chords respectively, or will exceed their several and respective chords in very small ratios of majority. Therefore the sum of all the chords of the several little loxodromick arches BC , CD , DE , EF , &c, continued to the point A , may be considered as equal to the sum of all those arches themselves, or to the whole loxodromick arch AB ; and the sum of all the chords of the several little circular arches Cc , DG , EK , FN , &c, corresponding to the loxodromick arches BC , CD , DE , EF , &c, may be considered as equal to the sum of all the said arches themselves, and consequently as being equal to the circular arch AQ of the meridian circle PAL , or to the difference of the latitudes of the points A and B . We must therefore now endeavour to shew that the sum of all the chords of the little loxodromick arches BC , CD , DE , EF , &c, continued to the point A , is to the sum of all the chords of the corresponding little circular arches Cc , DG , EK , FN , &c, in the proportion of the radius of any circle to the co-sine of an angle equal to the rhumb-angle LAB , in the same circle.

Art. 7. We must now consider the several little triangles BCc , CDG , DEK , FEN , &c, (continued to the point A ;) formed by the chords of the several little loxodromick arches BC , CD , DE , EF , &c, and the chords of the several corresponding little circular arches Cc , DG , EK , FN , &c, of the meridian circles Pc , Pg , Pk , Pn , &c, and the chords of the several other corresponding little circular arches Bc , CG , DK , EN , &c, of the parallels of latitude BQ , CR , DS , ET , &c; which triangles will all be right-lined triangles, because they will be formed by the chords of the said several little loxodromick arches and circular arches, and not by the said loxodromick and circular arches themselves.

Now, because the little circular arch Cc is extremely small, the angle contained

tained between its chord and a tangent to it in the point C will be extremely small; and consequently the said chord may be considered as co-inciding with the said tangent: and, in like manner, because the little loxodromick arch BC is extremely small, the angle contained between its chord and a tangent to it in the point C will be extremely small; and consequently the said chord may be considered as co-inciding with the said tangent. Therefore the angle contained between the chord of the circular arch Cc and the chord of the loxodromick arch BC, will be equal to the angle contained between the tangent of the circular arch Cc in the point C and the tangent of the loxodromick arch BC in the same point C, that is, to the angle made by the loxodromick curve CB with the meridian circle P c, or to the angle of the rhumb. Therefore, in the little right-lined triangle BCc, formed by the chords of the loxodromick arch BC and the two circular arches Cc and Bc, the angle BCc will be equal to the angle of the rhumb. Further, the angle BcC, in the same right-lined triangle, which is formed by the two little circular arches Bc and Cc, will be a right angle; as may be shewn in the manner following. The plane of every parallel of latitude, or circle parallel to the Equator, is at right angles to the plane of every meridian circle that it crosses. Therefore, if a tangent be drawn to the parallel of latitude Bc, or BQ, in the point c, in which it cuts the meridian circle P Cc, the said tangent will be at right angles to a tangent drawn to the said meridian circle P Cc in the same point of intersection c. But the angle made by the chord of the arch Bc with the tangent to the said arch in the point c is extremely small, on account of the extrem smallness of the said arch; and therefore the said chord may be considered as co-inciding with the said tangent. And, for a like reason, the chord of the arch Cc may be considered as co-inciding with the tangent to the meridian circle P Cc in the same point c. Therefore the angle contained between the chord of the arch Bc, and the chord of the arch Cc, will be equal to the angle made by the two tangents of the said arches Bc and Cc in the same point c. But the angle made by the said two tangents is a right angle. Therefore the angle made by the said two chords will also be a right angle; that is, the angle BcC in the right-lined triangle BcC, formed by the chords of the three little arches BC, Cc, and Bc, will be a right angle. Therefore, if the chord BC (which subtends the right angle BcC,) be made the radius of a circle, the chord Bc will be the sine of the opposite angle BCc, that is, of the angle of the rhumb, in the same circle; and Cc will be the sine of the angle CBc, or of the complement of the angle BCc, or the rhumb-angle, to a right angle, or will be the co-sine of the said angle BCc, or the rhumb-angle, in the same circle.

And in the same manner it may be shewn that in the next little right-lined triangle CDG, formed by the chords of the little arches CD, DG, and GC, the angle DGC will be a right angle, and that the chord of the arch CD will be to the chord of the arch DG as the radius of a circle to the co-sine of the angle CDG, which is equal to BCc, or the angle of the rhumb; and that in the third little right-lined triangle DEK, formed by the chords of the arches DE, EK, KD, the angle EKD will be a right angle, and

that the chord of the arch DE will be to the chord of the arch EK as the radius of a circle to the co-sine of the angle DEK, which is equal to BCr, or the angle of the rhumb; and, in like manner, that in every following little right-lined triangle near the following points of division of the loxodromick line AB, formed by the chord of the little loxodromick arch and the chords of the two corresponding little circular arches, the chord of the loxodromick arch will be to the chord of the corresponding little circular arch that is a portion of a meridian line, as the radius of a circle to the co-sine of the angle of the rhumb. Therefore, by El. 5, 12, the sum of all the chords of the said little loxodromick arches, lying between the points A and B, will be to the sum of all the chords of the corresponding little circular arches that are portions of the several meridian lines lying between the same points A and B, in the same proportion of the radius of a circle to the co-sine of the rhumb-angle in the same circle.

But the sum of all the chords of the said little loxodromick arches BC, CD, DE, EF, &c, lying between the points A and B, is equal to the sum of all the said loxodromick arches themselves, or to the whole loxodromick arch AB; and the sum of all the chords of the little circular arches, or portions of meridian lines, Cr, DG, EK, FN, &c, lying between the same points A and B, is equal to the sum of all those arches themselves, and consequently to the sum of the corresponding arches QR, RS, ST, TO, &c, (continued to the point A,) of the meridian line P A L, (which are, respectively, equal to the arches Cr, DG, EK, FN, &c,) or to the whole arch A Q of the said meridian line P A L, or to the difference of the latitudes of the two points A and B. Therefore the whole loxodromick arch, or rhumb line, AB will be to the whole circular arch A Q, or to the difference of the latitudes of the two points A and B, in the same proportion of the radius of a circle to the co-sine of the angle of the rhumb, or of the ship's course, or the angle LAB, in the same circle.

Q. E. D.

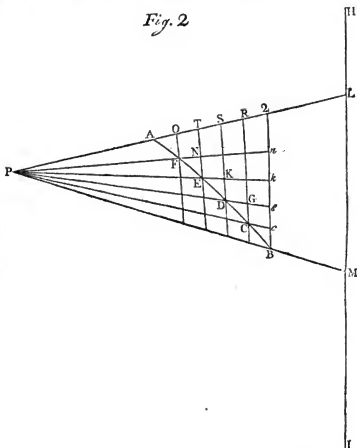
ART. 8. COROLL. Therefore, when the arch AL, or the latitude of the first point A, is given; and the length of the loxodromick arch, or rhumb-line, AB, or the distance run by the ship in its passage from A to B, is also given; and likewise the rhumb-angle LAB, or the angle of the ship's course; the arch BM, or the latitude of the second point B, may be found in the following manner.

Find a fourth proportional to the radius of any circle, the co-sine of the rhumb-angle LAB, or angle of the ship's course, and the length of the rhumb-line AB, or the distance run by the ship in her passage from A to B. And the said fourth proportional will be equal to the arch A Q, or the difference of the arches AL and BM, or of the latitudes of the places A and B. And, having found this fourth proportional, (which is equal to the arch A Q,) subtract it from the arch AL, or the latitude of the first point A. And the remainder will be the length of the arch BM, or the latitude of the second point B.

Q. E. I.

Fig. 2.

Fig. 2



Art. 9. Having shewn, in the foregoing Corollary, how, from the latitude AL of the first point A , and the rhumb-angle, or angle of the ship's course, LAB , and the length of the loxodromick arch, or rhumb-line, AB , or the distance run by the ship in its passage from A to B , we may find the arch BM , or the latitude of the second point B ; we will now proceed to shew how, from the two arches AL and BM , or the latitudes of the two points A and B , the rhumb-angle LAB , and the rhumb-line, or distance run, AB , we may find the arch LM in the circumference of the Equator, or the difference of the longitudes of the two points A and B . In order to this it will be necessary,

in the first place, to suppose the ship to continue her voyage in the same course, or in the same loxodromick curve, or rhumb-line, beyond the point B, till she arrives at the Equator, and to find the difference of the longitudes of the first point A, and the point in which the said loxodromick curve, or line of the ship's passage, so continued on, will meet HI, or the circumference of the Equator. This will be the subject of the following Problem.

LEMMA II. A PROBLEM.

Art. 10. Let P (in Fig. 3,) represent one of the poles of the Earth; HI a part of the circumference of the Equator; A a point on the surface of the Earth between the Pole P and the Equator HI; and B another point on the surface of the Earth between the Pole P and the Equator HI, that is nearer to the Equator than the former point A, and is situated in a different meridian; and let PAL be a meridian circle passing through the point A, and meeting HI, or the circumference of the Equator, in the point L; and let PBM be another meridian circle passing through the point B, and meeting HI, or the circumference of the Equator, in the point M; and let AB be a loxodromick curve, or rhumb-line, along which a ship sails from the point A to the point B, keeping always the same course, or cutting all the meridian lines, over which it passes, in the same angle LAB; and let BV be a continuation of the same loxodromick curve, or rhumb-line, AB, to the point V in the line HI, or the circumference of the Equator; and let the ship be supposed, after it has arrived at the point B, to continue its voyage further from B to the point V, or to the Equator, along the loxodromick arch BV, which cuts the several meridian lines, over which it passes, in the same angle LAB in which the loxodromick arch AB cut the several meridian lines over which it passed. And let the circular arch AL of the meridian line PAL, or the latitude of the first point A, be supposed to be known; and likewise the course of the ship, or the rhumb-angle LAB; and the length of the whole loxodromick line ABV from the first point A to the Equator, or the whole distance run by the ship in passing from A to the Equator. It is required to find the length of the arch LV of the circumference of the Equator, or the difference of the longitudes of the points A and V.

SOLUTION.

S O L U T I O N.

Art. 11. Let r be put for the radius of the Earth, or of any one of its great circles; and τ for the tangent of the circular arch AL of the meridian circle PAL , or of the latitude of the first point A , or, in general, for the tangent of the latitude of the ship's place, during its passage from A to V , in a great circle of the Earth; and y for the secant of the same arch, or of the latitude of the ship's place, in the same circle: and let t be put for the tangent of the rhumb-angle LAB , or the angle of the ship's course, in the same circle.

Then, since the angle LAB is supposed to be known, its tangent t in the circle of which r is the radius, will be known also.

Let the whole arch LV , (which is the difference of the longitudes of the points A and V , or the quantity whose magnitude we are to find,) be conceived to be divided into some very great number of small and equal parts, as, for example, into ten thousand millions of such parts. And let the point of division that is nearest to the point M (in which the meridian line which passes through the point B cuts the equatorial arch HI .) of all the points of division that lie between L and M , be the point m ; so that mM shall be one of those very small parts of the arch LV . And through m draw the meridian line Pm cutting the rhumb-line AV , or ABV , in the point C . And through the point B draw a circle parallel to the Equator HI , cutting the meridian circle PCm in the point c . And let the chords of the loxodromick arch BC and of the three circular arches Cc , Bc , and Mm be supposed to be drawn.

Art. 12. Then it is evident, that, since the arch Mm of the equatorial circle HI is extremely small, the three arches BC , Cc , and Bc , (which are derived from it by drawing the meridian circle PCm through the point m , and the parallel of latitude Bc through the point B till it cuts the said meridian circle PCm in the point c .) will also be extremely small: and consequently the chords of these four arches Mm , BC , Cc , and Bc , will be very nearly equal to the said arches themselves respectively, and therefore may be substituted for them in our investigation of the magnitude of the whole circular arch LV , which is the object of this Problem.

Further, it is evident, that, while the ship is passing from the point A to the point V in the Equator, the loxodromick curve ABV will increase from a to ABV ; and the circular arch LV of the Equator, or the difference of the longitudes of the points A and V , or, rather, of the points A and C (supposing C to represent the several successive places of the ship in its passage from A to V .) will at the same time increase from a to LV ; and the circular arch AL , or, rather, Cm , or the latitude of the ship, will decrease at the same time from AL to a . And the little circular arch mM will be the increment of the

the circular arch Lm , or of the difference of the longitudes of the points A and C in a very small portion of time; and the little loxodromick arch CB will be the contemporary increment of the loxodromick arch AC; and the little circular arch $C\epsilon$ will be the contemporary decrement of the circular arch Cm , or of the latitude of the ship's place.

Art. 13. Now it will be rather more convenient for our investigation of the length of the arch LV, that these three lines should all increase together than that two of them should increase, and the third should decrease. And therefore, in order to make them all increase together from ϕ at the same time, we will suppose the ship to sail back again from the point V in the circumference of the Equator to the first point A along the same loxodromick line VBA. And then the circular arch Vm , or the difference of the longitudes of the ship's successive places, will increase gradually from ϕ to VL; and the circular arch Cm , or the latitude of the ship's place, will at the same time increase gradually from ϕ to LA, or AL; and the loxodromick arch VC will also increase gradually from ϕ to VBCA: and the little arch Mm will be the increment of the circular arch Vm , or of the difference of the longitudes of the ship's places, in a very small portion of time; and the little arch $C\epsilon$ will be the increment of the circular arch Cm , or of the latitude of the ship's place, in the same time; and the little arch BC will be the contemporary increment of the loxodromick curve VC. And consequently, if we substitute the chords of these little arches instead of the arches themselves, the chord of the little arch Mm will be the increment of the circular arch Vm , or of the difference of the longitudes of the ship's places, in a very small portion of time; and the chord of the little arch $C\epsilon$ will be the increment of the circular arch Cm , or of the latitude of the ship's place, in the same time; and the chord of the little arch BC will be the contemporary increment of the loxodromick curve VC.

Art. 14. Let the circular arch Cm , or mC , or the varying latitude of the ship's place in her passage from V to A along the loxodromick curve VBCA, be denoted by the letter l , and its little increment $C\epsilon$ be denoted by $\dot{l}$, or the same letter with a point placed over it; and let Vm , or the varying difference of the longitudes of the ship's places, at her first departure from V, and in the following parts of her passage from V to A, be denoted by the Greek letter λ , and its little increment, Mm , contemporary with the increment $C\epsilon$, or $\dot{l}$, be denoted by $\dot{\lambda}$, or the same Greek letter with a point placed over it; and let $\dot{y}$ denote the increment of y , or of the secant of the circular arch Cm , or of the latitude of the point C, or the ship's place, contemporary with the increment $C\epsilon$, or $\dot{l}$, of the said arch Cm , or l , itself; and let $\dot{T}$ denote the increment of T , or of the tangent of the circular arch Cm , or of the latitude of C, or the ship's place, contemporary with $\dot{l}$ and $\dot{y}$.

Then will $\dot{y}$, or the increment of y , the secant of the arch Cm , be to $\dot{l}$, or $C\epsilon$,

Cc , or the increment of the said arch Cm itself, as the rectangle under the said secant y and the tangent T of the same arch Cm , to the square of the radius of the circle to which the arch Cm belongs, that is, to the square of the radius r of a great circle of the Earth; as will be evident from my Elements of Plane Trigonometry, Prop. 30, Coroll. 4, art. 165, pages 184, 185. Therefore we shall have Cc , or i , $= \frac{rr \times j}{y \times r}$.

But T^2 , or the square of the tangent T , is equal to $yy - rr$, or the excess of the square of the secant y above the square of the radius r . And consequently T will be $= \sqrt{yy - rr}$. Therefore $\frac{rr \times j}{y \times r}$ will be $= \frac{rr \times j}{y \times \sqrt{yy - rr}}$; and consequently Cc , or i , (which is equal to $\frac{rr \times j}{y \times r}$) will be $= \frac{rr \times j}{y \times \sqrt{yy - rr}}$.

Further, the little right-lined triangle, formed by the chords of the three little arches BC , Cc , and Bc , will be a right-angled triangle having a right angle at the point c ; as has been shewn above in art. 7. Therefore, if C be made the center, and the chord Cc be made the radius, of a circle, the line Bc will be tangent of the angle BCc in the said circle. But the angle BCc is equal to the angle LAb , or the rhumb angle, of which (by art. 11,) t is the tangent in a great circle of the Earth, or in a circle of which r , or the radius of the Earth, is the radius. Therefore the small tangent Bc will be to the small radius Cc , or the chord of the little arch Cc , as the great tangent t of the same angle in a circle of which r is the radius, is to the radius r ; and consequently Bc will be $= \frac{t \times cc}{r}$.

But Cc has been shewn to be $= \frac{rr \times j}{y \times \sqrt{yy - rr}}$.

Therefore Bc will be $= \frac{t}{r} \times \frac{rr \times j}{y \times \sqrt{yy - rr}} = \frac{tr \times j}{y \times \sqrt{yy - rr}}$.

Art. 15. Further, because the little arch Bc of the circle drawn parallel to the Equator through the point B , is contained between the same two meridian circles PCm and PBM as the little arch Mm of the circumference of the Equator, it follows that Bc will be to the whole circumference of the circle to which it belongs in the same proportion as Mm is to the whole circumference of the Equator. Therefore, *permutando*, Bc will be to Mm in the same proportion as the whole circumference of the former circle, or of the parallel of latitude passing through the point B , to the whole circumference of the Equator, and consequently as the radius of the former circle, or of the parallel of latitude passing through the point B , to the radius of the Equator, or to r . But the radius of the parallel of latitude that passes through the point B , is the sine of the arch PB in the meridian circle PBM ; or (because the points B and C are extremely near to each other, and consequently the arch PC in the meridian

ridian circle PCm may be considered as equal to the arch PB in the meridian circle PBM ,) it is the sine of the arch PC in the meridian circle PCm , and consequently is the co-sine of the arch Cm in the same meridian circle, or the co-sine of the latitude of the point C . Therefore the little arch Bc will be to the little arch Mm as the co-sine of the arch Cm , or of the latitude of the point C , is to the radius r .

But the co-sine of the arch Cm is to the radius r as the radius r is to the secant of the same arch, or to y .

Therefore the little arch Bc will be to the little arch Mm as r is to y . And consequently the chord of the little arch Bc (which may be considered as equal to the said little arch itself,) will also be to the little arch Mm as r is to y . Therefore the little arch Mm will be $= \frac{y}{r} \times$ the chord of the little arch Bc .

But the said chord has been shewn to be equal to $\frac{tr \times j}{y \times \sqrt{yy - rr}}$.

Therefore the little arch Mm will be $= \frac{y}{r} \times \frac{tr \times j}{y \times \sqrt{yy - rr}} = \frac{t \times j}{\sqrt{yy - rr}}$;
that is, $\dot{\lambda}$ will be $= \frac{t \times j}{\sqrt{yy - rr}}$.

But $\frac{t \times j}{\sqrt{yy - rr}}$ is $= \frac{tr \times j}{r \times \sqrt{yy - rr}} = \frac{t}{r} \times \frac{r \times j}{\sqrt{yy - rr}}$.

Therefore $\dot{\lambda}$ will be $= \frac{t}{r} \times \frac{rj}{\sqrt{yy - rr}}$.

Art. 16. But, by my Elements of Plane Trigonometry, Prop. 30, Coroll. 5, page 185, the very small increment of the tangent T of the arch Cm , or of the latitude of the ship's place, is to the contemporary increment of the secant y of the same arch as the secant y is to the tangent T , or $\sqrt{yy - rr}$; that is, $\dot{T}$ is to $\dot{y}$ as y is to T , or $\sqrt{yy - rr}$. Therefore $y \times \dot{y}$ will be $= \dot{T} \times \sqrt{yy - rr}$, and consequently $\dot{T}$ will be $= \frac{y \times \dot{y}}{\sqrt{yy - rr}}$, and $r \times \dot{T}$ will be $= \frac{ryj}{\sqrt{yy - rr}}$, and $\frac{r \times \dot{t}}{j}$ will be $= \frac{rj}{\sqrt{yy - rr}}$, and $\frac{t}{r} \times \frac{r \times \dot{t}}{j}$ will be $= \frac{t}{r} \times \frac{rj}{\sqrt{yy - rr}}$.

But it has been shewn in the last article, that $\dot{\lambda}$ is $= \frac{t}{r} \times \frac{rj}{\sqrt{yy - rr}}$.

Therefore $\dot{\lambda}$ will be $= \frac{t}{r} \times \frac{r \times \dot{t}}{j}$.

Art. 17.

Art. 17. But, by my Elements of Plane Trigonometry, art. 267, if a circular arch and its tangent are supposed to increase from ϕ at the same time, and its secant is supposed to increase at the same time from its first, or least, magnitude, which it has when the arch is equal to ϕ and begins to increase, that is, from the radius, (for the secant is equal to the radius, when the arch is equal to ϕ ;) and during the increase of these three quantities, to wit, the arc, tangent, and secant, a fourth quantity increases from ϕ , and in the several very small portions of time into which the time of its increase may be supposed to be divided, receives increments which are to the contemporary small increments of the tangent in the proportion of the radius to the corresponding secant of the said arch, the said fourth quantity will be equal to the portion of the axis, or asymptote, of a logarithmick curve of which the subtangent is equal to the radius of the said circle, that is intercepted between two ordinates to the axis, or asymptote, of the said curve, of which the lesser is equal to the radius of the circle, or the subtangent of the said logarithmick curve, and the greater is equal to the sum of the secant and tangent of the said arch; or, in other words, the said fourth quantity will be equal to the logarithm of the ratio of the sum of the said secant and tangent of the said arch to the radius of the said circle, taken on the axis, or asymptote, of a logarithmick curve of which the subtangent is equal to the radius of the same circle. Therefore the

quantity generated by the little increment $\frac{r \times \tau}{y}$, or the sum of all the very numerous little increments denoted by the successive values of the fraction $\frac{r \times \tau}{y}$, will be equal to the logarithm of the ratio of the sum of the secant y and tangent T of the arch Cm , or of the latitude of the point C , or of the ship's place, to the radius r , taken on the axis, or asymptote, of a logarithmick curve of which r , or the radius of the earth, is the subtangent. And consequently the quantity λ , or the quantity generated by the little increment $\frac{r}{y}$

$\times \frac{r \times \tau}{y}$, (which is to the little increment $\frac{r \times \tau}{y}$ in the constant proportion of r , or the tangent of the rhumb-angle LAB in a circle of which r is the radius, to r ;) will be equal to $\frac{r}{y} \times$ the logarithm of the said ratio of $y + T$

to r , or of $y + \sqrt{yy - rr}$ to r , in a logarithmick curve of which the subtangent is equal to r ; that is, the arch Vm of the circumference of the Equator, or the difference of the longitudes of the points V and C , will be equal

to the product of the multiplication of the fraction $\frac{r}{y}$ into the logarithm of the ratio of the sum of the secant y and tangent T , or $\sqrt{yy - rr}$, of the arch Cm , or of the latitude of the point C , to the radius r of a great circle of the earth, in a logarithmick curve of which r , or the radius of the earth, is the subtangent. And consequently, if we suppose the point C to coincide with the

point A, or the ship to have passed over the whole loxodromick curve VBCA from the point V in the Equator to the point A, (in which case the arch CA will become equal to the arch AL, and the arch Vm will become equal to the arch VL,) the arch VL, or the difference of the longitudes of the points V and A, will be equal to the product of the multiplication of the fraction $\frac{t}{r}$ into the logarithm of the ratio of the sum of the secant and tangent of the arch AL, or the latitude of the point A, to the radius r of a great circle of the earth, taken on the axis, or asymptote, of a logarithmick curve of which the subtangent is equal to r , or the radius of a great circle of the earth.

Q. E. I.

Coroll. 1. The ratio of the sum of the secant and tangent of a circular arch to the radius is equal to the ratio of the radius to the tangent of half the complement of the said arch to the arch of a quadrant, or an arch of 90 degrees: as is shewn in my Elements of Plane Trigonometry, Prop. 23, page 64. Therefore the arch VL, or the difference of the longitudes of the points V and A (of which the former point V is in the Equator,) will be equal to the product of the multiplication of the fraction $\frac{t}{r}$ into the logarithm of the ratio of the radius r of a great circle of the earth to the tangent of half the complement of the arch AL, or the latitude of the point A, to the arch PL of the meridian circle PAL, (which is an arch of 90 degrees,) taken on the axis, or asymptote, of a logarithmick curve, of which the subtangent is equal to r , or the radius of the earth; or, if, for the sake of brevity, we denote the logarithm of the said ratio of the radius to the tangent of half the complement of the latitude, in the said logarithmick curve, by the capital letter L, we shall have λ , or the arch VL, or the difference of the longitudes of the points V and A, $= \frac{t}{r} \times L$.

Coroll. 2. The arch PA of the meridian circle PAL, is the complement of the arch AL, or of the latitude of the point A, to the arch PL, or the arch of a quadrant. And consequently the arch VL, or the difference of the longitudes of the points V and A, will be $= \frac{t}{r} \times \text{Log.} \frac{r}{\text{tangent of } \frac{PA}{2}}$, in a logarithmick curve of which the subtangent is equal to r , or the radius of the Earth.

Coroll. 3. If the angle LAB, or the rhumb-angle, or the angle of the ship's course, is an angle of 45 degrees, or half a right angle, its tangent t will be equal to the radius r , and consequently the quantity $\frac{t}{r} \times L$, or $\frac{t}{r} \times \text{log.} \frac{r}{\text{tangent of } \frac{PA}{2}}$, will be equal to $\frac{r}{r} \times L$, or $\frac{r}{r} \times \text{log.} \frac{r}{\text{tangent of } \frac{PA}{2}}$, or to L,

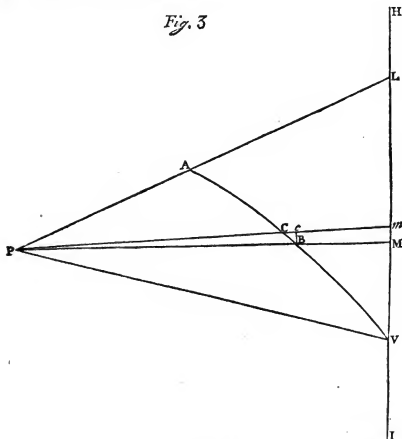
or

or $\log. \frac{r}{\text{tangent of } \frac{PA}{2}}$. Therefore, in this case, the arch VL, or the difference of

the longitudes of the points V and A, will be equal to L, or $\log. \frac{r}{\text{tangent of } \frac{PA}{2}}$,

or the logarithm of the ratio of the radius r of a great circle of the earth to the tangent of half the arch PA, or of half the complement of the latitude AL, taken on the axis, or asymptote of a logarithmick curve, of which the subtangent is equal to the radius of the earth.

Fig. 3



D 2

LEMMA

LEMMA III. A PROBLEM.

Art. 18. Let the ship be supposed to sail only from the point A (in Fig. 3,) to the point B, which is not in the Equator, but at some distance from it on the same side as the point A, and in a different meridian. And let both the arches AL and BM, or the latitudes of both the points A and B, be supposed to be known; and likewise the rhumb-angle LAB, or the angle of the ship's course, and the length of the rhumb-line AB, or the distance run by the ship in her passage from A to B along the loxodromick curve AB. It is required to determine the length of the arch LM in the circumference of the Equator, or the difference of the longitudes of the points A and B.

SOLUTION.

It appears from the foregoing Problem and its Corollaries, that if the ship had continued her course along the line BV till she had cut the Equator in the point V, the arch LV would have been equal to the quantity $\frac{t}{r} \times \log. \frac{r}{\text{tangent of } \frac{PA}{2}}$, in a logarithmick curve of which the subtangent is equal to r ,

or the radius of the earth. And, for the same reason, if she had begun her voyage from the point B, and had gone to the point V in the Equator along the loxodromick curve BV, which cuts the several meridian lines, over which it passes, in the same angle as the loxodromick curve AB cut the meridian lines lying between the points A and B, or in an angle equal to the angle LAB, (of which t is the tangent in a circle of which r is the radius,) the arch MV would have been equal to the quantity $\frac{t}{r} \times \log. \frac{r}{\text{tangent of } \frac{PB}{2}}$ in the same logarithmick

curve. Therefore the arch LM, (which is the excess of the arch LV above the arch MV,) will be equal to the quantity $\frac{t}{r} \times \log. \frac{r}{\text{tangent of } \frac{PA}{2}} - \frac{t}{r} \times \log. \frac{r}{\text{tangent of } \frac{PB}{2}}$, or to $\frac{t}{r} \times \left[\log. \frac{r}{\text{tangent of } \frac{PA}{2}} - \log. \frac{r}{\text{tangent of } \frac{PB}{2}} \right]$; that is, the

arch

arch LM, or the difference of the longitudes of the points A and B, will be equal to the product of the multiplication of the fraction $\frac{t}{r}$ into the excess of the logarithm of the ratio of the radius r to the tangent of half the arch PA, or of half the complement of the latitude of the first point A to an arch of 90 degrees, above the logarithm of the ratio of the radius r to the tangent of half the arch PB, or of half the complement of the latitude of the second point B to an arch of 90 degrees, in a logarithmick curve, of which the subtangent is equal to r , or the radius of the Earth. Q. E. I.

Coroll. 1. To shorten the foregoing expression of the value of the arch LM, let a be put for the logarithm of the ratio of the radius r to the tangent of half the arch PA, and let b be put for the logarithm of the ratio of the radius r to the tangent of half the arch PB, in a logarithmick curve of which the subtangent is equal to r , or the radius of the Earth. And we shall then have $LM = \frac{t}{r} \times \overline{a - b}$, or the difference of the longitudes of the two points A and B will be equal to the product of the multiplication of the fraction $\frac{t}{r}$ into the excess of the logarithm a above the logarithm b .

Coroll. 2. If LAB, or the angle of the ship's course, be 45 degrees, or half a right angle, the tangent t will be equal to the radius r , and consequently $\frac{t}{r} \times \overline{a - b}$ will be $= \frac{r}{r} \times \overline{a - b} = \overline{a - b}$. Therefore in this case the arch LM, or the difference of the longitudes of the two points A and B, will be equal to $a - b$, or to the excess of the logarithm a above the logarithm b .

Art. 19. By the help of these three Lemmas, we may now proceed to solve the Problem proposed by Dr. HALLEY, which is as follows.

THE PROBLEM PROPOSED BY DR. HALLEY.

Let P, in Fig. 1, be one of the Poles of the Earth; HI a portion of the circumference of the Equator; A and B two points on the surface of the Earth, both on the same side of the Equator HI, but at different distances from it, and in different meridian circles PAL and PBM, which cut the Equatorial arch HI in the points L and M. And let the point A be more distant from the Equator than the point B. Further, let a ship be supposed to sail from the point

point A to the point B along the rhumb-line, or loxodromick curve, AB, which cuts all the meridians, over which it passes, in the same angle, to wit, in an angle equal to the angle LAB. And let the arch LM, or the difference of the longitudes of the two points A and B, and likewise the length of the rhumb-line, or loxodromick curve, AB, or the number of miles run over by the ship in her passage from A to B, and the arch AL, or the latitude of the first point A, from which the ship begins her voyage, be supposed to be known; but not the arch BM, or the latitude of the second point B, at which the ship arrives after passing over the line AB, nor the angle LAB, which the line of the ship's course makes with the several meridian-lines over which it passes. It is required to find the said angle LAB.

THE SOLUTION.

Art. 20. In the foregoing Problem, or Lemma 3d, the two circular arches AL and BM, or the latitudes of the points A and B, were supposed to be known; as were likewise the rhumb angle LAB and the length of the rhumb-line AB: and the length of the Equatorial arch LM, or the difference of the longitudes of the two points A and B, was supposed to be unknown, and was derived from the former quantities in the Solution of that Problem, and was shewn in Coroll. 1 of the said Problem, to be equal to $\frac{t}{r} \times \overline{a-b}$, the letters a and b being put for the logarithms of the ratios of the radius r to the tangent of half the arch PA, or half the complement of the latitude of the first point A, to an arch of 90 degrees, and to the tangent of half the arch PB, or half the complement of the latitude of the second point B, to an arch of 90 degrees, in a logarithmick curve of which the subtangent is equal to r , or the radius of the Earth. But in the present Problem the latter of these two ratios, (which depends on the arch BM, or the latitude of the second point B,) and consequently its logarithm b , is unknown. Nor can the arch BM, or the latitude of the second point B, be derived from the arch AL, or the latitude of the first point A, and the length of the rhumb-line AB, (which are both supposed to be known,) in the manner directed in the first Lemma; because in that Lemma the rhumb-angle LAB was supposed to be known, and consequently its tangent t (in the circle of which r is the radius,) was also known, as well as the length of the rhumb-line AB, and the arch AL, or the latitude of the first point A; whereas in the present Problem the rhumb-angle LAB is supposed to be unknown, and is the very quantity we are required to find. Therefore in the equation $LM = \frac{t}{r} \times \overline{a-b}$

(obtained

(obtained in the first Corollary of Lemma 3d,) there will be two unknown quantities, to wit, the tangent t of the unknown angle $L A B$, and the logarithm b of the ratio of r to the tangent of half the arch $P B$. We cannot therefore consider the equation $L M = \frac{t}{r} \times \overline{a - b}$ as a simple equation containing only one unknown quantity t , (in which case we should have $r \times L M = t \times \overline{a - b}$, and $t = \frac{L M \times r}{\overline{a - b}}$;) but must investigate the relation between the two unknown quantities t and b , which are mutually dependant on each other, and must reduce them to one, before we can discover the value of t , or the tangent of the unknown angle $L A B$. And this investigation is a matter of great nicety and difficulty.

Art. 21. Let x be put for the co-sine of the rhumb-angle $L A B$, of which t is the tangent, in the circle of which r , or the radius of the Earth, is the radius. Then will the square of the sine of the said angle be $rr - xx$, and the said sine itself will be $= \sqrt{rr - xx}$. But the cosine of any arc is to its sine as the radius is to the tangent. Therefore x will be to $\sqrt{rr - xx}$ as r is to t ; and consequently t will be $= \frac{r \times \sqrt{rr - xx}}{x}$. But it has been shewn above, in Lemma 3d, art. 18, that the arch $L M$ of the circumference of the Equator, or the difference of the longitudes of the points A and B , is $= \frac{t}{r} \times$

$\left[\log. \frac{r}{\text{tangent of } \frac{PA}{2}} - \log. \frac{r}{\text{tangent of } \frac{PB}{2}} \right]$. Therefore the said arch $L M$, or the

said difference of longitudes, will be $= \frac{r \times \sqrt{rr - xx}}{r \times x} \times$

$\log. \frac{r}{\text{tangent of } \frac{PA}{2}} - \log. \frac{r}{\text{tangent of } \frac{PB}{2}} = \frac{\sqrt{rr - xx}}{x} \times$

$\log. \frac{r}{\text{tangent of } \frac{PA}{2}} - \log. \frac{r}{\text{tangent of } \frac{PB}{2}}$; or (because a is $= \log. \frac{r}{\text{tangent of } \frac{PA}{2}}$,

and b is $= \log. \frac{r}{\text{tangent of } \frac{PB}{2}}$;) $L M$ will be $= \frac{\sqrt{rr - xx}}{x} \times \overline{a - b}$.

Let c be put for the arch $L M$, or the difference of the longitudes of the points A and B , which in the present Problem is supposed to be known. And we shall then have $c = \frac{\sqrt{rr - xx}}{x} \times \overline{a - b}$; in which equation the quantity b is unknown, as well as the quantity x , but is dependant on it and may be derived from it, though not without great difficulty.

Art. 22.

Art. 22. Let d be put for the length of the loxodromick curve AB , or the distance run by the ship in her passage from A to B .

Then, since, by Lemma 1, the length of the loxodromick curve AB is to AQ , or $AL - BM$, as r is to x , and l is $= AL$, we shall have d to $l - BM$ as r is to x ; and consequently dx will be $= (l - BM) \times r$, and $l - BM$ will be $= \frac{dx}{r}$. Therefore (adding BM to both sides, we shall have $l = \frac{dx}{r} + BM$, and (subtracting $\frac{dx}{r}$ from both sides,) we shall have $BM = l - \frac{dx}{r}$; that is, the circular arch BM , or the latitude of the second point B , will be equal to $l - \frac{dx}{r}$.

Art. 23. Now let q be put for the length of the arch PBM , or the arch of a whole quadrant of the meridian circle which passes through the point B , and of which r , or the radius of the circle, is the radius. Then will PB , or $PBM - BM$, be $= q - BM$; or (because BM is $= l - \frac{dx}{r}$), PB will be $(= q - (l - \frac{dx}{r})) = q - l + \frac{dx}{r}$; and consequently $\frac{PB}{2}$ will be equal to $(\frac{q - l + \frac{dx}{r}}{2}) = \frac{q}{2} - \frac{l}{2} + \frac{dx}{2r}$. Therefore b , or the logarithm of the ratio of the radius r to the tangent of the arch $\frac{PB}{2}$, will be equal to the logarithm of the ratio of the radius r to the tangent of the arch $\frac{q}{2} - \frac{l}{2} + \frac{dx}{2r}$.

Art. 24. Further, because in every circle the radius is a mean proportional between the tangent of any arch less than 90 degrees, or the arch of a quadrant, and the co-tangent of the same arch, or the tangent of its complement to the arch of a quadrant, it follows that the ratio of the radius r to the tangent of the arch $\frac{q}{2} - \frac{l}{2} + \frac{dx}{2r}$ will be equal to the ratio of the tangent of the arch which is the complement of the arch $\frac{q}{2} - \frac{l}{2} + \frac{dx}{2r}$ to an arch of a quadrant, or to q , to the radius r . But the complement of the arch $\frac{q}{2} - \frac{l}{2} + \frac{dx}{2r}$ to the arch q is $(= \text{the arch } q - \sqrt{\frac{q}{2} - \frac{l}{2} + \frac{dx}{2r}} = q - \frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}) = \text{the arch } \frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$. Therefore the ratio of the tangent of the arch $\frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$ to the radius r will be equal to the ratio of the radius r to the tangent

of the arch $\frac{q}{2} - \frac{l}{2} + \frac{dx}{2r}$; and consequently the logarithm of the former ratio will be equal to the logarithm of the latter ratio. But the logarithm of the latter ratio is equal to b . Therefore the logarithm of the former ratio will also be equal to b ; or b will be equal to the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$ to the radius r . Therefore $a - b$ will be =

$a - \log.$ of $\frac{\text{tangent of } \frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}}{r}$, and consequently $L M$, or c , which is equal to $\frac{\sqrt{rr - xx}}{x} \times a - b$, will be = $\frac{\sqrt{rr - xx}}{x} \times$

$a - \log.$ of $\frac{\text{tangent of } \frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}}{r}$. We must therefore now endeavour to find an expression for the logarithm of the ratio of the tangent of the arch of $\frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$ to the radius r , that shall consist of a set of simple terms involving either known quantities, or the powers of the unknown quantity x combined with known quantities. This may be done in the manner following.

ART. 25. The arch PB is less than the arch PM , or than the arch of a whole quadrant of a circle. And consequently the arch $\frac{PB}{2}$ must be less than an arch of 45 degrees, or half the arch of a quadrant. Therefore the complement of the arch $\frac{PB}{2}$ to an arch of 90 degrees, must be greater than an arch of 45 degrees. But it has been shewn that the arch $\frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$ is the complement of the arch $\frac{PB}{2}$ to an arch of 90 degrees. Therefore the arch $\frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$ will be greater than an arch of 45 degrees. And this arch $\frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$, (being the complement of the arch $\frac{PB}{2}$ to an arch of 90 degrees,) must be less than 90 degrees. Therefore it will be greater than 45 degrees, but less than 90 degrees. But, when an arch is greater than 45 degrees, but less than 90 degrees, or the arch of a quadrant, Mr. JAMES GREGORY has given us a series that will express the logarithm of the ratio of its tangent to the radius of the circle, (which logarithm he calls its *artificial, or logarithmick, tangent*;) in terms involving the powers of the excess of twice the said arch above the arch of a quadrant and the powers of the radius of the circle; which series is as follows. If the radius of a circle be called R , and an arch that is less than an arch of 90 degrees, but greater than an arch of 45

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degrees, in the said circle, be called A, and an arch of 90 degrees in the said circle be called Q, and an arch that is equal to $2A - Q$ be called E, the logarithm of the ratio of the tangent of the arch A to the radius R in a logarithmick curve of which the subtangent is equal to the radius R, will be equal to the infinite series $E + \frac{E^2}{6R^2} + \frac{E^3}{24R^4} + \frac{61E^4}{5040R^6} + \frac{277E^5}{72,576R^8} + \&c.$ * Therefore, if, in the circle of which r , or the radius of the Earth, is the radius, and in which q is the arch of a quadrant, the arch $\frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$ (which has been shewn to be greater than an arch of 45 degrees in the said circle, but at the same time is less than the arch of a whole quadrant, being the complement of the arch $\frac{r^2}{2}$ to q , or the arch of a quadrant,) be doubled, and from the double of it, which will be $= q + l - \frac{dx}{r}$, we subtract the quadrantal arch q , and substitute the remaining arch $l - \frac{dx}{r}$ instead of E, and r instead of R, in the terms of the foregoing series $E + \frac{E^2}{6R^2} + \frac{E^3}{24R^4} + \frac{61E^4}{5040R^6} + \frac{277E^5}{72,576R^8} + \&c.$ *ad infinitum*, the complicated series thence arising, to wit, the series $l - \frac{dx}{r} + \frac{\left(l - \frac{dx}{r}\right)^2}{6rr} + \frac{\left(l - \frac{dx}{r}\right)^3}{24r^4} + \frac{61 \times \left(l - \frac{dx}{r}\right)^4}{5040r^6} + \frac{277 \times \left(l - \frac{dx}{r}\right)^5}{72,576r^8} + \&c.$ *ad infinitum*, will be equal to the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$ to the radius r , in a logarithmick curve of which the subtangent is equal to the radius r ; or the said series will be equal to b . And consequently the equatorial arch LM, or c , (which is equal to $\frac{\sqrt{rr - xx}}{x} \times a - b$;) will be $= \frac{\sqrt{rr - xx}}{x} \times$ the excess of a above the said complicated series $l - \frac{dx}{r} + \frac{\left(l - \frac{dx}{r}\right)^2}{6rr} + \frac{\left(l - \frac{dx}{r}\right)^3}{24r^4} + \frac{61 \times \left(l - \frac{dx}{r}\right)^4}{5040r^6} + \frac{277 \times \left(l - \frac{dx}{r}\right)^5}{72,576r^8} + \&c.$ This equation must now be reduced into a proper form for a resolution of it; which may be done in the manner following.

* Mr. James Gregory has given us only these five terms of this series; and these are all that I shall make use of in the present solution. But the next two terms of the series will be found to be $+\frac{50,521E^{10}}{359,916,800R^{10}} + \frac{41,581E^{12}}{95,800,320R^{12}}$ as is shewn in the 3d volume of this Collection of Tracts, called *Scriptures Logarithmici*, page 453.

Art. 26. The second term, $\frac{l - \frac{dx}{r}}{6rr}$, of the foregoing series is

$$l - \frac{3^{1/2} dx}{r} + \frac{3^{1/2} d^2 x^2}{6rr} - \frac{d^3 x^3}{r^3} = \frac{r^3 l^3 - \frac{3r^{1/2} dx}{r^3} + \frac{3r^{1/2} d^2 x^2}{r^3} - \frac{d^3 x^3}{r^3}}{6rr} =$$

$$\frac{r^3 l^3 - 3r^{1/2} dx + 3r^{1/2} d^2 x^2 - d^3 x^3}{6r^3}.$$

And the third term $\frac{l - \frac{dx}{r}}{24r^4}$ is = $\frac{l - \frac{5^{1/2} dx}{r} + \frac{10^{1/2} d^2 x^2}{r^2} - \frac{10^{1/2} d^3 x^3}{r^3} + \frac{5^{1/2} d^4 x^4}{r^4} - \frac{d^5 x^5}{r^5}}{24r^4}$

$$= \frac{r^5 l^5 - \frac{5r^{1/2} dx}{r^5} + \frac{10r^{1/2} d^2 x^2}{r^5} - \frac{10r^{1/2} d^3 x^3}{r^5} + \frac{5r^{1/2} d^4 x^4}{r^5} - \frac{d^5 x^5}{r^5}}{24r^4} =$$

$$\frac{r^5 l^5 - 5r^{1/2} dx + 10r^{1/2} d^2 x^2 - 10r^{1/2} d^3 x^3 + 5r^{1/2} d^4 x^4 - d^5 x^5}{24r^9}.$$

And the fourth term, $\frac{61 \times l - \frac{dx}{r}}{5040r^5}$ is

$$= \frac{61}{5040r^5} \times \left\{ l^5 - \frac{7^{1/2} dx}{r} + \frac{21^{1/2} d^2 x^2}{r^2} - \frac{35^{1/2} d^3 x^3}{r^3} + \frac{35^{1/2} d^4 x^4}{r^4} - \frac{21^{1/2} d^5 x^5}{r^5} + \right.$$

$$\left. = \frac{61}{5040r^5} \times \left\{ \frac{r^7 l^7}{r^7} - \frac{7r^{1/2} dx}{r^7} + \frac{21r^{1/2} d^2 x^2}{r^7} - \frac{35r^{1/2} d^3 x^3}{r^7} + \frac{35r^{1/2} d^4 x^4}{r^7} - \right.$$

$$\left. \frac{21r^{1/2} d^5 x^5}{r^7} + \frac{7r^{1/2} d^6 x^6}{r^7} - \frac{d^7 x^7}{r^7} \right\}$$

$$= \frac{61r^7 l^7 - 45r^{1/2} dx + 1281r^{1/2} d^2 x^2 - 215r^{1/2} d^3 x^3 + 2135r^{1/2} d^4 x^4 - 1281r^{1/2} d^5 x^5 + 427r^{1/2} d^6 x^6 - 61d^7 x^7}{5040r^{12}}.$$

And the fifth term, $\frac{277 \times l - \frac{dx}{r}}{72576r^6}$, is = $\frac{277}{72576r^6} \times$ the compound quantity

$$l^6 - \frac{9^{1/2} dx}{r} + \frac{36^{1/2} d^2 x^2}{r^2} - \frac{84^{1/2} d^3 x^3}{r^3} + \frac{236^{1/2} d^4 x^4}{r^4} - \frac{1256^{1/2} d^5 x^5}{r^5} + \frac{84^{1/2} d^6 x^6}{r^6} - \frac{36^{1/2} d^7 x^7}{r^7}$$

$$+ \frac{9^{1/2} d^8 x^8}{r^8} - \frac{d^9 x^9}{r^9} = \frac{277}{72576r^6} \times$$

$$\frac{r^9 l^9 - \frac{9r^{1/2} dx}{r^9} + \frac{36r^{1/2} d^2 x^2}{r^9} - \frac{84r^{1/2} d^3 x^3}{r^9} + \frac{236r^{1/2} d^4 x^4}{r^9} - \frac{1256r^{1/2} d^5 x^5}{r^9} + \frac{84r^{1/2} d^6 x^6}{r^9} - \frac{36r^{1/2} d^7 x^7}{r^9} + \frac{9r^{1/2} d^8 x^8}{r^9} - \frac{d^9 x^9}{r^9}}{72576r^{12}}$$

$$= \frac{277r^9 l^9 - 2493r^{1/2} dx + 9972r^{1/2} d^2 x^2 - 23268r^{1/2} d^3 x^3 + 34902r^{1/2} d^4 x^4 - 34902r^{1/2} d^5 x^5 + 23268r^{1/2} d^6 x^6 - 9972r^{1/2} d^7 x^7 + 2493r^{1/2} d^8 x^8 - 277d^9 x^9}{72576r^{12}}.$$

Therefore the said complicated series $l - \frac{dx}{r} + \frac{l - \frac{dx}{r}}{6rr} + \frac{l - \frac{dx}{r}}{24r^4} + \frac{l - \frac{dx}{r}}{61 \times}$

E 2

$$\begin{aligned}
& \frac{61 \times \sqrt{1 - \frac{dx}{r}}}{5040r^6} + \frac{277 \times 1 - \frac{dx}{r}}{72,576r^3} + \&c, \text{ will be equal to the series } 1 - \frac{dx}{r} + \\
& \frac{r^{12} - 3r^{10}dx + 3r^{10}d^2x - d^3x^3}{6r^3} + \frac{r^{12} - 5r^{10}dx + 10r^{10}d^2x^2 - 10r^{10}d^3x^3 + 5r^{10}d^4x^4 - d^5x^5}{24r^9} \\
& + \frac{61r^{12} - 427r^{10}dx + 1281r^{10}d^2x^2 - 2135r^{10}d^3x^3 + 2135r^{12}d^4x^4 - 1281r^{10}d^5x^5}{42,720r^6} - \frac{5040r^6}{72,576r^{12}} \\
& + \frac{277r^{12} - 2493r^{10}dx + 9972r^{10}d^2x^2 - 23,268r^{10}d^3x^3 + 34,902r^{10}d^4x^4 - 34,902r^{10}d^5x^5}{72,576r^{12}} \\
& + \&c = \text{the series } \frac{r^{12}}{r} + \frac{r^{12}}{6r^3} + \frac{r^{12}}{24r^9} + \frac{61r^{12}}{5040r^{12}} + \frac{277r^{12}}{72,576r^{12}} + \&c \\
& - \text{the series } \frac{dx}{r} + \frac{3r^{10}dx}{6r^3} + \frac{5r^{10}dx}{24r^9} + \frac{427r^{10}dx}{5040r^{12}} + \frac{2493r^{10}dx}{72,576r^{12}} + \&c \\
& + \text{the series } \frac{3r^{10}d^2x^2}{6r^3} + \frac{10r^{10}d^2x^2}{24r^9} + \frac{1281r^{10}d^2x^2}{5040r^{12}} + \frac{9972r^{10}d^2x^2}{72,576r^{12}} + \&c \\
& - \text{the series } \frac{d^3x^3}{6r^3} + \frac{10r^{10}d^3x^3}{24r^9} + \frac{2135r^{10}d^3x^3}{5040r^{12}} + \frac{23,268r^{10}d^3x^3}{72,576r^{12}} + \&c \\
& + \text{the series } \frac{5r^{10}d^4x^4}{24r^9} + \frac{2135r^{10}d^4x^4}{5040r^{12}} + \frac{34,902r^{10}d^4x^4}{72,576r^{12}} + \&c \\
& - \text{the series } \frac{d^5x^5}{24r^9} + \frac{1281r^{10}d^5x^5}{5040r^{12}} + \frac{34,902r^{10}d^5x^5}{72,576r^{12}} + \&c \\
& + \text{the series } \frac{427r^{10}d^6x^6}{5040r^{12}} + \frac{23,268r^{10}d^6x^6}{72,576r^{12}} + \&c \\
& - \text{the series } \frac{61d^7x^7}{5040r^{12}} + \frac{9972r^{10}d^7x^7}{72,576r^{12}} + \&c \\
& + \text{the series } \frac{2493r^{10}d^8x^8}{72,576r^{12}} + \&c \\
& - \text{the series } \frac{277r^{10}d^9x^9}{72,576r^{12}} + \&c \\
& = \text{the series } 1 + \frac{r^2}{6r^3} + \frac{r^6}{24r^9} + \frac{61r^2}{5040r^6} + \frac{277r^6}{72,576r^6} + \&c \\
& - \text{the series } \frac{dx}{r} + \frac{r^2dx}{2r^2} + \frac{5r^4dx}{24r^9} + \frac{61r^4dx}{720r^2} + \frac{277r^4dx}{8064r^9} + \&c \\
& + \text{the series } \frac{d^2x^2}{2r^4} + \frac{5r^4d^2x^2}{12r^9} + \frac{61r^4d^2x^2}{240r^3} + \frac{277r^4d^2x^2}{2016r^{10}} + \&c \\
& - \text{the series } \frac{d^3x^3}{6r^3} + \frac{5r^4d^3x^3}{12r^2} + \frac{61r^4d^3x^3}{144r^9} + \frac{277r^4d^3x^3}{864r^{12}} + \&c \\
& + \text{the series } \frac{5r^4d^4x^4}{24r^3} + \frac{61r^4d^4x^4}{144r^{10}} + \frac{277r^4d^4x^4}{576r^{12}} + \&c \\
& - \text{the series } \frac{d^5x^5}{24r^9} + \frac{61r^4d^5x^5}{240r^{12}} + \frac{277r^4d^5x^5}{576r^{12}} + \&c \\
& + \text{the series } \frac{61d^6x^6}{720r^{12}} + \frac{277r^4d^6x^6}{864r^{16}} + \&c
\end{aligned}$$

— the

$$\begin{aligned}
 & - \text{the series } \frac{61d^3x^2}{5040r^{13}} + \frac{277d^3d^2x^2}{2016r^{13}} + \&c \\
 & + \text{the series } \frac{277d^3d^2x^2}{8064r^{16}} + \&c \\
 & - \text{the series } \frac{277d^3d^2x^2}{72576r^{17}} + \&c
 \end{aligned}$$

Art. 27. Now let the capital letter B be put = the infinite series $l + \frac{l^3}{6rr}$
 $+ \frac{l^3}{24r^4} + \frac{61l^3}{5040r^6} + \frac{277l^3}{72576r^8} + \&c$; and let the capital letter C be put = the
 infinite series $d + \frac{l^2d}{2r^3} + \frac{5l^2d}{24r^4} + \frac{61l^2d}{720r^6} + \frac{277l^2d}{8064r^8} + \&c$; and let D be = the
 infinite series $\frac{ld^2}{2r^3} + \frac{5l^2d^2}{12r^4} + \frac{61l^2d^2}{240r^6} + \frac{277l^2d^2}{2016r^8} + \&c$; and F be = the infinite
 series $\frac{d^3}{6r^3} + \frac{5l^2d^3}{12r^4} + \frac{61l^2d^3}{144r^6} + \frac{277l^2d^3}{864r^8} + \&c$; and G be = the infinite series
 $\frac{5ld^4}{24r^4} + \frac{61l^2d^4}{144r^6} + \frac{277l^2d^4}{576r^8} + \&c$; and H be = the infinite series $\frac{d^5}{24r^4} + \frac{61l^2d^5}{240r^6}$
 $+ \frac{277l^2d^5}{576r^8} + \&c$; and I be = the infinite series $\frac{61ld^5}{720r^6} + \frac{277l^2d^5}{864r^8} + \&c$; and
 K be = the infinite series $\frac{61d^5}{5040r^6} + \frac{277l^2d^5}{2016r^8} + \&c$; and L be = the infinite
 series $\frac{277ld^5}{8064r^8} + \&c$; and M be = the infinite series $\frac{277d^5}{72576r^8} + \&c$. And the

foregoing complicated series $l - \frac{dx}{r} + \frac{l - \frac{dx}{r}}{6rr} + \frac{l - \frac{dx}{r}}{24r^4} + \frac{61 \times l - \frac{dx}{r}}{5040r^6} +$
 $\frac{277 \times l - \frac{dx}{r}}{72576r^8} + \&c$, *ad infinitum*, will be equal to the series $B - \frac{Cx}{r} + \frac{Dx^2}{r^2} -$
 $\frac{Ex^3}{r^3} + \frac{Gx^4}{r^4} - \frac{Hx^5}{r^5} + \frac{Ix^6}{r^6} - \frac{Kx^7}{r^7} + \frac{Lx^8}{r^8} - \frac{Mx^9}{r^9} + \&c$, *ad infinitum*.

Art. 28. But, by art. 25, the complicated series $l - \frac{dx}{r} + \frac{l - \frac{dx}{r}}{6rr} +$
 $\frac{l - \frac{dx}{r}}{24r^4} + \frac{61 \times l - \frac{dx}{r}}{5040r^6} + \frac{277 \times l - \frac{dx}{r}}{72576r^8} + \&c$, *ad infinitum*, is equal to b , or
 the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{l}{2} - \frac{dx}{2r}$ to the
 radius r in a logarithmick curve of which the subtangent is equal to the radius r .
 Therefore the series $B - \frac{Cx}{r} + \frac{Dx^2}{r^2} - \frac{Ex^3}{r^3} + \frac{Gx^4}{r^4} - \frac{Hx^5}{r^5} + \frac{Ix^6}{r^6} - \frac{Kx^7}{r^7} + \frac{Lx^8}{r^8} -$

the tangent of the arch $\frac{PA}{2}$ in a logarithmick curve of which r is the subtangent.

But a is equal to this logarithm, as appears from Coroll. 1, to Lemma 3d.

Therefore B will be equal to a . And consequently the series $a = B + \frac{Cx}{r} - \frac{Dx^3}{r^3} + \frac{Ex^5}{r^5} - \frac{Gx^7}{r^7} + \frac{Hx^9}{r^9} - \frac{Ix^{11}}{r^{11}} + \frac{Kx^{13}}{r^{13}} - \frac{Lx^{15}}{r^{15}} + \frac{Mx^{17}}{r^{17}} - \&c$, *ad infinitum*, will be equal to the same series without the two first terms $a = B$, that is, to the series $\frac{Cx}{r} - \frac{Dx^3}{r^3} + \frac{Ex^5}{r^5} - \frac{Gx^7}{r^7} + \frac{Hx^9}{r^9} - \frac{Ix^{11}}{r^{11}} + \frac{Kx^{13}}{r^{13}} - \frac{Lx^{15}}{r^{15}} + \frac{Mx^{17}}{r^{17}} - \&c$, *ad infinitum*.

But the series $a = B + \frac{Cx}{r} - \frac{Dx^3}{r^3} + \frac{Ex^5}{r^5} - \frac{Gx^7}{r^7} + \frac{Hx^9}{r^9} - \frac{Ix^{11}}{r^{11}} + \frac{Kx^{13}}{r^{13}} - \frac{Lx^{15}}{r^{15}} + \frac{Mx^{17}}{r^{17}} - \&c$, *ad infinitum*, has been shewn to be equal to $a = b$.

Therefore the series $\frac{Cx}{r} - \frac{Dx^3}{r^3} + \frac{Ex^5}{r^5} - \frac{Gx^7}{r^7} + \frac{Hx^9}{r^9} - \frac{Ix^{11}}{r^{11}} + \frac{Kx^{13}}{r^{13}} - \frac{Lx^{15}}{r^{15}} + \frac{Mx^{17}}{r^{17}} - \&c$, *ad infinitum*, will also be equal to $a = b$.

Art. 31. But c , or the equatorial arch LM , or the difference of the longitudes of the points A and B , is $= \frac{\sqrt{rr - xx}}{x} \times a = b$. Therefore c , or LM , will also be $= \frac{\sqrt{rr - xx}}{x} \times$ the series $\frac{Cx}{r} - \frac{Dx^3}{r^3} + \frac{Ex^5}{r^5} - \frac{Gx^7}{r^7} + \frac{Hx^9}{r^9} - \frac{Ix^{11}}{r^{11}} + \frac{Kx^{13}}{r^{13}} - \frac{Lx^{15}}{r^{15}} + \frac{Mx^{17}}{r^{17}} - \&c$, *ad infinitum*; and consequently (multiplying both sides of the equation by x) we shall have $cx = \sqrt{rr - xx} \times$ the series $\frac{Cx}{r} - \frac{Dx^3}{r^3} + \frac{Ex^5}{r^5} - \frac{Gx^7}{r^7} + \frac{Hx^9}{r^9} - \frac{Ix^{11}}{r^{11}} + \frac{Kx^{13}}{r^{13}} - \frac{Lx^{15}}{r^{15}} + \frac{Mx^{17}}{r^{17}} - \&c$, *ad infinitum*; and (by extracting the square-root of the residual quantity $rr - xx$, and substituting the infinite series to which it is equal, to wit, the infinite series $r - \frac{xx}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7} - \frac{7x^{10}}{256r^9} - \&c$, instead of $\sqrt{rr - xx}$ in this last equation,) we shall have $cx =$ the product of the multiplication of the infinite series $r - \frac{xx}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7} - \frac{7x^{10}}{256r^9} - \&c$, into the infinite series $\frac{Cx}{r} - \frac{Dx^3}{r^3} + \frac{Ex^5}{r^5} - \frac{Gx^7}{r^7} + \frac{Hx^9}{r^9} - \frac{Ix^{11}}{r^{11}} + \frac{Kx^{13}}{r^{13}} - \frac{Lx^{15}}{r^{15}} + \frac{Mx^{17}}{r^{17}} - \&c$, and consequently to the following complicated infinite series, to wit,

$$\left\{ \begin{aligned} Cx - \frac{Dx^3}{r} + \frac{FA^3}{r^3} - \frac{Gx^4}{r^3} + \frac{Hx^5}{r^4} - \frac{Ix^6}{r^5} + \frac{Kx^7}{r^6} - \frac{Lx^8}{r^7} + \frac{Mx^9}{r^8} - \&c \\ - \frac{Cx^3}{2r^2} + \frac{Dx^4}{2r^3} - \frac{FA^4}{2r^4} + \frac{Gx^5}{2r^5} - \frac{Hx^6}{2r^6} + \frac{Ix^7}{2r^7} - \frac{Kx^8}{2r^8} + \&c \\ - \frac{Cx^5}{8r^4} + \frac{Dx^6}{8r^5} - \frac{FA^6}{8r^6} + \frac{Gx^7}{8r^7} - \frac{Hx^8}{8r^8} + \&c \\ - \frac{Cx^7}{16r^6} + \frac{Dx^8}{16r^7} - \frac{FA^8}{16r^8} + \&c \\ - \frac{5Cx^9}{128r^8} + \&c \end{aligned} \right.$$

and consequently (by dividing all the terms of the last equation by x), we shall have the complicated series

$$\left\{ \begin{aligned} C - \frac{Dx^2}{r} + \frac{FA^3}{r^3} - \frac{Gx^4}{r^3} + \frac{Hx^5}{r^4} - \frac{Ix^6}{r^5} + \frac{Kx^7}{r^6} - \frac{Lx^8}{r^7} + \frac{Mx^9}{r^8} - \&c \\ - \frac{Cx^3}{2r^2} + \frac{Dx^4}{2r^3} - \frac{FA^4}{2r^4} + \frac{Gx^5}{2r^5} - \frac{Hx^6}{2r^6} + \frac{Ix^7}{2r^7} - \frac{Kx^8}{2r^8} + \&c \\ - \frac{Cx^5}{8r^4} + \frac{Dx^6}{8r^5} - \frac{FA^6}{8r^6} + \frac{Gx^7}{8r^7} - \frac{Hx^8}{8r^8} + \&c \\ - \frac{Cx^7}{16r^6} + \frac{Dx^8}{16r^7} - \frac{FA^8}{16r^8} + \&c \\ - \frac{5Cx^9}{128r^8} + \&c \end{aligned} \right.$$

$= \varepsilon$; in which equation there is only one unknown quantity, to wit, x , or the co-sine of the rhumb-angle L A B. Therefore by resolving this equation we shall find the value of the said co-sine of the rhumb-angle L A B, and consequently shall, by the help of a table of sines and tangents, discover the magnitude of the said angle L A B itself. Q_4 E. 1.

Art. 32. This equation will be true only when the several infinite series denoted by the capital letters C, D, F, G, H, I, K, L, M, &c, are converging series. But this will happen only when l , or the latitude of the point A, or of the first place of the ship, from which it begins its voyage, is less than r , or the radius of the Earth, and when d , or the loxodromick curve A B, or the distance run by the ship in going from A to B, is also less than r , or the radius of the Earth; because the quantities l and d occur in the numerators of the terms of the said infinite series, and the quantity r occurs in their denominators. Therefore the said equation will be true only when l and d , or the said latitude of the point A and the distance run by the ship in going from A to B, are less than the radius of the Earth.

Art. 33. In order to simplify the notation of the foregoing equation, let r be substituted instead of r , or the radius of the Earth; or let the radius of the Earth

Earth be called 1. And then we shall have $r^2 = 1$, and $r^3 = 1$, and $r^4 = 1$, and every following power of $r = 1$; and consequently the foregoing equation will become

$$\left\{ \begin{array}{l} C - Dx + Fx^2 - Gx^3 + Hx^4 - Ix^5 + Kx^6 - Lx^7 + Mx^8 - \&c \\ \quad - \frac{Cx^2}{2} + \frac{Dx^3}{2} - \frac{Fx^4}{2} + \frac{Gx^5}{2} - \frac{Hx^6}{2} + \frac{Ix^7}{2} - \frac{Kx^8}{2} + \&c \\ \quad \quad - \frac{Cx^4}{8} + \frac{Dx^5}{8} - \frac{Fx^6}{8} + \frac{Gx^7}{8} - \frac{Hx^8}{8} + \&c \\ \quad \quad \quad - \frac{Cx^6}{16} + \frac{Dx^7}{16} - \frac{Fx^8}{16} + \&c \\ \quad \quad \quad \quad - \frac{5Cx^8}{128} + \&c \end{array} \right.$$

= 6.

An Example of the foregoing Solution.

Art. 34. Let the point P, in Fig. 1, denote the North Pole of the Earth, and the point A, from which the ship begins its voyage, be in North latitude $51^\circ, 18'$, and in West longitude, reckoned from London, $22^\circ, 6'$. Let the ship sail, in a direction partly tending to the South and partly to the West, along the loxodromick curve, or rhumb-line, AB, which cuts all the meridian-lines, over which it passes, in the same angle LAB, which is supposed to be unknown, and of which α is the co-sine in a circle of which r , or 1, or the radius of the Earth, is the radius. And let the length of the said rhumb-line AB, or the distance run by the ship when she arrives at B, be 564 miles; and the length of the equatorial arch LM, or the difference of the longitudes of the points A and B, be 786 miles. It is required to find, from these three things so given, the angle LAB of the ship's course, by means of the foregoing solution.

Art. 35. A degree of a great circle of the Earth is about 69 miles and a half. Therefore the arch AL, or the latitude of the point A, which is equal to $51^\circ, 18'$, (or $51^\circ + \frac{18'}{60}$, or $51^\circ + \frac{3}{10}$, or $51^\circ + 0.3^\circ$,) or 51.3° , will be $= 51.3 \times 69.5$ miles $= 3565.35$ miles. But the radius of the Earth is equal to 4000 miles. Therefore AL will be $= \frac{3565.35}{4000}$ of the radius of the Earth, or 0.891,337 of the radius of the Earth; or (because the radius of the Earth is called 1,) AL will be $= 0.891,337$. But l is the arch AL. Therefore l will be $= 0.891,337$.

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In the next place the loxodromick arch A B, or the distance run by the ship in passing from A to B, is supposed to be 564 miles, which is $= \frac{564}{4000}$ of the radius of the Earth, or 0.141 of the radius of the Earth; or (because the radius of the Earth is called 1,) the loxodromick arch A B will be $= 0.141$. But d is $=$ the loxodromick arch A B. Therefore d will be $= 0.141$.

Thirdly, by art. 27, the capital letter C is equal to the infinite series $d + \frac{r^2 d}{2r^4} + \frac{5r^4 d}{720r^6} + \frac{61r^6 d}{8064r^8} + \&c$, and therefore (as r is $= 1$) will be equal to the infinite series $d + \frac{r^2 d}{2} + \frac{5r^4 d}{720} + \frac{61r^6 d}{8064} + \frac{277r^8 d}{8064} + \&c$, $= d \times$ the infinite series $1 + \frac{r^2}{2} + \frac{5r^4}{720} + \frac{61r^6}{8064} + \frac{277r^8}{8064} + \&c$. But l is $= 0.891,337$. Therefore l^n will be $(= 0.891,337)^1 = 0.804,481$, and l^2 will be $(= l^1 \times l^1 = 0.804,481 \times 0.804,481) = 0.647,188$, and l^3 will be $(= l^1 \times l^2 = 0.647,188 \times 0.804,481) = 0.520,650$, and l^4 will be $(= l^1 \times l^3 = 0.520,650 \times 0.804,481) = 0.418,853$; and consequently the series $1 + \frac{r^2}{2} + \frac{5r^4}{720} + \frac{61r^6}{8064} + \frac{277r^8}{8064}$ will be $= 1.000,000 + \frac{0.804,481}{2} + \frac{5 \times 0.647,188}{720} + \frac{61 \times 0.520,650}{8064} + \frac{277 \times 0.418,853}{8064} = 1.000,000 + 0.402,240 + \frac{3.235,940}{720} + \frac{31.759,650}{8064} + \frac{116,012,181}{8064} = 1.000,000 + 0.402,240 + 0.134,831 + 0.044,110 + 0.014,387 = 1.595,558$. Therefore $d \times$ the infinite series $1 + \frac{r^2}{2} + \frac{5r^4}{720} + \frac{61r^6}{8064} + \frac{277r^8}{8064} + \&c$ will be $= d \times 1.595,558 = 0.141 \times 1.595,558 = 0.224,973$. Therefore the capital letter C will, in this case, be $= 0.224,973$.

Fourthly, the small letter e , which is equal to the equatorial arch L M, will be 786 miles, or $\frac{786}{4000}$ of the radius of the Earth, or $\frac{786}{4000} \times 1$, or 0.1965 $\times 1$, or 0.1965.

Therefore the equation set down above in art. 32, will, in this case, become

$$\left\{ \begin{aligned} 0.224,973 - dx + rx^3 - ex^5 + hx^7 - lx^9 + \frac{rx^8}{2} - \frac{ex^7}{2} + \frac{hx^6}{2} - \frac{lx^5}{2} + \frac{mx^4}{2} - \&c \\ - \frac{cx^6}{2} + \frac{dx^5}{2} - \frac{fx^4}{2} + \frac{gx^3}{2} - \frac{hx^2}{2} + \frac{ix^1}{2} - \frac{kx^0}{2} + \&c \\ - \frac{cx^6}{8} + \frac{dx^5}{8} - \frac{fx^4}{8} + \frac{gx^3}{8} - \frac{hx^2}{8} + \frac{ix^1}{8} - \frac{kx^0}{8} + \&c \\ - \frac{cx^6}{16} + \frac{dx^5}{16} - \frac{fx^4}{16} + \frac{gx^3}{16} - \frac{hx^2}{16} + \frac{ix^1}{16} - \frac{kx^0}{16} + \&c \\ - \frac{5cx^6}{128} + \&c \end{aligned} \right.$$

$= 0.1965$.

Art. 36.

Art. 36. We must next find the values of the following capital letters, D, F, G, H, I, K, L, and M; which may be done as follows.

The capital letter D is equal to the infinite series $\frac{ld^3}{2r^3} + \frac{5l^2d^3}{12r^4} + \frac{61l^2d^3}{240r^5} + \frac{277l^2d^3}{2016r^6} + \&c$, or (because r is = 1,) to the infinite series $\frac{ld^3}{2} + \frac{5l^2d^3}{12} + \frac{61l^2d^3}{240} + \frac{277l^2d^3}{2016} + \&c$, or to $d^3 \times$ the infinite series $\frac{l}{2} + \frac{5l^2}{12} + \frac{61l^2}{240} + \frac{277l^2}{2016} + \&c$, or to $d^3 \times l \times$ the infinite series $\frac{1}{2} + \frac{5l}{12} + \frac{61l^2}{240} + \frac{277l^2}{2016} + \&c$. But l is = 0.891,337, and l^2 is = 0.804,481, and l^3 is = 0.647,188, and l^4 is = 0.520,650. Therefore $d^3 \times l \times$ the infinite series $\frac{1}{2} + \frac{5l}{12} + \frac{61l^2}{240} + \frac{277l^2}{2016} + \&c$, will be = $d^3 \times 0.891,337 \times$ the infinite series $\frac{1}{2} + \frac{5 \times 0.804,481}{12} + \frac{61 \times 0.647,188}{240} + \frac{277 \times 0.520,650}{2016} + \&c$, = $d^3 \times 0.891,337 \times$ the infinite series $\frac{1}{2} + \frac{4.022,405}{12} + \frac{39.478,468}{240} + \frac{144.230,050}{2016} + \&c$, = $d^3 \times 0.891,337 \times$ the infinite series 0.500,000 + 0.335,200 + 0.164,493 + 0.071,537 + $\&c$ = $d^3 \times 0.891,337 \times 1.071,230$, $\&c$, = $d^3 \times 0.954,827$. But d is = 0.141. Therefore $d^3 \times 0.954,827$ will be = 0.141 $\times$ 0.141 $\times$ 0.954,827 = 0.141 $\times$ 0.134,630 = 0.018,983. Therefore the capital letter D is = 0.018,983, and consequently dx is = 0.018,983 $\times x$.

The capital letter F is = the infinite series $\frac{d^3}{6r^3} + \frac{5l^2d^3}{12r^4} + \frac{61l^2d^3}{144r^5} + \frac{277l^2d^3}{864r^6} + \&c$, = the infinite series $\frac{d^3}{6} + \frac{5l^2d^3}{12} + \frac{61l^2d^3}{144} + \frac{277l^2d^3}{864} + \&c$, = $d^3 \times$ the infinite series $\frac{1}{6} + \frac{5l}{12} + \frac{61l^2}{144} + \frac{277l^2}{864} + \&c$, = $d^3 \times$ the infinite series $\frac{1}{6} + \frac{5 \times 0.804,481}{12} + \frac{61 \times 0.647,188}{144} + \frac{277 \times 0.520,650}{864} + \&c$, = $d^3 \times$ the infinite series $\frac{1}{6} + \frac{4.022,405}{12} + \frac{39.478,468}{144} + \frac{144.230,050}{864} + \&c$, = $d^3 \times$ the infinite series 0.166,666 + 0.335,200 + 0.276,156 + 0.166,921 + $\&c$, = $d^3 \times 0.944,943$, $\&c$, = 0.141 $\times$ 0.944,943 = 0.002,688. Therefore fx is = 0.002,688 $\times x^2$.

The capital letter G is = the infinite series $\frac{5ld^4}{24r^4} + \frac{61l^2d^4}{144r^5} + \frac{277l^2d^4}{576r^6} + \&c$, = the infinite series $\frac{5ld^4}{24} + \frac{61l^2d^4}{144} + \frac{277l^2d^4}{576} + \&c$ = $d^4 \times$ the infinite series $\frac{5l}{24} + \frac{61l^2}{144} + \frac{277l^2}{576} + \&c$ = $d^4 \times$ the infinite series $\frac{5}{24} + \frac{61 \times 0.804,481}{144} +$

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+

finite series $0.034,350 + \&c = d^3 \times 0.891,337 \times 0.034,350$, &c $= d^3 \times 0.030,617 = 0.141^3 \times 0.030,617 = 0.000,000,004$. Therefore Lx^3 will be $= 0.000,000,004 \times x^3$.

And the capital letter M is equal to the infinite series $\frac{277d^3}{72,576} + \&c =$ the infinite series $\frac{277d^3}{72,576} + \&c = d^3 \times$ the infinite series $\frac{277}{72,576} + \&c = d^3 \times$ the infinite series $0.003,816 + \&c = 0.141^3 \times 0.003,816 = 0.000,000,000,083$, &c. Therefore Mx^3 will be $= 0.000,000,000,083$, &c $\times x^3$.

Therefore, if we carry the computation of these capital letters only to six places of decimal fractions, we may consider the letters K, L, and M, as being, each of them, equal to 0, and may therefore omit the several terms in the above complicated equation, set down in art. 35, that involve them. And then the said equation will be as follows; to wit,

$$\left\{ \begin{array}{l} 0.224,973 - Dx + Fx^3 - Gx^5 + Hx^7 - Ix^9 + 0 + 0 + 0 + \&c \\ - \frac{Cx^3}{2} + \frac{Dx^5}{2} - \frac{Ex^7}{2} + \frac{Gx^9}{2} - \frac{Hx^{11}}{2} + \frac{Ix^{13}}{2} - 0 - \&c \\ - \frac{Cx^9}{8} + \frac{Dx^{11}}{8} - \frac{Ex^{13}}{8} + \frac{Gx^{15}}{8} - \frac{Hx^{17}}{8} + \&c \\ - \frac{Cx^{15}}{16} + \frac{Dx^{17}}{16} - \frac{Ex^{19}}{16} + \&c \\ - \frac{5Cx^{18}}{128} + \&c \end{array} \right.$$

$= 0.1965$.

Art. 37. In this equation we must now reduce the several compound co-efficients of $x^3, x^5, x^7, x^9, x^{11}, x^{13}, x^{15}$, to wit, the compound quantities $+ P - \frac{c}{2}, - G + \frac{D}{2}, + H - \frac{F}{2} - \frac{c}{8}, - I + \frac{G}{2} + \frac{D}{8}, - \frac{H}{2} - \frac{F}{8} - \frac{c}{16} + \frac{I}{2} + \frac{G}{8} + \frac{D}{16}$, and $- \frac{H}{8} - \frac{F}{16} - \frac{5c}{128}$, into single co-efficients: which may be done as follows.

Since c is $= 0.224,973$, and F is $= 0.002,688$, we shall have $\frac{c}{2}$ ($= \frac{0.224,973}{2}$) $= 0.112,486$, and $+ P - \frac{c}{2}$ ($= + 0.002,688 - 0.112,486$) $= - 0.109,798$.

And, since D is $= 0.018,983$, and G is $= 0.000,263$, we shall have $- G + \frac{D}{2} = - 0.000,263 + \frac{0.018,983}{2}$ ($= - 0.000,263 + 0.009,491$) $= + 0.009,228$.

And,

And, since H is = 0.000,029, we shall have $+H - \frac{v}{2} - \frac{c}{8} = +$
 $0.000,029 - \frac{0.002,688}{2} - \frac{0.224,973}{8} (= +0.000,029 - 0.001,344 - 0.028,122$
 $= +0.000,029 - 0.029,466) = -0.029,437.$

And, since I is = 0.000,002, we shall have $-I + \frac{a}{2} + \frac{b}{8} = -$
 $0.000,002 + \frac{0.000,263}{2} + \frac{0.018,983}{8} (= -0.000,002 + 0.000,131 + 0.002,373$
 $= -0.000,002 + 0.002,504) = +0.002,502.$

And $-\frac{H}{2} - \frac{v}{8} - \frac{c}{16}$ will be $= -\frac{0.000,029}{2} - \frac{0.002,688}{8} - \frac{0.224,973}{16} (=$
 $-0.000,014 - 0.000,336 - 0.014,061) = -0.014,411.$

And $+\frac{I}{2} + \frac{a}{8} + \frac{b}{16}$ will be $= +\frac{0.000,002}{2} + \frac{0.000,263}{8} + \frac{0.018,983}{16} (=$
 $+0.000,001 + 0.000,033 + 0.001,186) = 0.001,220.$

And $-\frac{H}{8} - \frac{v}{16} - \frac{5c}{128}$ will be $= -\frac{0.000,029}{8} - \frac{0.002,688}{16} - \frac{5 \times 0.224,973}{128}$
 $(= -0.000,003 - 0.000,168 - \frac{1.124,865}{128} = -0.000,003 - 0.000,168 -$
 $0.008,788) = 0.008,959.$

Therefore the foregoing complicated equation will now become as follows, to
 wit, $0.224,973 - 0.018,983 \times x - 0.109,798 \times x^2 + 0.009,228 \times x^3$
 $- 0.029,437 \times x^4 + 0.002,502 \times x^5 - 0.014,411 \times x^6$
 $+ 0.001,220 \times x^7 - 0.008,959 \times x^8 \&c = 0.1965.$

Art. 38. Now let all the terms on the left-hand side of this equation that
 have the sign $-$ prefixed to them, or are subtracted from the other terms on
 the same side, be added to both sides of the equation. And we shall then have

$0.224,973 + 0.009,228 \times x^3 + 0.002,502 \times x^5 + 0.001,220 \times x^7$
 $= 0.1965 + 0.018,983 \times x + 0.109,798 \times x^2 + 0.029,437 \times x^4$
 $+ 0.014,411 \times x^6 + 0.008,959 \times x^8.$ Therefore, (subtracting
 0.1965 from both sides,) we shall have

$0.028,473 + 0.009,228 \times x^3 + 0.002,502 \times x^5 + 0.001,220 \times x^7 =$
 $0.018,983 \times x + 0.109,798 \times x^2 + 0.029,437 \times x^4$
 $+ 0.014,411 \times x^6 + 0.008,959 \times x^8;$ and (subtracting
 $0.009,228 \times x^3 + 0.002,502 \times x^5 + 0.001,220 \times x^7$ from both sides,) we
 shall have $0.018,983 \times x + 0.109,798 \times x^2$

$- 0.009,228 \times x^3 + 0.029,437 \times x^4$
 $- 0.002,502 \times x^5 + 0.014,411 \times x^6$
 $- 0.001,220 \times x^7 + 0.008,959 \times x^8 \&c = 0.028,473.$

Art. 39.

Art. 39. In this equation it is remarkable that 0.109,798, the co-efficient of x^0 , is greater than 0.018,983, the co-efficient of x ; and that 0.029,437, the co-efficient of x^2 , is greater than 0.009,228, the co-efficient of x^3 ; and that 0.014,411, the co-efficient of x^4 , is greater than 0.002,502, the co-efficient of x^5 ; and that 0.008,959, the co-efficient of x^6 , is greater than 0.001,220, the co-efficient of x^7 ; but the co-efficients of x , x^3 , x^5 , and x^7 , or the odd powers of x , (to wit, 0.018,983, and 0.009,228, and 0.002,502, and 0.001,220,) are every one less than the next before it, or decrease when the powers of x increase; and likewise that the co-efficients of x^2 , x^4 , x^6 , and x^8 , or the even powers of x , (to wit, 0.109,798, and 0.029,437, and 0.014,411, and 0.008,959,) are every one less than the next before it, or decrease when the powers of x increase; as was the case with the co-efficients of x , x^3 , x^5 , and x^7 , or of the odd powers of x .

Art. 40. We must now endeavour to resolve this last equation obtained in art. 38, to wit, the equation $0.018,983 \times x + 0.109,798 \times x^2 - 0.009,228 \times x^3 + 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6 - 0.001,220 \times x^7 + 0.008,959 \times x^8 \&c = 0.028,473$. Now this can only be done by approximation; and I take Mr. RAPHSON's method of resolving equations by approximation to be the best method that can be employed for this purpose. I will therefore now endeavour to find a first near value of x that shall differ from its true value by only a tenth, or less than a tenth, part of the said true value, in order to make the said first near value of x the basis of a further approach to its true value, according to the directions of Mr. RAPHSON's method.

Art. 41. Now, in order to find this first near value of x , I begin by observing, that, since r , or 1, is the radius of a great circle of the Earth, and x is the sine of a certain angle in the said circle, (to wit, the sine of the angle which is the complement of the rhumb-angle LAB to a right angle,) x must be less than 1; and consequently that xx must be less than x , and x^3 than xx , and x^4 than x^3 , and every following power of x less than that which precedes it. And hence we may conclude that the two first terms of the left-hand side of the foregoing equation, to wit, the two terms $0.018,983 \times x$, and $0.109,798 \times xx$, will be greater than the two next terms, or than any other two terms, of the same side of the equation, and (because those following terms are marked alternately with the signs + and -, and consequently tend to counterbalance each other, or to diminish each other's magnitudes,) may be reasonably supposed to be pretty nearly equal to the whole infinite series that forms the left-hand side of this equation. Let us therefore suppose these two terms to be equal to the said whole series, and consequently to the absolute term 0.028,473 of the said equation. And then we shall have the following quadratic equation to resolve, to wit, $0.018,983 \times x + 0.109,798 \times xx = 0.028,473$. This equation may be resolved as follows.

Let both sides of it be divided by the co-efficient of xx , to wit, 0.109,798.
And

And we shall have $xx + \frac{0.018,983}{0.109,798} \times x = \frac{0.028,473}{0.109,798}$, or $xx + 0.172,890 \times x = 0.259,321$.

Add now to both sides of this equation the square of half the co-efficient of x , or of half $0.172,890$, or the square of $0.086,445$; which square is $0.007,472$. And we shall have $xx + 0.172,890 \times x + 0.007,472 (= 0.259,321 + 0.007,472) = 0.266,793$. Therefore (extracting the square-roots of both sides,) we shall have $x + 0.086,445 (= \sqrt{0.266,793}) = 0.516,520$, and consequently $x (= 0.516,520 - 0.086,445) = 0.430,075$. Therefore the first near value of x in the foregoing high equation will be $0.430,075$; of which we may reasonably suppose at least the first figure, $.4$, to be exact.

Art. 42. We will therefore now substitute 0.4 , instead of x , in the compound quantity which forms the left-hand side of the equation set down in art. 38, to wit, the equation

$$\begin{aligned} &0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3 \\ &+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6 \\ &- 0.001,220 \times x^7 + 0.008,959 \times x^8 \text{ \&c.} = 0.028,473, \end{aligned}$$

in order to discover whether the value of the said compound quantity resulting from such substitution will be greater, or less, than the absolute term, $0.028,473$, of the said equation, and whether its difference from the said absolute term will be great, or small.

Now, if we suppose x to be $= 0.4$, we shall have

$$\begin{aligned} &0.018,983 \times x = 0.018,983 \times 0.4 = 0.007,593, \\ &\text{and } 0.109,798 \times xx = 0.109,798 \times 0.16 = 0.017,567, \\ &\text{and } 0.009,228 \times x^3 = 0.009,228 \times 0.064 = 0.000,590, \\ &\text{and } 0.029,437 \times x^4 = 0.029,437 \times 0.0256 = 0.000,753, \\ &\text{and } 0.002,502 \times x^5 = 0.002,502 \times 0.01024 = 0.000,025, \\ &\text{and } 0.014,411 \times x^6 = 0.014,411 \times 0.004,096 = 0.000,059, \\ &\text{and } 0.001,220 \times x^7 = 0.001,220 \times 0.001,638,4 = 0.000,002, \\ &\text{and } 0.008,959 \times x^8 = 0.008,959 \times 0.000,655,36 = 0.000,005. \end{aligned}$$

Therefore the compound quantity

$$\begin{aligned} &0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3 \\ &+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6 \\ &- 0.001,220 \times x^7 + 0.008,959 \times x^8 \text{ will be } = \\ &0.007,593 + 0.017,567 - 0.000,590 + 0.000,753 - 0.000,025 \\ &+ 0.000,059 - 0.000,002 + 0.000,005 - \text{\&c} = 0.025,977 - 0.000,617 \\ &= 0.025,360; \text{ which is less than } 0.028,473, \text{ or the absolute term of the equation} \end{aligned}$$

0.018,

$0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3$
 $+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6$
 $- 0.001,220 \times x^7 + 0.008,959 \times x^8 - \&c = 0.028,473.$ Therefore
 0.4 will be less than the true value of x in the said equation.

Art. 43. We will therefore, in the next place, suppose x to be equal to 0.43, or the two first figures of the number 0.430,075, which was obtained in art. 41, by the resolution of the quadratick equation $0.018,983 \times x + 0.109,798 \times xx = 0.028,473$, and will substitute 0.43 instead of x in the compound quantity which forms the left hand side of the equation

$$\begin{aligned}
 &0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3 \\
 &+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6 \\
 &- 0.001,220 \times x^7 + 0.008,959 \times x^8 = 0.028,473.
 \end{aligned}$$

Now, if x is supposed to be = 0.43, we shall have

$$\begin{aligned}
 xx (= 0.43^2) &= 0.1849, \\
 \text{and } x^3 (= 0.43^3) &= 0.079,507, \\
 \text{and } x^4 (= 0.43^4) &= 0.034,188, \\
 \text{and } x^5 (= 0.43^5) &= 0.014,700, \\
 \text{and } x^6 (= 0.43^6) &= 0.006,321, \\
 \text{and } x^7 (= 0.43^7) &= 0.002,718, \\
 \text{and } x^8 (= 0.43^8) &= 0.001,168; \text{ and consequently we shall have}
 \end{aligned}$$

$$\begin{aligned}
 &0.018,983 \times x (= 0.018,983 \times 0.43) = 0.008,162, \\
 \text{and } &0.109,798 \times xx (= 0.109,798 \times 0.1849) = 0.020,302, \\
 \text{and } &0.009,228 \times x^3 (= 0.009,228 \times 0.079,507) = 0.000,733, \\
 \text{and } &0.029,437 \times x^4 (= 0.029,437 \times 0.034,188) = 0.001,006, \\
 \text{and } &0.002,502 \times x^5 (= 0.002,502 \times 0.014,700) = 0.000,036, \\
 \text{and } &0.014,411 \times x^6 (= 0.014,411 \times 0.006,321) = 0.000,091, \\
 \text{and } &0.001,220 \times x^7 (= 0.001,220 \times 0.002,718) = 0.000,003, \\
 \text{and } &0.008,959 \times x^8 (= 0.008,959 \times 0.001,168) = 0.000,010.
 \end{aligned}$$

Therefore the compound quantity

$$\begin{aligned}
 &0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3 \\
 &+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6 \\
 &- 0.001,220 \times x^7 + 0.008,959 \times x^8 \text{ will be } = 0.008,162 + 0.020,302 \\
 &- 0.000,733 + 0.001,006 - 0.000,036 + 0.000,091 - 0.000,003 \\
 &+ 0.000,010 = 0.029,571 - 0.000,772 = 0.028,799; \text{ which is a little} \\
 &\text{greater than } 0.028,473, \text{ or the absolute term of the equation}
 \end{aligned}$$

$$\begin{aligned}
 &0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3 \\
 &+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6
 \end{aligned}$$

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$- 0.001,220 \times x^7 + 0.008,959 \times x^8 - \&c = 0.028,473$. Therefore 0.43 will be a little greater than the true value of x in that equation.

But it has been shewn above that 0.4 is less than the true value of x in that equation.

Therefore the true value of x in that equation will be of an intermediate magnitude between 0.4 and 0.43, and but little less than 0.43, because the absolute term 0.028,473 is but little less than 0.028,799.

Art. 44. In order to obtain the value of x in the foregoing equation.

$0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3$
 $+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6$
 $- 0.001,220 \times x^7 + 0.008,959 \times x^8 - \&c = 0.028,473$ to a greater degree of exactness, let us suppose x to be equal to 0.43 = z , and substitute the powers of this residual quantity 0.43 = z instead of the powers of x in the said equation, omitting all the terms of those powers that involve any higher power of z than its simple power, or z itself. This may be done in the manner following.

Since x is = 0.43 = z , we shall have

$$\begin{aligned} xx (= 0.43 - 2)^2 &= 0.43^2 - 2 \times 0.43 \times z + \&c = 0.1849 - 0.86 \\ &\quad \times z + \&c, \\ \text{and } x^3 (= 0.43 - 2)^3 &= 0.43^3 - 3 \times 0.43^2 \times z + \&c = 0.43^3 - 3 \times \\ &\quad 0.1849 \times z + \&c = 0.43^3 - 0.5547 \times z + \&c = 0.079,507 \\ &\quad - 0.5547 \times z + \&c, \\ \text{and } x^4 (= 0.43 - 2)^4 &= 0.43^4 - 4 \times 0.43^3 \times z + \&c = 0.43^4 - 4 \times \\ &\quad 0.079,507 \times z + \&c = 0.43^4 - 0.318,028 \times z + \&c = \\ &\quad 0.034,188 - 0.318,028 \times z + \&c, \\ \text{and } x^5 (= 0.43 - 2)^5 &= 0.43^5 - 5 \times 0.43^4 \times z + \&c = 0.43^5 - \\ &\quad 5 \times 0.034,188 \times z + \&c = 0.43^5 - 0.170,940 \times z + \\ &\quad \&c = 0.014,700 - 0.170,940 \times z + \&c, \\ \text{and } x^6 (= 0.43 - 2)^6 &= 0.43^6 - 6 \times 0.43^5 \times z + \&c = 0.43^6 - 6 \times \\ &\quad 0.014,700 \times z + \&c = 0.43^6 - 0.088,200 \times z + \&c = \\ &\quad 0.006,321 - 0.088,200 \times z + \&c, \\ \text{and } x^7 (= 0.43 - 2)^7 &= 0.43^7 - 7 \times 0.43^6 \times z + \&c = 0.43^7 - 7 \times \\ &\quad 0.006,321 \times z + \&c = 0.43^7 - 0.044,247 \times z + \&c = \\ &\quad 0.002,718 - 0.044,247 \times z + \&c, \\ \text{and } x^8 (= 0.43 - 2)^8 &= 0.43^8 - 8 \times 0.43^7 \times z + \&c = 0.43^8 - 8 \times \\ &\quad 0.002,718 \times z + \&c = 0.43^8 - 0.021,744 \times z + \&c = \\ &\quad 0.001,168 - 0.021,744 \times z + \&c. \end{aligned}$$

Therefore

Therefore the compound quantity

$$\begin{aligned}
 & 0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3 \\
 & + 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6 \\
 & - 0.001,220 \times x^7 + 0.008,959 \times x^8 \text{ will be } = \\
 & \frac{0.018,983 \times 0.43 - z + 0.109,798 \times 0.1849 - 0.86 \times z + \&c}{- 0.009,228 \times 0.079,507 - 0.5547 \times z + \&c} \\
 & + \frac{0.029,437 \times 0.034,188 - 0.318,028 \times z + \&c}{- 0.002,502 \times 0.014,700 - 0.170,940 \times z + \&c} \\
 & + \frac{0.014,411 \times 0.006,321 - 0.088,200 \times z + \&c}{- 0.001,220 \times 0.002,718 - 0.044,247 \times z + \&c} \\
 & + \frac{0.008,959 \times 0.001,168 - 0.021,744 \times z + \&c}{- \&c}
 \end{aligned}$$

$$\begin{aligned}
 & = \left\{ \begin{array}{l} 0.008,162 - 0.018,983 \times z \\ + 0.020,302 - 0.094,426 \times z + \&c \\ - 0.000,733 + 0.005,118 \times z - \&c \\ + 0.001,006 - 0.009,361 \times z + \&c \\ - 0.000,036 + 0.000,427 \times z - \&c \\ + 0.000,091 - 0.001,271 \times z + \&c \\ - 0.000,003 + 0.000,054 \times z - \&c \\ + 0.000,010 - 0.000,194 \times z + \&c \end{array} \right\} \\
 & = \left\{ \begin{array}{l} 0.029,571 - 0.124,235 \times z + \&c \\ - 0.000,772 + 0.005,599 \times z - \&c \\ 0.028,799 - 0.118,636 \times z + \&c. \end{array} \right\}
 \end{aligned}$$

But the said compound quantity is equal to the absolute term 0.028,473.

Therefore 0.028,799 - 0.118,636 $\times$ z will also be equal to 0.028,473; and consequently 0.028,799 will be = 0.028,473 + 0.118,636 $\times$ z, and 0.118,636 $\times$ z will be (= 0.028,799 - 0.028,473) = 0.000,326, and z will be = $\frac{0.000,326}{0.118,636}$ = 0.002,747. Therefore 0.43 - z, or x, will be (=

0.430,000 - 0.002,747) = 0.427,253, or (neglecting the two last figures,) 0.4272; or the more accurate value of x in the equation

$$\begin{aligned}
 & 0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3 \\
 & + 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6 \\
 & - 0.001,220 \times x^7 + 0.008,959 \times x^8 - \&c = 0.028,473 \text{ will be } = \\
 & 0.4272. \qquad \qquad \qquad \text{Q. E. I.}
 \end{aligned}$$

Art. 45. We will now substitute this last value of x, to wit, 0.4272, instead of x, in the compound quantity

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0.018,

$0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3$
 $+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6$
 $- 0.001,220 \times x^7 + 0.008,959 \times x^8$, (which forms the first, or left-hand side of the foregoing equation,) in order to discover how near the value of the said compound quantity resulting from the said substitution will approach to the true value of the said compound quantity, or to the value of its equal, the absolute term, 0.028,473, of the said equation; and consequently to find how near the said number 0.4272 will approach to the true value of x in the said equation. This may be done in the manner following.

If we suppose x to be equal to 0.4272, we shall have

$$xx (= 0.4272^2) = 0.182,499,$$

$$\text{and } x^3 (= 0.4272^3 = 0.4272^2 \times 0.4272)$$

$$= 0.182,499 \times 0.4272 = 0.077,963,$$

$$\text{and } x^4 (= 0.077,963 \times 0.4272) = 0.033,305,$$

$$\text{and } x^5 (= 0.033,305 \times 0.4272) = 0.014,228,$$

$$\text{and } x^6 (= 0.014,228 \times 0.4272) = 0.006,078,$$

$$\text{and } x^7 (= 0.006,078 \times 0.4272) = 0.002,596,$$

$$\text{and } x^8 (= 0.002,596 \times 0.4272) = 0.001,109,$$

$$\text{and consequently } 0.018,983 \times x (= 0.018,983 \times 0.4272) = 0.008,109,$$

$$\text{and } 0.109,798 \times xx (= 0.109,798 \times 0.182,499) = 0.020,038,$$

$$\text{and } 0.009,228 \times x^3 (= 0.009,228 \times 0.077,963) = 0.000,719,$$

$$\text{and } 0.029,437 \times x^4 (= 0.029,437 \times 0.033,305) = 0.000,980,$$

$$\text{and } 0.002,502 \times x^5 (= 0.002,502 \times 0.014,228) = 0.000,035,$$

$$\text{and } 0.014,411 \times x^6 (= 0.014,411 \times 0.006,078) = 0.000,087,$$

$$\text{and } 0.001,220 \times x^7 (= 0.001,220 \times 0.002,596) = 0.000,003,$$

$$\text{and } 0.008,959 \times x^8 (= 0.008,959 \times 0.001,109) = 0.000,010.$$

Therefore the whole compound quantity

$$\begin{aligned}
 &0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3 \\
 &+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6 \\
 &- 0.001,220 \times x^7 + 0.008,959 \times x^8 - \&c \text{ will be } (=
 \end{aligned}$$

$$\left\{ \begin{array}{rcl}
 &0.008,109 & \\
 + &0.020,038 & - 0.000,719 \\
 + &0.000,980 & - 0.000,035 \\
 + &0.000,087 & - 0.000,003 \\
 + &0.000,010 & - \&c
 \end{array} \right\} =$$

+ 0.029,224 - 0.000,757) = 0.028,467; which is very nearly equal to 0.028,473, or the absolute term of the equation

$0.018,983 \times x + 0.109,798 \times xx - 0.009,228 \times x^3$
 $+ 0.029,437 \times x^4 - 0.002,502 \times x^5 + 0.014,411 \times x^6$
 $- 0.001,220 \times x^7 + 0.008,959 \times x^8 - \&c.$ And consequently 0.4272 will be very nearly equal to the true value of x in the said equation.

Q. E. I.

Art. 46.

Art. 46. Now it appears from the Tables of Sines and Tangents, that 0.427,357,9 (which differs but little from 0.4272, or x .) is the sine of an arch of $25^{\circ}, 18'$, or the co-sine of an arch of $64^{\circ}, 42'$, in a circle of which the radius is called 1. Therefore the arch of which x , or 0.4272, is the sine in the circle of which 1, or the radius of the Earth, is the radius, is an arch of $25^{\circ}, 18'$ very nearly, and the arch of which it is the co-sine is very nearly $64^{\circ}, 42'$. Therefore the angle L A B, or the rhumb-angle, or the angle of the course, will be, very nearly, an angle of $64^{\circ}, 42'$. Q₂ E. 1.

And, because $\frac{d}{r}$ is = $\frac{564}{4000}$ = 0.141, we shall have $\frac{dx}{r}$ ($= 0.141 \times r = 0.141 \times 0.4272$) = 0.060,235,2, and consequently $l - \frac{dx}{r}$ ($= l - 0.060,235 = 0.891,337 - 0.060,235$) = 0.831,102; that is, the length of the arch B M of the meridian line P B M, or the latitude of the second point B, will be = 0.831,102; which may be converted into degrees and minutes in the following manner.

Since 0.891,337 is the length of the arch A L, or the latitude of the first point A, expressed in decimal parts of 1, or the radius of the Earth, and 0.831,102 is the length of the arch B M, or the latitude of the second point B, expressed in decimal parts of the same radius, and $(51^{\circ}, 18', \text{ or } 51^{\circ} + \frac{18^{\circ}}{60}, \text{ or } 51^{\circ} + \frac{3^{\circ}}{10}, \text{ or } 51^{\circ} + 0.3, \text{ or } 51.3^{\circ})$ is the length of the former arch expressed in degrees and decimal parts of a degree, it follows that the length of the latter arch, B M, expressed likewise in degrees and decimal parts of a degree, will be a fourth proportional to the three numbers 0.891,337, 0.831,102, and 51.3° , and consequently will be ($= \frac{51.3 \times 0.831,102}{0.891,337} = \frac{42.615,532,6}{0.891,337}$) = $47.83^{\circ} = 47^{\circ} + \frac{83^{\circ}}{100} = 47^{\circ} + \frac{49^{\circ}}{60} = 47^{\circ}, 49'$. Therefore the arch B M, or the latitude of the second point B, will be 47 degrees and 49 minutes.

Q₂ E. 1.

A Proof of the Truth of the Conclusions obtained in the preceding Articles, by the Application of the foregoing Solution of Dr. HALLEY's Problem, to the Example given of it in Art. 34.

Art. 47. Having thus found the angle L A B of the ship's course to be an angle of $64^{\circ}, 42'$, and the arch B M, or the latitude of the second point B, to be an arch of $47^{\circ}, 49'$, we will now try the truth of these conclusions by supposing these values to be such as we have here found them, and deriving from

from them the value of the equatorial arch LM , or the difference of the longitudes of the points A and B , (which before was supposed to be given and to be equal to 786 miles, or $\frac{786}{4000}$ of the radius of the Earth, or $\frac{1965}{10,000}$ of the radius of the Earth, or $= 0.1965$, if the radius of the Earth be called 1,) by means of the equation $LM = \frac{t}{r} \times \overline{a-b}$, obtained above in Lemma 3d, Coroll. 1. For, if the value we shall thereby obtain for the equatorial arch LM shall appear to be, nearly, equal to 786 miles, or 0.1965 of the radius of the Earth, we may justly conclude that the foregoing values of the rhumb-angle LAB and the arch BM , or latitude of the second point B , have been rightly assigned. Now this investigation of the value of the arch LM may be made in the following manner.

If the rhumb-angle LAB be supposed to be $64^\circ, 42'$, its tangent t in the circle of which 1, or the radius of the Earth, is the radius, will be $2.115,516,5$, and consequently the fraction $\frac{t}{r}$ will be $= \frac{2.115,516,5}{1}$, or $2.115,516,5$.

Further, if the arch BM , or the latitude of the second point B , is supposed to be $47^\circ, 49'$, the arch PB (which is its complement to PM , or 90 degrees,) will be $42^\circ, 11'$, and consequently $\frac{PB}{r}$ will be $(= \frac{42^\circ, 11'}{2}) = 21^\circ, 5', 30''$, the tangent of which in the circle of which 1, or the radius of the Earth, is the radius, will be $0.385,667,3$.

Lastly, since the arch AL , or the latitude of the first point A , is $51^\circ, 18'$, the arch PA (which is its complement to PL , or 90 degrees,) will be $38^\circ, 42'$, and consequently $\frac{PA}{r}$ will be $(= \frac{38^\circ, 42'}{2}) = 19^\circ, 21'$, the tangent of which in the circle of which 1, or the radius of the Earth, is the radius, will be $0.351,175,0$.

Now a is the logarithm of the ratio of the radius 1 to the tangent of $\frac{PA}{2}$, and b is the logarithm of the ratio of the radius 1 to the tangent of $\frac{PB}{2}$, in a logarithmick curve of which the radius 1, or the radius of the Earth is the subtangent. Therefore a will be the logarithm of the ratio of 1 to $0.351,175,0$, and b will be the logarithm of the ratio of 1 to $0.385,667,3$, in a logarithmick curve, of which r , or 1, or the radius of the Earth, is the subtangent.

Now, if the subtangent of this logarithmick curve was called $0.434,294,481,903,251,827,651,128$, &c, and the logarithm of the ratio of 10 to 1 in it was called 1, that is, if the abscisses of the axis, or asymptote, of this curve, were denominated according to BRIGGS'S system of logarithms, the logarithm of the ratio of 1 to $0.351,175,0$, (which is equal to the ratio of $2847,573,1$ to

to 1,) would be $0.454,387,4$, and the logarithm of the ratio of 1 to $0.385,667,3$ (which is equal to the ratio of $2.592,908,4$ to 1,) would be $0.413,802,5$. Therefore, when the subtangents of this curve is called 1, and consequently the logarithm of the ratio of 10 to 1, taken on the axis or asymptote of the said curve, is called $2.302,585,092,994,045,684,017,991$, &c., the logarithm of the ratio of 1 to $0.351,175,0$ will be $= 2.302,585$, &c. $\times 0.454,387,4 = 1.046,265,6$, and the logarithm of the ratio of 1 to $0.385,667,3$ will be $= 2.302,585$, &c. $\times 0.413,802,5 = 0.952,815,4$. Therefore a will be $= 1.046,265,6$, and b will be $= 0.952,815,4$, and consequently $a - b$ will be $= 1.046,265,6 - 0.952,815,4 = 0.093,450,2$.

But we before found that $\frac{t}{r}$, or $\frac{t}{1}$, was $= 2.115,516,5$.

Therefore $\frac{t}{r} \times a - b$ will be $= 2.115,516,5 \times 0.093,450,2 = 0.197,695,4$, or (neglecting the three last figures,) 0.1976 ; that is, the equatorial arch L M, or the difference of the longitudes of the points A and B, will be, nearly, $= 0.1976$, or 0.1976 of 1, or $\frac{1976}{10,000}$ of 1, or of the radius of the Earth, or $\frac{790.4}{4000}$ of the radius of the Earth, or 790.4 miles. Now this number of miles, 790.4 , is very nearly equal to 786 miles, which was the given length of the said equatorial arch L M in the foregoing Problem of Dr. HALLEY. And therefore we may conclude that the values found above for the rhumb-angle L A B and the circular arch B M, or the latitude of the second point B, in the foregoing Solution of Dr. HALLEY's Problem, are nearly equal to the true values of the said angle and arch, or that the said angle L A B is nearly equal to 64° , $42'$, and that the said arch B M is nearly equal to 47° , $49'$. Q. E. D.

Art. 48. As there are 32 points in the compass, and consequently the direction of every point makes an angle of $(\frac{360^\circ}{32})$, or 11° , $15'$, with the direction of the point immediately preceeding it, the direction in which the ship begins to move at its first setting out from the point A will be $(64^\circ, 42', \text{ or } 56^\circ, 15' + 8^\circ, 27'$ to the West of the line A L, or of a direction due South, or will be $8^\circ, 27'$ to the West of a direction South-West by West, or will be S. W. by W. with 8 degrees and 27 minutes, or about three quarters of a point, more to the Westward, or will be within a quarter of a point of West South-West.

Art. 49. The truth of the foregoing determination of the rhumb-angle L A B. in the example that has been here given of the foregoing Solution of Dr. HALLEY's Problem, has been confirmed to me by a determination of the same angle (obtained by a different process from that of the foregoing Solution,) which has been sent me by the learned Mr. ANDREW MACKAY, M. A. of Aberdeen in Scotland, and which agrees with the foregoing determination of it to about half a degree. Mr. MACKAY makes the angle L A B amount to $65^\circ, 13', \frac{7}{10}$ instead of $64^\circ, 42'$; and he makes the arch B M, or the latitude

of

of the second point B, amount to 47° , $21' \frac{8}{10}$, instead of 47° , $49'$. The differences of these values of LAB and BM, from those obtained above, are only $31' + \frac{7}{10}$ and $27' + \frac{2}{10}$, which must be considered as but trifling, if we reflect that the former values were derived from the resolution of an equation, in which not only the number of terms involving the unknown quantity x was infinite, but each of the co-efficients of the powers of x in the said equation was the result of the addition or subtraction of several infinite series denoted by the capital letters C, D, F, G, H, I, K, L, M, &c, and which therefore could not be resolved with perfect accuracy.

Art. 50. We have hitherto considered only one case of Dr. HALLEY's Problem, namely, that in which the point A, from which the ship sets out, and the point B, at which it arrives, are both on the same side of the Equator, and the first latitude AL (which is supposed to be known,) is greater than the second latitude BM. We will therefore now proceed to consider another case of this Problem, in which the ship shall be supposed to sail back again from B to A along the same loxodromick curve, or rhumb-line, BA, which it had before described in sailing from A to B, and the first latitude (which is supposed to be known,) will be that of the point B. This case of Dr. HALLEY's Problem may be stated and resolved in the following manner.

PROPOSITION V; A PROBLEM;

BEING

ANOTHER CASE OF DOCTOR HALLEY'S PROBLEM.

Art. 51. Let P (in Fig. 1,) be one of the Poles of the Earth; HI a portion of the circumference of the Equator; A and B two points on the surface of the Earth, both on the same side of the Equator HI, but at different distances from it, and in different meridian circles PAL and PBM, which cut the equatorial arch HI in the points L and M. And let the point A be more distant from the Equator than the point B. Further, let a ship be supposed to sail from the point B to the point A along the rhumb-line, or loxodromick curve, BA, which cuts all the meridian-lines, over which it passes, in the same angle, to wit, in an angle equal to the angle LAB. And let the arch LM, or the difference of the longitudes of the two points B and A, and likewise the length of the rhumb-line, or loxodromick curve, BA, or the number of miles run over by the ship in her passage from B to A, and the arch BM, or the latitude of the first point B, from which the ship begins her voyage, be supposed to be known; but not the arch AL, or the latitude of the second point A,

at

at which the ship arrives after passing over the line BA , nor the angle LAB , which the line of the ship's course makes with the several meridian-lines over which it passes. It is required to find the said angle LAB .

THE SOLUTION.

ART. 52. In Lemma 3d the two circular arches AL and BM , or the latitudes of the points A and B , were supposed to be known; as were likewise the rhumb-angle LAB and the length of the rhumb-line AB : and the length of the equatorial arch LM , or the difference of the longitudes of the two points A and B , was supposed to be unknown, and was derived from the former quantities in the solution of that Lemma, and was shewn, in Coroll. 1 of the said Lemma, to be equal to $\frac{t}{r} \times a - b$, the letter r being put for the radius

of the Earth, and t for the tangent of the angle LAB in a circle of which r is the radius, and a for the logarithm of the ratio of the radius r to the tangent of half the arch PA , or half the complement of the latitude of the point A to the arch PL , or an arch of 90 degrees, and b for the logarithm of the ratio of the radius r to the tangent of half the arch PB , or half the complement of the latitude of the point B to the arch PM , or an arch of 90 degrees, in a logarithmick curve of which the subtangent is equal to r , or the radius of the Earth. But in the present Problem the former of these two ratios, (which depends on the arch AL , or the latitude of the point A ,) and consequently its logarithm a , is unknown. Nor can the arch AL , or the latitude of the point A , be derived from the arch BM , or the latitude of the other point B , and the length of the rhumb-line BA , (which are both supposed to be known,) in the manner directed in the first Lemma; because in that Lemma the rhumb-angle LAB was supposed to be known, and consequently its tangent t (in the circle of which r is the radius,) was also known, as well as the length of the rhumb-line BA , and the arch BM , or the latitude of the point B , whereas in the present Problem the rhumb-angle LAB is supposed to be unknown, and is the very quantity we are required to find. Therefore

in the equation $LM = \frac{t}{r} \times a - b$ (obtained in the first Corollary of Lemma 3d,) there will be two unknown quantities, to wit, the tangent t of the unknown angle LAB , and the logarithm a of the ratio of r to the tangent of half the arch PA . We cannot therefore consider the equation $LM = \frac{t}{r} \times a - b$ as a simple equation containing only one unknown quantity t (in
Vot. IV. H which

which case we should have $r \times LM = t \times \overline{a-b}$, and $t = \frac{r \times LM}{a-b}$,) but must investigate the relation between the two unknown quantities t and a , (which are mutually dependant on each other,) and must reduce them to one, before we can discover the value of t , or the tangent of the unknown angle $L A B$. And this investigation is a matter of great nicety and difficulty, as we have seen in the foregoing solution of the first case of Dr. HALLEY's Problem.

Art. 53. Let x be put (as before,) for the co-sine of the rhumb-angle $L A B$, of which t is the tangent, in the circle of which r , or the radius of the Earth, is the radius. Then will the square of the sine of the said angle be $rr - xx$, and the said sine itself will be $\sqrt{rr - xx}$. But the co-sine of any arc is to its sine as the radius is to the tangent. Therefore x will be to $\sqrt{rr - xx}$ as r is to t ; and consequently t will be $= \frac{r \times \sqrt{rr - xx}}{x}$. But it has been shewn above in Lemma 3d, Coroll. 1, that the arch LM of the circumference of the Equator, or the difference of the longitudes of the points A and B , is $= \frac{t}{r} \times \overline{a-b}$. Therefore the said arch LM will be $= \frac{r \times \sqrt{rr - xx}}{x \times r} \times \overline{a-b} = \frac{\sqrt{rr - xx}}{x} \times \overline{a-b}$.

Now let c be put, as before, for the arch LM , or the difference of the longitudes of the points A and B , which in the present Problem is supposed to be known. And we shall then have $c = \frac{\sqrt{rr - xx}}{x} \times \overline{a-b}$; in which equation the quantity a is unknown, as well as the quantity x , but is dependant on it, and may be derived from it, though not without great difficulty.

Art. 54. Let d be put (as before,) for the length of the loxodromick curve BA , or the distance run by the ship in her passage from B to A ; and let m be put for the arch BM , or the latitude of the point B , which in this case is supposed to be known.

Then, since, by Lemma 1, the length of the loxodromick curve BA is to AQ , or $AL - BM$, as r is to x , and BM is $= m$, we shall have d to $AL - m$ as r is to x ; and consequently dx will be $= r \times \overline{AL - m}$, and $AL - m$ will be $= \frac{dx}{r}$. Therefore AL , or the latitude of the point A , will be equal to $\frac{dx}{r} + m$, or $m + \frac{dx}{r}$.

Art. 55. Now let q be put for the length of the arch PAL , or the arch of a whole quadrant of the meridian-circle which passes through the point A , and of which r , or the radius of the Earth, is the radius. Then will PA , or $PAL - AL$,

$PAL - AL$, be $= q - AL$; or (because AL is $= m + \frac{dx}{r}$), PA will be
 $(= q - \left[m + \frac{dx}{r} \right]) = q - m - \frac{dx}{r}$, and consequently $\frac{PA}{2}$ will be $(=$
 $\frac{q - m - \frac{dx}{r}}{2}) = \frac{q}{2} - \frac{m}{2} - \frac{dx}{2r}$. Therefore a , or the logarithm of the ratio of
 the radius r to the tangent of the arch $\frac{PA}{2}$, will be equal to the logarithm of the
 ratio of the radius r to the tangent of the arch $\frac{q}{2} - \frac{m}{2} - \frac{dx}{2r}$.

Art. 56. Further, because in every circle the radius is a mean proportional
 between the tangent of any arch less than 90 degrees, or the arch of a quadrant,
 and the co-tangent of the same arch, or the tangent of its complement to the
 arch of a quadrant, it follows that the ratio of the radius r to the tangent of
 the arch $\frac{q}{2} - \frac{m}{2} - \frac{dx}{2r}$ will be equal to the ratio of the tangent of the arch
 which is the complement of the arch $\frac{q}{2} - \frac{m}{2} - \frac{dx}{2r}$ to the arch of a quadrant,
 or to q , to the radius r . But the complement of the arch $\frac{q}{2} - \frac{m}{2} - \frac{dx}{2r}$
 to the arch q is $(= q - \left[\frac{q}{2} - \frac{m}{2} - \frac{dx}{2r} \right]) = q - \frac{q}{2} + \frac{m}{2} + \frac{dx}{2r} = \frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}$.
 Therefore the ratio of the tangent of the arch $\frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}$
 to the radius r will be equal to the ratio of the radius r to the tangent of
 the arch $\frac{q}{2} - \frac{m}{2} - \frac{dx}{2r}$; and consequently the logarithm of the former ra-
 tio will be equal to the logarithm of the latter ratio. But the logarithm of
 the latter ratio is equal to a . Therefore the logarithm of the former ratio will
 also be equal to a ; or a will be equal to the logarithm of the ratio of the tan-
 gent of the arch $\frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}$ to the radius r . Therefore $a - b$ will be
 $= \log.$ of $\frac{\text{tangent of } \frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}}{r} - b$; and consequently LM , or e , (which
 is equal to $\frac{\sqrt{rr - xx}}{x} \times \overline{a - b}$), will be $= \frac{\sqrt{rr - xx}}{x} \times$
 $\left[\log. \text{ of } \frac{\text{tangent of } \frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}}{r} - b \right]$. We must therefore now endeavour to
 find an expression for the logarithm of the ratio of the tangent of the arch

H 2

 $\frac{q}{2}$

$\frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}$ to the radius r , that shall consist of a set of simple terms involving either known quantities, or the powers of the unknown quantity x combined with known quantities. This may be done in the manner following.

Art. 57. The arch PA is less than the arch PL, or than the arch of a whole quadrant of the circle. And consequently the arch $\frac{PA}{2}$ must be less than an arch of 45 degrees, or half the arch of a quadrant. Therefore the complement of the arch $\frac{PA}{2}$ to an arch of 90 degrees, must be greater than an arch of 45 degrees. But it has been shewn (in art. 56,) that the arch $\frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}$ is the complement of the arch $\frac{q}{2} - \frac{m}{2} - \frac{dx}{2r}$, or $\frac{PA}{2}$, to an arch of 90 degrees. Therefore the arch $\frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}$ will be greater than an arch of 45 degrees. And, as it is the complement of the arch $\frac{PA}{2}$ to an arch of 90 degrees, it must be less than an arch of 90 degrees. Therefore it will be both greater than an arch of 45 degrees, and less than an arch of 90 degrees.

But, when an arch is greater than 45 degrees, but less than 90 degrees, or the arch of a quadrant, Mr. JAMES GREGORY has given us a series that will express the logarithm of the ratio of its tangent to the radius of the circle, (which logarithm he calls its *artificial, or logarithmick, tangent,*) in terms involving the powers of the excess of twice the said arch above the arch of a quadrant, and the powers of the radius of the circle; which series is as follows. If the radius of a circle be called R, and an arch that is less than an arch of 90 degrees, but greater than an arch of 45 degrees, be called A, and an arch of 90 degrees in the said circle be called Q, and an arch that is equal to $2A - Q$ be called E, the logarithm of the ratio of the tangent of the arch A to the radius R, in a logarithmick curve of which the subtangent is equal to the radius R, will be equal to the infinite series $x +$

$\frac{E^3}{6R^3} + \frac{x^3}{24R^4} + \frac{61E^5}{5040R^5} + \frac{277E^7}{72,576R^7} + \&c.$ Therefore, if, in the circle of which r , or the radius of the Earth, is the radius, and in which q is the arch of a quadrant, the arch $\frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}$ (which has been shewn to be greater than an arch of 45 degrees in the said circle, but at the same time is less than the arch of a whole quadrant, being the complement of the arch $\frac{PA}{2}$ to q , or the arch of a quadrant,) be doubled, and from the double of it,

which

which will be $= q + m + \frac{dx}{r}$, we subtract the quadrantal arch q , and substitute the remaining arch $m + \frac{dx}{r}$ instead of E , and r instead of R , in the terms of the foregoing series $x + \frac{x^3}{6rk} + \frac{x^5}{24rk^3} + \frac{61x^7}{5040rk^5} + \frac{277x^9}{72576rk^7} + \&c$, *ad infinitum*, the complicated series thence arising, to wit, the series $m + \frac{dx}{r} + \frac{m + \frac{dx}{r}}{6rr} + \frac{m + \frac{dx}{r}}{24r^3} + \frac{61 \times m + \frac{dx}{r}}{5040r^5} + \frac{277 \times m + \frac{dx}{r}}{72576r^7} + \&c$, *ad infinitum*, will be equal to the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{m}{2} + \frac{dx}{r}$ to the radius r , in a logarithmick curve of which the subtangent is equal to the radius r , or the said series will be equal to a . And consequently the equatorial arch LM , or c , (which is equal to $\frac{\sqrt{rr - xx}}{x} \times a - b$) will be $= \frac{\sqrt{rr - xx}}{x} \times$ the excess of the said complicated series $m + \frac{dx}{r} + \frac{m + \frac{dx}{r}}{6rr} + \frac{m + \frac{dx}{r}}{24r^3} + \frac{61 \times m + \frac{dx}{r}}{5040r^5} + \frac{277 \times m + \frac{dx}{r}}{72576r^7} + \&c$ above the known logarithm b . This equation must now be reduced into a proper form for a resolution of it; which may be done in the manner following.

Art. 58. The second term, $\frac{m + \frac{dx}{r}}{6rr}$, of the foregoing series is $=$

$$\frac{m^3 + \frac{3m^2dx}{r} + \frac{3md^2x^2}{rr} + \frac{d^3x^3}{r^3}}{6rr} = \frac{\frac{r^3m^3}{r^3} + \frac{3r^2m^2dx}{r^3} + \frac{3rmd^2x^2}{r^3} + \frac{d^3x^3}{r^3}}{6rr} =$$

$$\frac{r^3m^3 + 3r^2m^2dx + 3rmd^2x^2 + d^3x^3}{6r^3}.$$

And the third term, $\frac{m + \frac{dx}{r}}{24r^3}$ is $=$

$$\frac{m^4 + \frac{5m^3dx}{r} + \frac{10m^2d^2x^2}{r^2} + \frac{10md^3x^3}{r^3} + \frac{5md^4x^4}{r^4} + \frac{d^5x^5}{r^5}}{24r^3} =$$

$$\frac{\frac{r^4m^4}{r^4} + \frac{5r^3m^3dx}{r^4} + \frac{10r^2m^2d^2x^2}{r^4} + \frac{10r^2m^2d^3x^3}{r^4} + \frac{5rmd^4x^4}{r^4} + \frac{d^5x^5}{r^5}}{24r^3} =$$

$$\frac{r^4m^4 + 5r^3m^3dx + 10r^2m^2d^2x^2 + 10r^2m^2d^3x^3 + 5rmd^4x^4 + d^5x^5}{24r^3}.$$

And

And the fourth term, $\frac{61 \times \sqrt{m + \frac{dx}{r}}}{5040r^5}$ is $= \frac{61}{5040r^5} \times \sqrt{m + \frac{dx}{r}} = \frac{61}{5040r^5} \times$ the compound quantity $\left\{ m^2 + \frac{7m^2 dx}{r} + \frac{21m^2 d^2 x^2}{r^2} + \frac{35m^2 d^3 x^3}{r^3} + \frac{35m^2 d^4 x^4}{r^4} + \frac{21m^2 d^5 x^5}{r^5} + \frac{7m^2 d^6 x^6}{r^6} + \frac{d^7 x^7}{r^7} \right\}$

$$= \frac{61}{5040r^5} \times \text{the compound quantity}$$

$$\frac{\frac{r^2 m^2}{r^7} + \frac{7r^2 m^2 dx}{r^1} + \frac{21r^2 m^2 d^2 x^2}{r^2} + \frac{35r^2 m^2 d^3 x^3}{r^3} + \frac{35r^2 m^2 d^4 x^4}{r^4} + \frac{21r^2 m^2 d^5 x^5}{r^5} + \frac{7r^2 m^2 d^6 x^6}{r^6} + \frac{d^7 x^7}{r^7}}{5040r^{12}}$$

$$= \frac{61r^7 m^2 + 427r^6 m^2 dx + 1281r^5 m^2 d^2 x^2 + 2135r^4 m^2 d^3 x^3 + 2135r^3 m^2 d^4 x^4 + 1281r^2 m^2 d^5 x^5 + 427r m^2 d^6 x^6 + 61d^7 x^7}{5040r^{12}}$$

And the fifth term, $\frac{277 \times \sqrt{m + \frac{dx}{r}}}{72,576r^5}$ is $= \frac{277}{72,576r^5} \times$ the compound quantity $m^2 + \frac{9m^2 dx}{r} + \frac{36m^2 d^2 x^2}{r^2} + \frac{84m^2 d^3 x^3}{r^3} + \frac{126m^2 d^4 x^4}{r^4} + \frac{126m^2 d^5 x^5}{r^5} + \frac{84m^2 d^6 x^6}{r^6} + \frac{36m^2 d^7 x^7}{r^7} + \frac{9m^2 d^8 x^8}{r^8} + \frac{d^9 x^9}{r^9} = \frac{277}{72,576r^5} \times$ the compound quantity $\frac{r^9 m^2}{r^0} + \frac{9r^8 m^2 dx}{r^0} + \frac{36r^7 m^2 d^2 x^2}{r^0} + \frac{84r^6 m^2 d^3 x^3}{r^0} + \frac{126r^5 m^2 d^4 x^4}{r^0} + \frac{126r^4 m^2 d^5 x^5}{r^0} + \frac{84r^3 m^2 d^6 x^6}{r^0} + \frac{36r^2 m^2 d^7 x^7}{r^0} + \frac{9r m^2 d^8 x^8}{r^0} + \frac{d^9 x^9}{r^0} = \frac{277r^9 m^2 + 2493r^8 m^2 dx + 9972r^7 m^2 d^2 x^2 + 23,268r^6 m^2 d^3 x^3 + 34,902r^5 m^2 d^4 x^4 + 34,902r^4 m^2 d^5 x^5 + 23,268r^3 m^2 d^6 x^6 + 9972r^2 m^2 d^7 x^7 + 2493r m^2 d^8 x^8 + 277d^9 x^9}{72,576r^{12}}$

Therefore the said complicated series $m + \frac{dx}{r} + \frac{\sqrt{m + \frac{dx}{r}}}{6rr} + \frac{\sqrt{m + \frac{dx}{r}}}{24r^4} + \frac{61 \times \sqrt{m + \frac{dx}{r}}}{5040r^5} + \frac{277 \times \sqrt{m + \frac{dx}{r}}}{72,576r^5} + \&c$, will be equal to the series $m + \frac{dx}{r} + \frac{r^3 m^3 + 3r^2 m^2 dx + 3rm^2 d^2 x^2 + d^3 x^3}{6r^3} + \frac{r^5 m^5 + 5r^4 m^4 dx + 10r^3 m^3 d^2 x^2 + 10r^2 m^2 d^3 x^3 + 5rm^2 d^4 x^4 + d^5 x^5}{24r^5} + \frac{61r^7 m^7 + 427r^6 m^6 dx + 1281r^5 m^5 d^2 x^2 + 2135r^4 m^4 d^3 x^3 + 2135r^3 m^3 d^4 x^4 + 1281r^2 m^2 d^5 x^5 + 427r m^2 d^6 x^6 + 61d^7 x^7}{5040r^{11}} + \frac{277r^9 m^9 + 2493r^8 m^8 dx + 9972r^7 m^7 d^2 x^2 + 23,268r^6 m^6 d^3 x^3 + 34,902r^5 m^5 d^4 x^4 + 34,902r^4 m^4 d^5 x^5 + 23,268r^3 m^3 d^6 x^6 + 9972r^2 m^2 d^7 x^7 + 2493r m^2 d^8 x^8 + 277d^9 x^9}{72,576r^{11}} + \&c =$ the series $\frac{rm}{r} + \frac{r^3 m^3}{6r^3} + \frac{r^5 m^5}{24r^5} + \frac{61r^7 m^7}{5040r^{11}} + \frac{277r^9 m^9}{72,576r^{11}} + \&c$ *ad infinitum*, + the

$$\begin{aligned}
 & + \text{the series } \frac{dx}{r} + \frac{3r^3 m^3 dx}{6r^3} + \frac{5r^5 m^5 dx}{24r^5} + \frac{427r^7 m^7 dx}{5040r^{13}} + \frac{2493r^9 m^9 dx}{72,576r^{13}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{3r^3 m^3 d^2 x}{6r^3} + \frac{10r^5 m^5 d^2 x}{24r^5} + \frac{1381r^7 m^7 d^2 x}{5040r^{13}} + \frac{9972r^9 m^9 d^2 x}{72,576r^{13}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{d^2 x}{6r^3} + \frac{10r^5 m^5 d^2 x}{24r^5} + \frac{2135r^7 m^7 d^2 x}{5040r^{13}} + \frac{23,688r^9 m^9 d^2 x}{72,576r^{13}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{5r^3 m^3 d^4 x}{24r^3} + \frac{2135r^5 m^5 d^4 x}{5040r^{13}} + \frac{84,902r^7 m^7 d^4 x}{72,576r^{13}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{d^4 x}{24r^3} + \frac{1281r^5 m^5 d^4 x}{5040r^{13}} + \frac{34,902r^7 m^7 d^4 x}{72,576r^{13}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{427r^3 m^3 d^6 x}{5040r^{13}} + \frac{21,268r^5 m^5 d^6 x}{72,576r^{13}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{61d^2 x^2}{5040r^{13}} + \frac{9972r^2 m^2 d^2 x^2}{72,576r^{13}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{2493r^3 m^3 d^4 x^2}{72,576r^{13}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{277d^6 x^2}{72,576r^{13}} + \&c \text{ ad infinitum,} + \&c \\
 & = \text{the series } m + \frac{m^3}{6r^4} + \frac{m^5}{24r^4} + \frac{61m^7}{5040r^6} + \frac{277m^9}{72,576r^8} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{dx}{r} + \frac{m^3 dx}{2r^3} + \frac{5m^5 dx}{24r^3} + \frac{61m^7 dx}{720r^5} + \frac{277m^9 dx}{8064r^7} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{m^3 d^2 x}{2r^4} + \frac{5m^5 d^2 x}{12r^4} + \frac{61m^7 d^2 x}{240r^4} + \frac{277m^9 d^2 x}{2016r^{10}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{d^2 x^2}{6r^3} + \frac{5m^3 d^2 x^2}{12r^3} + \frac{61m^5 d^2 x^2}{144r^5} + \frac{277m^7 d^2 x^2}{864r^{12}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{5m^3 d^4 x^2}{24r^3} + \frac{61m^5 d^4 x^2}{144r^{10}} + \frac{277m^7 d^4 x^2}{576r^{12}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{d^4 x^2}{24r^3} + \frac{61m^3 d^4 x^2}{240r^{12}} + \frac{277m^5 d^4 x^2}{576r^{12}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{61m^3 d^6 x^2}{720r^{12}} + \frac{277m^5 d^6 x^2}{864r^{14}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{61d^2 x^2}{5040r^{13}} + \frac{277m^2 d^2 x^2}{2016r^{13}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{277m^3 d^4 x^2}{8064r^{16}} + \&c \text{ ad infinitum,} \\
 & + \text{the series } \frac{277d^6 x^2}{72,576r^{17}} + \&c \text{ ad infinitum,} \\
 & + \&c \text{ ad infinitum.}
 \end{aligned}$$

Art. 59. Now let the capital letter B be put = the infinite series $m + \frac{m^3}{6rr}$
 $+ \frac{m^5}{24r^4} + \frac{61m^7}{5040r^6} + \frac{277m^9}{72,576r^8} + \&c$; and let the capital letter C be put = the
 infinite series $d + \frac{m^3 d}{2r^2} + \frac{5m^5 d}{24r^4} + \frac{61m^7 d}{720r^6} + \frac{277m^9 d}{8064r^8} + \&c$; and let D be = the
 infinite

infinite series $\frac{md^2}{2r^2} + \frac{5m^3d^2}{12r^4} + \frac{61m^5d^2}{240r^6} + \frac{277m^7d^2}{2016r^8} + \&c$; and F be = the infinite series $\frac{d^3}{6r^3} + \frac{5m^3d^3}{12r^4} + \frac{61m^5d^3}{144r^6} + \frac{277m^7d^3}{864r^8} + \&c$; and G be = the infinite series $\frac{5md^4}{24r^4} + \frac{61m^3d^4}{144r^6} + \frac{777m^5d^4}{576r^8} + \&c$; and H be = the infinite series $\frac{d^5}{24r^4} + \frac{61m^3d^5}{240r^6} + \frac{277m^5d^5}{576r^8} + \&c$; and I be = the infinite series $\frac{61md^6}{720r^6} + \frac{277m^3d^6}{864r^8} + \&c$; and K be = the infinite series $\frac{61d^7}{5040r^6} + \frac{277m^3d^7}{2016r^8} + \&c$; and L be = the infinite series $\frac{277md^8}{8064r^8} + \&c$; and M be = the infinite series $\frac{277d^9}{72,576r^8} + \&c$. And the foregoing complicated series $m + \frac{dx}{r} + \frac{m + \frac{dx}{r}}{6rr} + \frac{m + \frac{dx}{r}}{24r^4} + \frac{61 \times m + \frac{dx}{r}}{5040r^6} + \frac{277 \times m + \frac{dx}{r}}{72,576r^8} + \&c$, *ad infinitum*, will be equal to the series $a + \frac{cx}{r} + \frac{dx^2}{r^2} + \frac{7x^3}{r^3} + \frac{6x^4}{r^4} + \frac{7x^5}{r^5} + \frac{1x^6}{r^6} + \frac{8x^7}{r^7} + \frac{1x^8}{r^8} + \frac{8x^9}{r^9} + \&c$, *ad infinitum*.

Art. 60. But, by art. 57, the complicated series $m + \frac{dx}{r} + \frac{m + \frac{dx}{r}}{6rr} + \frac{m + \frac{dx}{r}}{24r^4} + \frac{61 \times m + \frac{dx}{r}}{5040r^6} + \frac{277 \times m + \frac{dx}{r}}{72,576r^8} + \&c$, *ad infinitum*, is equal to a , or the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{m}{2} + \frac{dx}{2r}$ to the radius r in a logarithmick curve of which the subtangent is equal to the radius r . Therefore the series $a + \frac{cx}{r} + \frac{dx^2}{r^2} + \frac{7x^3}{r^3} + \frac{6x^4}{r^4} + \frac{7x^5}{r^5} + \frac{1x^6}{r^6} + \frac{8x^7}{r^7} + \frac{1x^8}{r^8} + \frac{8x^9}{r^9} + \&c$, *ad infinitum*, will also be equal to the said logarithm, or to a .

Therefore $a - b$ will be equal to the excess of the series $a + \frac{cx}{r} + \frac{dx^2}{r^2} + \frac{7x^3}{r^3} + \frac{6x^4}{r^4} + \frac{7x^5}{r^5} + \frac{1x^6}{r^6} + \frac{8x^7}{r^7} + \frac{1x^8}{r^8} + \frac{8x^9}{r^9} + \&c$, *ad infinitum* above b , or will be equal to the series $a - b + \frac{cx}{r} + \frac{dx^2}{r^2} + \frac{7x^3}{r^3} + \frac{6x^4}{r^4} + \frac{7x^5}{r^5} + \frac{1x^6}{r^6} + \frac{8x^7}{r^7} + \frac{1x^8}{r^8} + \frac{8x^9}{r^9} + \&c$, *ad infinitum*.

Art. 61. But, because m is the excess of the arch $q + m$, or twice the arch $\frac{q}{2} + \frac{m}{2}$, above q , or the arch of a quadrant, it follows from art. 57, that

$m +$

$m + \frac{m^3}{6rr} + \frac{m^5}{24r^3} + \frac{61m^7}{5040r^5} + \frac{277m^9}{72576r^7} + \&c$, *ad infinitum*, will be equal to the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{m}{2}$ to the radius r in a logarithmick curve of which the subtangent is equal to r . Therefore B, (which is equal to the said series $m + \frac{m^3}{6rr} + \frac{m^5}{24r^3} + \frac{61m^7}{5040r^5} + \frac{277m^9}{72576r^7} + \&c$, *ad infinitum*,) will be equal to the logarithm of the said ratio of the tangent of the arch $\frac{q}{2} + \frac{m}{2}$ to the radius r in the said logarithmick curve.

But the ratio of the tangent of the arch $\frac{q}{2} + \frac{m}{2}$, or $\frac{q+m}{2}$, to the radius r is equal to the ratio of the radius r to the tangent of the arch $\frac{q-m}{2}$, which is the complement of the arch $\frac{q+m}{2}$ to q , or the arch of a quadrant; and consequently the logarithm of the former ratio in a logarithmick curve of which r is the subtangent is equal to the logarithm of the latter ratio in the same logarithmick curve.

Therefore B (which has just now been shewn to be equal to the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{m}{2}$, or $\frac{q+m}{2}$, to the radius r in the logarithmick curve of which r is the subtangent,) will also be equal to the logarithm of the ratio of the radius r to the tangent of the arch $\frac{q-m}{2}$ in the same logarithmick curve.

Art. 62. But this arch $\frac{q-m}{2}$ is equal to $\frac{r-m}{2}$, or $\frac{r}{2}$.

Therefore B will be equal to the logarithm of the ratio of the radius r to the tangent of the arch $\frac{r}{2}$ in a logarithmick curve of which r is the subtangent.

But b is equal to this logarithm, as appears from Coroll. 1 to Lemma 3d.

Therefore B will be equal to b . And consequently the series $B - b + \frac{bx^3}{r} + \frac{7bx^5}{r^3} + \frac{61bx^7}{r^5} + \frac{277bx^9}{r^7} + \frac{bx^3}{r^3} + \frac{bx^5}{r^5} + \frac{bx^7}{r^7} + \frac{bx^9}{r^9} + \&c$, *ad infinitum*, will be equal to the same series without the two first terms B and $-b$, (which counterbalance and destroy each other,) or will be equal to the series $\frac{bx^3}{r} + \frac{7bx^5}{r^3} + \frac{61bx^7}{r^5} + \frac{277bx^9}{r^7} + \frac{bx^3}{r^3} + \frac{bx^5}{r^5} + \frac{bx^7}{r^7} + \frac{bx^9}{r^9} + \&c$, *ad infinitum*.

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But

But the series $B = b + \frac{cx}{r} + \frac{dx^2}{r^2} + \frac{ex^3}{r^3} + \frac{fx^4}{r^4} + \frac{gx^5}{r^5} + \frac{hx^6}{r^6} + \frac{ix^7}{r^7} + \frac{kx^8}{r^8} + \frac{lx^9}{r^9} + \&c$, has been shewn to be equal to $a - b$.

Therefore the series $\frac{cx}{r} + \frac{dx^2}{r^2} + \frac{ex^3}{r^3} + \frac{fx^4}{r^4} + \frac{gx^5}{r^5} + \frac{hx^6}{r^6} + \frac{ix^7}{r^7} + \frac{kx^8}{r^8} + \frac{lx^9}{r^9} + \frac{mx^{10}}{r^{10}} + \&c$, *ad infinitum*, will also be equal to $a - b$.

Art. 63. But e , or the equatorial arch LM, or the difference of the longitudes of the points A and B, is $= \frac{\sqrt{rr - xx}}{\pi} \times a - b$. Therefore e , or LM, will also be $= \frac{\sqrt{rr - xx}}{\pi} \times$ the series $\frac{cx}{r} + \frac{dx^2}{r^2} + \frac{ex^3}{r^3} + \frac{fx^4}{r^4} + \frac{gx^5}{r^5} + \frac{hx^6}{r^6} + \frac{ix^7}{r^7} + \frac{kx^8}{r^8} + \frac{lx^9}{r^9} + \frac{mx^{10}}{r^{10}} + \&c$, *ad infinitum*; and consequently (multiplying both sides of the equation by π) we shall have $\pi e = \sqrt{rr - xx} \times$ the series $\frac{cx}{r} + \frac{dx^2}{r^2} + \frac{ex^3}{r^3} + \frac{fx^4}{r^4} + \frac{gx^5}{r^5} + \frac{hx^6}{r^6} + \frac{ix^7}{r^7} + \frac{kx^8}{r^8} + \frac{lx^9}{r^9} + \frac{mx^{10}}{r^{10}} + \&c$; and (by extracting the square-root of the residual quantity $rr - xx$, and substituting the infinite series to which it is equal, to wit, the infinite series $r - \frac{\pi x}{2r} - \frac{x^2}{8r^3} - \frac{x^4}{16r^5} - \frac{5x^6}{128r^7} - \frac{7x^{10}}{256r^9} - \&c$, instead of $\sqrt{rr - xx}$ in this last equation,) we shall have $\pi e =$ the product of the multiplication of the infinite series $r - \frac{\pi x}{2r} - \frac{x^2}{8r^3} - \frac{x^4}{16r^5} - \frac{5x^6}{128r^7} - \frac{7x^{10}}{256r^9} - \&c$ into the infinite series $\frac{cx}{r} + \frac{dx^2}{r^2} + \frac{ex^3}{r^3} + \frac{fx^4}{r^4} + \frac{gx^5}{r^5} + \frac{hx^6}{r^6} + \frac{ix^7}{r^7} + \frac{kx^8}{r^8} + \frac{lx^9}{r^9} + \frac{mx^{10}}{r^{10}} + \&c$, and consequently to the following complicated infinite series, to wit,

$$\left\{ \begin{aligned} & cx + \frac{dx^2}{r} + \frac{ex^3}{r^2} + \frac{fx^4}{r^3} + \frac{gx^5}{r^4} + \frac{hx^6}{r^5} + \frac{ix^7}{r^6} + \frac{kx^8}{r^7} + \frac{lx^9}{r^8} + \frac{mx^{10}}{r^9} + \&c \\ & - \frac{cx^2}{2r^2} - \frac{dx^3}{2r^3} - \frac{ex^4}{2r^4} - \frac{fx^5}{2r^5} - \frac{gx^6}{2r^6} - \frac{hx^7}{2r^7} - \frac{ix^8}{2r^8} - \frac{kx^9}{2r^9} - \&c \\ & - \frac{cx^3}{8r^4} - \frac{dx^4}{8r^5} - \frac{ex^5}{8r^6} - \frac{fx^6}{8r^7} - \frac{gx^7}{8r^8} - \frac{hx^8}{8r^9} - \frac{ix^9}{8r^{10}} - \&c \\ & - \frac{cx^4}{16r^6} - \frac{dx^5}{16r^7} - \frac{ex^6}{16r^8} - \frac{fx^7}{16r^9} - \&c \\ & - \frac{5cx^6}{128r^8} - \&c \\ & - \&c; \end{aligned} \right.$$

and consequently (by dividing all the terms of this last equation by π ;) we shall have the complicated series

$c +$

$$\left\{ \begin{array}{l} C + \frac{Dx}{r} + \frac{Fx^2}{r^2} + \frac{Gx^3}{r^3} + \frac{Hx^4}{r^4} + \frac{Ix^5}{r^5} + \frac{Kx^6}{r^6} + \frac{Lx^7}{r^7} + \frac{Mx^8}{r^8} + \&c \\ - \frac{Cx^2}{2r^2} - \frac{Dx^3}{2r^3} - \frac{Fx^4}{2r^4} - \frac{Gx^5}{2r^5} - \frac{Hx^6}{2r^6} - \frac{Ix^7}{2r^7} - \frac{Kx^8}{2r^8} - \&c \\ - \frac{Cx^4}{8r^4} - \frac{Dx^5}{8r^5} - \frac{Fx^6}{8r^6} - \frac{Gx^7}{8r^7} - \frac{Hx^8}{8r^8} - \&c \\ - \frac{Cx^6}{16r^6} - \frac{Dx^7}{16r^7} - \frac{Fx^8}{16r^8} - \&c \\ - \frac{5Cx^8}{128r^8} - \&c \\ - \&c \end{array} \right.$$

$\equiv c$: in which equation there is only one unknown quantity, to wit, x , or the co-sine of the rhumb-angle L A B. Therefore by resolving this equation we shall find the value of the said co-sine of the rhumb-angle L A B, and consequently shall, by the help of a Table of Sines and Tangents, discover the magnitude of the said angle L A B itself. Q. E. D.

Art. 64. This equation will be true only when the several infinite serieses denoted by the capital letters C, D, F, G, H, I, K, L, M, &c, are converging serieses. But this will happen only when m , or the latitude of the point B, or of the first place of the ship, from which it begins its voyage, is less than r , or the radius of the Earth, and when d , or the loxodromick arch B A, or the distance run by the ship in going from B to A, is also less than r , or the radius of the Earth; because the quantities m and d occur in the numerators of the terms of the said infinite serieses, and the quantity r occurs in their denominators. Therefore the said equation will be true only when m and d , or the said latitude of the point B and the distance run by the ship in going from B to A, are less than the radius of the Earth.

Art. 65. In order to simplify the notation of the foregoing equation, let x be substituted in its terms instead of r , or the radius of the Earth; or let the radius of the Earth be called 1. And then we shall have $r^2 = 1$, and $r^3 = 1$, and every following power of $r = 1$; and consequently the foregoing equation will become

$$\left\{ \begin{array}{l} C + Dx + Fx^2 + Gx^3 + Hx^4 + Ix^5 + Kx^6 + Lx^7 + Mx^8 + \&c \\ - \frac{Cx^2}{2} - \frac{Dx^3}{2} - \frac{Fx^4}{2} - \frac{Gx^5}{2} - \frac{Hx^6}{2} - \frac{Ix^7}{2} - \frac{Kx^8}{2} - \&c \\ - \frac{Cx^4}{8} - \frac{Dx^5}{8} - \frac{Fx^6}{8} - \frac{Gx^7}{8} - \frac{Hx^8}{8} - \&c \\ - \frac{Cx^6}{16} - \frac{Dx^7}{16} - \frac{Fx^8}{16} - \&c \\ - \frac{5Cx^8}{128} - \&c \\ - \&c \end{array} \right.$$

$\equiv c$.

An Example of the foregoing Solution.

Art. 66. Let the point P, in Fig. 1, denote (as before,) the North Pole of the Earth, and let the point B, from which the ship begins its voyage, be in North latitude $40^{\circ}, 45'$, and in West longitude, reckoned from London, 15° . Let the ship sail, in a direction partly tending to the North and partly to the East, along the loxodromick curve, or rhumb-line, BA, which cuts all the meridian-lines, over which it passes, in the same angle, to wit, in the angle LAB, which is supposed to be unknown, and of which x is the co-sine in a circle of which r , or 1, or the radius of the Earth, is the radius. And let the length of the said rhumb-line BA, or the distance run by the ship when she arrives at A, be 60 leagues, or 180 miles. And let the length of the equatorial arch LM, or the difference of the longitudes of the points B and A, be $2^{\circ}, 15'$. It is required to find, from these three things so given, the angle LAB, of the ship's course, by means of the foregoing solution.

Art. 67. A degree of a great circle of the Earth is about 69 miles and a half in length. Therefore the arch BM, or the latitude of the point B, which is equal to $40^{\circ}, 45'$, (or $40^{\circ} + \frac{45'}{60}$, or $40^{\circ} + \frac{75}{100}$), or 40.75° , will be = 40.75×69.5 miles = 2832.125 miles. But the radius of the Earth is equal to 4000 miles, and consequently a mile will be one 4000th part of the said radius.

Therefore BM will be = $2832.125 \times \frac{1}{4000}$, or $\frac{2832.125}{4000}$, of the radius of the Earth, or $\frac{209.031}{1,000,000}$ of the radius of the Earth, or 0.708,031 of the radius of the Earth; or (because the radius of the Earth is called 1,) the arch BM will be = $0.708,031 \times 1$, or 0.708,031. But m is = the arch BM. Therefore m will be = 0.708,031.

In the next place the loxodromick arch BA, or the distance run by the ship in passing from B to A, is supposed to be 180 miles, which is = $180 \times \frac{1}{4000}$, or $\frac{180}{4000}$ of the radius of the Earth, or 0.045 of the radius of the Earth; or (because the radius of the Earth is called 1,) = 0.045×1 , or 0.045; so that the loxodromick arch BA will be = 0.045. But d is = the loxodromick arch BA. Therefore d will be = 0.045.

Thirdly, by art. 59, the capital letter C is equal to the infinite series $d + \frac{m^2 d}{2r^2} + \frac{5m^4 d}{24r^4} + \frac{61m^6 d}{720r^6} + \frac{277m^8 d}{8064r^8} + \&c$, and therefore (as r is = 1) it will be equal,

to the infinite series $d + \frac{m^2 d}{2} + \frac{5m^4}{24} + \frac{61m^6}{720} + \frac{277m^8}{8064} + \&c. = d \times$ the infinite series $1 + \frac{m^2}{2} + \frac{5m^4}{24} + \frac{61m^6}{720} + \frac{277m^8}{8064} + \&c.$ But m is $= 0.708,031$. Therefore m^2 will be $(= 0.708,031^2) = 0.501,307$, and m^4 will be $(= m^2 \times m^2 = 0.501,307 \times 0.501,307) = 0.251,308$, and m^6 will be $(= m^4 \times m^2 = 0.251,308 \times 0.501,307) = 0.125,982$, and m^8 will be $(= m^6 \times m^2 = 0.125,982 \times 0.501,307) = 0.063,155$; and consequently the series $1 + \frac{m^2}{2} + \frac{5m^4}{24} + \frac{61m^6}{720} + \frac{277m^8}{8064}$ will be $= 1.000,000 + \frac{0.501,307}{2} + \frac{5 \times 0.251,308}{24} + \frac{61 \times 0.125,982}{720} + \frac{277 \times 0.063,155}{8064} = 1.000,000 + 0.250,653 + \frac{1.256,540}{24} + \frac{7.684,502}{720} + \frac{17.493,935}{8064} = 1.000,000 + 0.250,653 + 0.052,356 + 0.010,673 + 0.002,169 = 1.315,851$. Therefore $d \times$ the infinite series $1 + \frac{m^2}{2} + \frac{5m^4}{24} + \frac{61m^6}{720} + \frac{277m^8}{8064} + \&c$ will be $= d \times 1.315,851 = 0.045 \times 1.315,851 = 0.059,213$. Therefore the capital letter C will, in this case, be $= 0.059,213$.

Fourthly, the small letter c , which is equal to the equatorial arch LM, or to 2 degrees and 15 minutes, or 2 degrees and a quarter of a degree, will be $(= 2\frac{1}{4} \times 65\frac{1}{2} \text{ miles} = 2.25 \times 65.5 \text{ miles}) = 147.375 \text{ miles} = 147.375 \times \frac{1}{4000}$ part of the radius of the Earth, $(= \frac{147.375}{4000}) = 0.036,843$ of the radius of the Earth; or (because the radius of the Earth is $= 1$), the said equatorial arch LM, or c , will be $= 0.036,843 \times 1$, or $0.036,843$.

Therefore the equation set down above in art. 65, will, in this case, become

$$\begin{aligned}
 &= \left\{ \begin{aligned} &0.059,213 + Dx + Fx^3 + Gx^5 + Hx^7 + Lx^9 + Kx^{11} + Lx^{13} + Mx^{15} + \&c \\ &\quad - \frac{Cx^3}{2} - \frac{Dx^5}{2} - \frac{Fx^7}{2} - \frac{Gx^9}{2} - \frac{Hx^{11}}{2} - \frac{Kx^{13}}{2} - \&c \\ &\quad \quad \quad - \frac{Cx^5}{8} - \frac{Dx^7}{8} - \frac{Fx^9}{8} - \frac{Gx^{11}}{8} - \frac{Hx^{13}}{8} - \&c \\ &\quad \quad \quad \quad \quad - \frac{Cx^7}{16} - \frac{Dx^9}{16} - \frac{Fx^{11}}{16} - \&c \\ &\quad \quad \quad \quad \quad \quad \quad - \frac{5Cx^9}{128} - \&c \end{aligned} \right. \\
 &= 0.036,843.
 \end{aligned}$$

[Art. 68. We must next find the values of the capital letters, D, F, G, H, I, K, L, and M; which may be done as follows.

The capital letter D is equal to the infinite series $\frac{md^3}{2r^3} + \frac{5m^3 d^5}{12r^5} + \frac{61m^5 d^7}{240r^7} + \frac{277m^7 d^9}{2016r^9} + \&c.$

$\frac{277m^3d^3}{2016r^3} + \&c$, or (because r is equal to 1,) and consequently r^3, r^4, r^5 , and r^6 , are, each of them, equal to 1,) it is equal to the infinite series $\frac{md^3}{2} + \frac{5m^2d^3}{12} + \frac{61m^4d^3}{240} + \frac{277m^6d^3}{2016} + \&c$; and consequently it will be $= d^3m \times$ the infinite series $\frac{1}{2} + \frac{5m^2}{12} + \frac{61m^4}{240} + \frac{277m^6}{2016} + \&c$, $= d^3m \times$ the infinite series $\frac{1}{2} + \frac{5 \times 0.501,307}{12} + \frac{61 \times 0.251,308}{240} + \frac{277 \times 0.125,982}{2016} + \&c$, $= d^3m \times$ the infinite series $\frac{1}{2} + \frac{2.506,535}{12} + \frac{15.329,788}{240} + \frac{34.897,014}{2016} + \&c$, $= d^3m \times$ the infinite series $0.500,000 + 0.208,878 + 0.063,874 + 0.017,310 + \&c = d^3m \times 0.790,062 = d^3 \times 0.790,031 \times 0.790,062 = d^3 \times 0.559,388 = 0.045^3 \times 0.559,388 = 0.045 \times 0.025,172 = 0.001,132$. Q. E. I.

The capital letter F is = the infinite series $\frac{d^3}{6r^3} + \frac{5m^2d^3}{12r^3} + \frac{61m^4d^3}{144r^3} + \frac{277m^6d^3}{864r^3} + \&c$, = the infinite series $\frac{d^3}{6} + \frac{5m^2d^3}{12} + \frac{61m^4d^3}{144} + \frac{277m^6d^3}{864} + \&c$, $= d^3 \times$ the infinite series $\frac{1}{6} + \frac{5m^2}{12} + \frac{61m^4}{144} + \frac{277m^6}{864} + \&c$, $= d^3 \times$ the infinite series $\frac{1}{6} + \frac{5 \times 0.501,307}{12} + \frac{61 \times 0.251,308}{144} + \frac{277 \times 0.125,982}{864} + \&c$, $= d^3 \times$ the infinite series $\frac{1}{6} + \frac{2.506,535}{12} + \frac{15.329,788}{144} + \frac{34.897,014}{864} + \&c$, $= d^3 \times$ the infinite series $0.166,666 + 0.208,878 + 0.106,456 + 0.040,390 + \&c = d^3 \times 0.522,390 = 0.045^3 \times 0.522,390 = 0.045^3 \times 0.023,507 = 0.045 \times 0.001,057 = 0.000,047$. Q. E. I.

The capital letter G is = the infinite series $\frac{5md^4}{24r^4} + \frac{61m^3d^4}{144r^4} + \frac{277m^5d^4}{576r^4} + \&c =$ the infinite series $\frac{5md^4}{24} + \frac{61m^3d^4}{144} + \frac{277m^5d^4}{576} + \&c = d^4m \times$ the infinite series $\frac{5}{24} + \frac{61m^3}{144} + \frac{277m^5}{576} + \&c = d^4m \times$ the infinite series $\frac{5}{24} + \frac{61 \times 0.501,307}{144} + \frac{277 \times 0.251,308}{576} + \&c = d^4m \times$ the infinite series $\frac{5}{24} + \frac{30.579,727}{144} + \frac{69,612,316}{576} + \&c = d^4m \times$ the infinite series $0.208,333 + 0.212,359 + 0.120,854 + \&c = d^4m \times 0.541,546$, $\&c = d^4 \times 0.790,031 \times 0.541,546 = d^4 \times 0.383,431 = 0.045^3 \times 0.383,431 = 0.045^3 \times 0.017,254 = 0.045^3 \times 0.000,776 = 0.045 \times 0.000,035 = 0.000,001$. Q. E. I.

The capital letter H is equal to the infinite series $\frac{d^5}{24r^4} + \frac{61m^3d^5}{240r^4} + \frac{277m^5d^5}{576r^4} + \&c$

&c = the infinite series $\frac{d^2}{24} + \frac{61m^2d^2}{240} + \frac{277m^4d^2}{576} + \&c = d^2 \times$ the infinite series $\frac{1}{24} + \frac{61 \times 0.501,307}{240} + \frac{277 \times 0.251,308}{576} + \&c = d^2 \times$ the infinite series $\frac{1}{24} + \frac{30.570,727}{240} + \frac{69.612,316}{576} + \&c = d^2 \times$ the infinite series $0.041,666 + 0.127,411 + 0.120,854 + \&c = d^2 \times 0.289,931 = 0.045^3 \times 0.289,931 = 0.045^4 \times 0.013,017 = 0.045^5 \times 0.000,585 = 0.045^6 \times 0.000,026 = 0.045 \times 0.000,001,170 = 0.000,000,052.$
Q₂ E. I.

The capital letter I is equal to the infinite series $\frac{61md^5}{720r^5} + \frac{277m^3d^5}{864r^5} + \&c =$ the infinite series $\frac{61md^5}{720} + \frac{277m^3d^5}{864} + \&c = d^5m \times$ the infinite series $\frac{61}{720} + \frac{277 \times 0.501,307}{864} + \&c = d^5m \times$ the infinite series $\frac{61}{720} + \frac{138.862,039}{864} + \&c = d^5m \times$ the infinite series $0.084,722 + 0.160,720 + \&c = d^5m \times 0.245,442, \&c = d^5 \times 0.708,031 \times 0.245,442 = d^5 \times 0.173,780 = 0.045^6 \times 0.173,780 = 0.045^7 \times 0.007,820 = 0.045^8 \times 0.000,352 = 0.045^9 \times 0.000,015,840 = 0.045^{10} \times 0.000,000,712 = 0.045 \times 0.000,000,032 = 0.000,000,001.$
Q₂ E. I.

The capital letter K is equal to the infinite series $\frac{61d^7}{5040r^6} + \frac{277m^3d^7}{2016r^6} + \&c =$ the infinite series $\frac{61d^7}{5040} + \frac{277m^3d^7}{2016} + \&c = d^7 \times$ the infinite series $\frac{61}{5040} + \frac{277m^3}{2016} + \&c = d^7 \times$ the infinite series $\frac{61}{5040} + \frac{138.862,039}{2016} + \&c = d^7 \times$ the infinite series $0.012,103 + 0.068,880 + \&c = d^7 \times 0.080,983 = 0.045^7 \times 0.080,983 = 0.045^8 \times 0.003,644 = 0.045^9 \times 0.000,164 = 0.045^{10} \times 0.000,007,380 = 0.045^{11} \times 0.000,000,332 = 0.045^{12} \times 0.000,000,015 = 0.045 \times 0.000,000,000,675 = 0.000,000,000,030.$
Q₂ E. I.

The capital letter L is equal to the infinite series $\frac{277md^8}{8064r^7} + \&c =$ the infinite series $\frac{277md^8}{8064} + \&c = d^8m \times$ the infinite series $\frac{277}{8064} + \&c = d^8m \times$ the infinite series $0.034,350 + \&c = d^8 \times 0.708,031 \times 0.034,350, \&c = d^8 \times 0.024,321 = 0.045^8 \times 0.024,321 = 0.045^9 \times 0.001,094 = 0.045^{10} \times 0.000,049 = 0.045^{11} \times 0.000,002,205 = 0.045^{12} \times 0.000,000,099 = 0.045^{13} \times$
X

$$\times 0.000,000,004,455 = 0.045^7 \times 0.000,000,000,200 = 0.045 \times 0.000,000,000,009,000 = 0.000,000,000,000,405. \quad Q_4 \text{ E. 1.}$$

And the capital letter M is equal to the infinite series $\frac{277d^9}{72,576^8} + \&c =$ the infinite series $\frac{277d^9}{72,576} + \&c = d^9 \times$ the infinite series $\frac{277}{72,576} + \&c = d^9 \times$ the infinite series $0.003,816 + \&c = 0.045^9 \times 0.001,816, \&c = 0.045^8 \times 0.000,171,720 = 0.045^7 \times 0.000,007,727,400 = 0.045^6 \times 0.000,000,347,733 = 0.045^5 \times 0.000,000,015,648 = 0.045^4 \times 0.000,000,000,704 = 0.045^3 \times 0.000,000,000,031,680 = 0.045^2 \times 0.000,000,000,001,425 = 0.045 \times 0.000,000,000,000,064 = 0.000,000,000,000,002. \quad Q_4 \text{ E. 1.}$

Therefore, if we carry the computation of the infinite series denoted by the said capital letters C, D, F, G, H, I, K, L, and M, only to six places of decimal fractions, we may consider the letters H, I, K, L, and M, as being, each of them, equal to 0, and may therefore omit the several terms in the above complicated equation, set down in art. 67, that involve them. And then the said equation will be as follows; to wit,

$$\begin{aligned} &0.059,213 + Dx + Fx^2 + Gx^3 + 0 + 0 + 0 + 0 + 0 + \&c \\ &\left\{ \begin{array}{l} - \frac{Cx^2}{2} - \frac{Dx^3}{2} - \frac{Fx^4}{2} - \frac{Gx^5}{2} - 0 - 0 - 0 - \&c \\ - \frac{Cx^4}{8} - \frac{Dx^5}{8} - \frac{Fx^6}{8} - \frac{Gx^7}{8} - 0 - \&c \\ - \frac{Cx^6}{16} - \frac{Dx^7}{16} - \frac{Fx^8}{16} - \&c \\ - \frac{5Cx^8}{128} - \&c \end{array} \right. \\ &= 0.036,843. \end{aligned}$$

Art. 69. In this equation we must now reduce the several compound coefficients of $x^2, x^3, x^4, x^5, x^6, x^7$, and x^8 , to wit, the compound quantities $+ F - \frac{c}{2} + G - \frac{D}{2} - \frac{F}{2} - \frac{c}{8} - \frac{G}{2} - \frac{D}{8} - \frac{F}{8} - \frac{c}{16} - \frac{G}{8} - \frac{D}{16} - \frac{F}{16} - \frac{5c}{128}$, into simple terms; which may be done as follows.

Since c is $= 0.059,213$, we shall have $\frac{c}{2} (= \frac{0.059,213}{2}) = 0.029,606$, and $\frac{c}{8} (= \frac{0.059,213}{8}) = 0.007,402$, and $\frac{c}{16} (= \frac{0.059,213}{16}) = 0.003,701$, and $\frac{5c}{128} (= \frac{5 \times 0.059,213}{128} = \frac{0.296,065}{128}) = 0.002,313$.

And, since D is $= 0.001,132$, we shall have $\frac{D}{2} (= \frac{0.001,132}{2}) = 0.000,566$, and $\frac{D}{8} (= \frac{0.001,132}{8}) = 0.000,141$, and $\frac{D}{16} (= \frac{0.001,132}{16}) = 0.000,070$. And,

And, since r is $= 0.000,047$, we shall have $\frac{r}{2} (= \frac{0.000,047}{2} = 0.000,023$, and $\frac{r}{8} (= \frac{0.000,047}{8} = 0.000,006$, and $\frac{r}{16} (= \frac{0.000,047}{16} = 0.000,003$.

And, since σ is $= 0.000,001$, we shall have $\frac{\sigma}{2} (= \frac{0.000,001}{2} = 0.000,0005$, and $\frac{\sigma}{8} = 0.000,000$.

Therefore $+ r - \frac{c}{2}$ will be $(= + 0.000,047 - 0.029,606) = - 0.029,559$; and $+ \sigma - \frac{b}{2}$ will be $(= + 0.000,001 - 0.000,566) = - 0.000,565$, and $-\frac{r}{2} - \frac{c}{8}$ will be $(= - 0.000,023 - 0.007,402) = - 0.007,425$; and $-\frac{\sigma}{2} - \frac{b}{8}$ will be $(= - 0.000,000 - 0.000,141) = - 0.000,141$; and $-\frac{r}{8} - \frac{c}{16}$ will be $(= - 0.000,006 - 0.003,701) = - 0.003,707$; and $-\frac{\sigma}{8} - \frac{b}{16}$ will be $(= - 0.000,000 - 0.000,070) = - 0.000,070$; and $-\frac{r}{16} - \frac{5c}{128}$ will be $(= - 0.000,003 - 0.002,316) = - 0.002,316$.

Therefore the foregoing complicated equation, set down in the preceeding article, will now become as follows, to wit,

$$\begin{aligned} 0.059,213 + 0.001,132 \times x - 0.029,559 \times x^2 - 0.000,565 \times x^3 \\ - 0.007,425 \times x^4 - 0.000,141 \times x^5 - 0.003,707 \times x^6 \\ - 0.000,070 \times x^7 - 0.002,316 \times x^8 - \&c = 0.036,843. \end{aligned}$$

Art. 70. Now let all the terms on the left-hand side of this equation that have the sign $-$ prefixed to them, or are subtracted from the other terms on the same side, be added to both sides of the equation. And we shall then have

$$\begin{aligned} 0.059,213 + 0.001,132 \times x = 0.036,843 + 0.029,559 \times x^2 \\ + 0.000,565 \times x^3 + 0.007,425 \times x^4 + 0.000,141 \times x^5 \\ + 0.003,707 \times x^6 + 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c. \end{aligned}$$

Therefore, (subtracting $0.036,843$ from both sides,) we shall have

$$\begin{aligned} 0.022,370 + 0.001,132 \times x = 0.029,559 \times x^2 + 0.000,565 \times x^3 \\ + 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ + 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c, \text{ and (subtracting } 0.001,132 \times x \text{ from both sides,)} \text{ we shall have} \end{aligned}$$

$$\begin{aligned} - 0.001,132 \times x + 0.029,559 \times x^2 + 0.000,565 \times x^3 \\ + 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ + 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c = 0.022,370. \end{aligned}$$

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Art. 71.

Art. 71. In this equation it is remarkable that 0.029,559, the co-efficient of x^2 , is greater than 0.001,132, the co-efficient of x ; and that 0.007,425, the co-efficient of x^4 , is greater than 0.000,565, the co-efficient of x^3 ; and that 0.003,707, the co-efficient of x^6 , is greater than 0.000,141, the co-efficient of x^5 ; and that 0.002,316, the co-efficient of x^8 , is greater than 0.000,070, the co-efficient of x^7 . But the co-efficients of x , x^3 , x^5 , and x^7 , or the odd powers of x , (to wit, 0.001,132, and 0.000,565, and 0.000,141, and 0.000,070,) are every one less than the next before it, or decrease when the powers of x increase; and the co-efficients of x^2 , x^4 , x^6 , and x^8 , or the even powers of x , (to wit, 0.029,559, and 0.007,425, and 0.003,707, and 0.002,316,) are also every one less than the next before it, or decrease when the powers of x increase; as well as the co-efficients of x , x^3 , x^5 , and x^7 , or the odd powers of x .

Art. 72. We must now endeavour to resolve this last equation obtained in art. 70, to wit, the equation — $0.001,132 \times x + 0.029,559 \times x^2 + 0.000,565 \times x^3 + 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 + 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c.$ = 0.022,370. Now this can only be done by approximation; and I take Mr. RAPHSON'S method of resolving equations by approximation to be the best method that can be employed for this purpose. I will therefore now endeavour to find a first near value of x that shall differ from its true value by only a tenth, or less than a tenth, part of the said true value, in order to make the said first near value of x the basis of a further approach to its true value, according to the directions of Mr. RAPHSON'S method.

Art. 73. Now, in order to find this first near value of x , we may begin by observing, that x , (being the co-sine of a certain angle in a circle of which r , or 1, or the radius of the Earth, is the radius,) must be less than the radius r , or 1; and consequently that xx must be less than x , and x^3 than xx , and x^4 than x^3 , and every following power of x must be less than that which precedes it. And hence we may conclude that the two first terms $0.001,132 \times x$, and $0.029,559 \times xx$, will be greater than the two next terms, or than any other two terms on the same side of the equation. We will therefore neglect all the terms on the left-hand side of the equation, except the said two first terms — $0.001,132 \times x + 0.029,559 \times xx$, and will suppose those two terms alone to be equal to the whole left-hand side of the equation, and consequently to the absolute term 0.022,370, and will resolve the quadratic equation resulting from this supposition, to wit, the quadratic equation — $0.001,132 \times x + 0.029,559 \times xx = 0.022,370$, or $0.029,559 \times xx - 0.001,132 \times x = 0.022,370$. This may be done in the manner following.

Divide all the terms by 0.029,559, or the co-efficient of xx . And we shall have $xx = \frac{0.001,132}{0.029,559} \times x = \frac{0.022,370}{0.029,559}$, that is, $xx = 0.038,296 \times x = 0.756,794$.

0.756,791. Divide the co-efficient 0.038,296 by 2; and the quotient will be 0.019,148, the square of which is 0.000,366,645,904. Let this square be added to both sides of the equation. And we shall have $xx - 0.038,296 \times x + 0.019,148^2 = 0.756,791 + 0.000,366,645,904 = 0.757,157,645,904$. Therefore (extracting the square-roots of both sides,) we shall have $x - 0.019,148 = 0.870$, &c, and consequently x will be $= 0.870 + 0.019$, &c $= 0.889$, &c. Therefore the first near value of x in the foregoing high equation, set down in art. 70, will be 0.889; of which it seems reasonable to suppose that the first two figures, 0.88, may be exact.

Art. 74. We will therefore now substitute 0.88, instead of x , in the compound quantity which forms the left-hand side of the equation set down in art. 70, to wit, the equation

$$\begin{aligned} &= 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3 \\ &+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ &+ 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c, = 0.022,370, \text{ in order} \end{aligned}$$

to discover whether the value of the said compound quantity resulting from such substitution will be greater, or less, than the absolute term, 0.022,370, of the said equation, and whether its difference from the said absolute term will be great, or small.

Now, if we suppose x to be $= 0.88$, we shall have

$$\begin{aligned} xx & (= 0.88 \times 0.88) = 0.7744, \\ \text{and } x^3 & (= 0.7744 \times 0.88) = 0.681,472, \\ \text{and } x^4 & (= 0.681,472 \times 0.88) = 0.599,695, \\ \text{and } x^5 & (= 0.599,695 \times 0.88) = 0.527,731, \\ \text{and } x^6 & (= 0.527,731 \times 0.88) = 0.464,403, \\ \text{and } x^7 & (= 0.464,403 \times 0.88) = 0.408,674, \\ \text{and } x^8 & (= 0.408,674 \times 0.88) = 0.359,633, \text{ and consequently} \end{aligned}$$

$$\begin{aligned} 0.001,132 \times x & (= 0.001,132 \times 0.88) = 0.000,996, \\ \text{and } 0.029,559 \times xx & (= 0.029,559 \times 0.7744) = 0.022,890, \\ \text{and } 0.000,565 \times x^3 & (= 0.000,565 \times 0.681,472) = 0.000,385, \\ \text{and } 0.007,425 \times x^4 & (= 0.007,425 \times 0.599,695) = 0.004,452, \\ \text{and } 0.000,141 \times x^5 & (= 0.000,141 \times 0.527,731) = 0.000,074, \\ \text{and } 0.003,707 \times x^6 & (= 0.003,707 \times 0.464,403) = 0.001,721, \\ \text{and } 0.000,070 \times x^7 & (= 0.000,070 \times 0.408,674) = 0.000,028, \\ \text{and } 0.002,316 \times x^8 & (= 0.002,316 \times 0.359,633) = 0.000,833. \end{aligned}$$

Therefore the compound quantity

$$\begin{aligned} &= 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3 \\ &+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ &\quad \quad \quad K 2 \end{aligned}$$

+

+ 0.000,070 $\times x^7$ + 0.002,316 $\times x^8$ will be (= - 0.000,996 + 0.022,890 + 0.000,385 + 0.004,452 + 0.000,074 + 0.001,721 + 0.000,028 + 0.000,833 = - 0.000,996 + 0.030,383) = + 0.029,387; which is greater than 0.022,370, or the absolute term of the equation

- 0.001,132 $\times x$ + 0.029,559 $\times xx$ + 0.000,565 $\times x^3$
 + 0.007,425 $\times x^4$ + 0.000,141 $\times x^5$ + 0.003,707 $\times x^6$
 + 0.000,070 $\times x^7$ + 0.002,316 $\times x^8$ = 0.022,370. Therefore 0.88 will be greater than the true value of x in the said equation.

Art. 75. We will therefore, in the next place, suppose x to be equal to 0.88; - z , and substitute 0.88 - z instead of x in the terms of the foregoing equation, but with an omission of all the terms that involve any higher powers of z than its simple power, or z itself, agreeably to the rules of Mr. RAPHSON'S method of approximation. This may be done in the manner following.

Since x is = 0.88 - z , we shall have

$$\begin{aligned}
 & x (= 0.88 - z)^1 = \overline{0.88}^1 - 2 \times \overline{0.88} \times z + \&c = 0.7744 - 1.76 \\
 & \quad \times z + \&c, \\
 \text{and } x^2 (= 0.88 - z)^2 = \overline{0.88}^2 - 3 \times \overline{0.88} \times z + \&c = \overline{0.88}^2 - 3 \times \\
 & \quad 0.7744 \times z + \&c = 0.681,472 - 2.3232 \times z + \&c, \\
 \text{and } x^3 (= 0.88 - z)^3 = \overline{0.88}^3 - 4 \times \overline{0.88}^2 \times z + \&c = \overline{0.88}^3 - 4 \times \\
 & \quad 0.681,472 \times z + \&c = 0.599,695 - 2.725,888 \times z + \&c, \\
 \text{and } x^4 (= 0.88 - z)^4 = \overline{0.88}^4 - 5 \times \overline{0.88}^3 \times z + \&c = \overline{0.88}^4 - \\
 & \quad 5 \times 0.599,695 \times z + \&c = 0.527,731 - 2.998,475 \times z + \&c, \\
 \text{and } x^5 (= 0.88 - z)^5 = \overline{0.88}^5 - 6 \times \overline{0.88}^4 \times z + \&c = \overline{0.88}^5 - 6 \times \\
 & \quad 0.527,731 \times z + \&c = 0.464,403 - 3.166,386 \times z + \&c, \\
 \text{and } x^6 (= 0.88 - z)^6 = \overline{0.88}^6 - 7 \times \overline{0.88}^5 \times z + \&c = \overline{0.88}^6 - 7 \times \\
 & \quad 0.464,403 \times z + \&c = 0.408,674 - 3.250,821 \times z + \&c, \\
 \text{and } x^7 (= 0.88 - z)^7 = \overline{0.88}^7 - 8 \times \overline{0.88}^6 \times z + \&c = \overline{0.88}^7 - 8 \times \\
 & \quad 0.408,674 \times z + \&c = 0.359,633 - 3.269,392 \times z + \&c.
 \end{aligned}$$

Therefore 0.001,132 $\times x$ will be (= 0.001,132 $\times \overline{0.88} - z$ = 0.001,132 $\times \overline{0.88} - 0.001,132 \times z$) = 0.000,996 - 0.001,132 $\times z$,

and 0.029,559 $\times xx$ will be (= 0.029,559 $\times \overline{0.7744} - 1.76 \times z$) = 0.022,890 - 0.052,023 $\times z$,

and

and $0.000,565 \times x^2$ will be $(= 0.000,565 \times \frac{0.681,472 - 2.3232 \times z}{0.000,565 \times 0.681,472 - 0.000,565 \times 2.3232 \times z}) = 0.000,385 - 0.001,312 \times z,$

and $0.007,425 \times x^4$ will be $(= 0.007,425 \times \frac{0.599,695 - 2.725,888 \times z}{0.007,425 \times 0.599,695 - 0.007,425 \times 2.725,888 \times z}) = 0.004,452 - 0.020,239 \times z,$

and $0.000,141 \times x^5$ will be $(= 0.000,141 \times \frac{0.527,731 - 2.998,475 \times z}{0.000,141 \times 0.527,731 - 0.000,141 \times 2.998,475 \times z}) = 0.000,074 - 0.000,422 \times z,$

and $0.003,707 \times x^6$ will be $(= 0.003,707 \times \frac{0.404,403 - 3.160,386 \times z}{0.003,707 \times 0.404,403 - 0.003,707 \times 3.160,386 \times z}) = 0.001,721 - 0.011,738 \times z,$

and $0.000,070 \times x^7$ will be $(= 0.000,070 \times \frac{0.408,674 - 3.250,821 \times z}{0.000,070 \times 0.408,674 - 0.000,070 \times 3.250,821 \times z}) = 0.000,028 - 0.000,227 \times z,$

and $0.002,316 \times x^8$ will be $(= 0.002,316 \times \frac{0.359,633 - 3.269,392 \times z}{0.002,316 \times 0.359,633 - 0.002,316 \times 3.269,392 \times z}) = 0.000,833 - 0.007,562 \times z;$

and consequently the compound quantity

$$\begin{aligned} & - 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3 \\ & + 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ & + 0.000,070 \times x^7 + 0.002,316 \times x^8 \text{ will be} \end{aligned}$$

$$\begin{aligned} & = \left\{ \begin{array}{l} - 0.000,996 + 0.001,132 \times z \\ + 0.022,890 - 0.052,023 \times z \\ + 0.000,385 - 0.001,312 \times z \\ + 0.004,452 - 0.020,239 \times z \\ + 0.000,074 - 0.000,422 \times z \\ + 0.001,721 - 0.011,738 \times z \\ + 0.000,028 - 0.000,227 \times z \\ + 0.000,833 - 0.007,562 \times z \end{array} \right\} \\ & = \left\{ \begin{array}{l} - 0.000,996 + 0.001,132 \times z \\ + 0.030,383 - 0.093,523 \times z \end{array} \right\} \\ & = + 0.029,387 - 0.092,391 \times z. \end{aligned}$$

But the compound quantity

$$\begin{aligned} & - 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3 \\ & + 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ & + 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c \text{ is } = 0.022,370. \end{aligned}$$

Therefore $0.029,387 - 0.092,391 \times z$ is also $= 0.022,370$. And consequently

frequently (adding $0.092,391 \times z$ to both sides,) we shall have $0.029,387 = 0.022,370 + 0.092,391 \times z$, and (subtracting $0.022,370$ from both sides,) $0.092,391 \times z = 0.007,017$, and $z = \frac{0.007,017}{0.092,391} = 0.0759$.

Therefore x , or $0.88 - z$, will be ($= 0.8800 - 0.0759$) $= 0.8041$; or the second near value of x in the equation

$$\begin{aligned} & - 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3 \\ & + 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ & + 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c = 0.022,370 \text{ will be} = \\ & 0.8041. \quad \text{Q. E. I.} \end{aligned}$$

Art. 76. We will now substitute this last value of x , to wit, 0.8041 , instead of x , in the compound quantity

$$\begin{aligned} & - 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3 \\ & + 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ & + 0.000,070 \times x^7 + 0.002,316 \times x^8, \text{ (which forms the first, or left-} \\ & \text{hand, side of the foregoing equation,)} \text{ in order to discover how near the value} \\ & \text{of the said compound quantity resulting from the said substitution will ap-} \\ & \text{proach to the true value of the said compound quantity, or to the value of its} \\ & \text{equal, the absolute term, } 0.022,370, \text{ of the said equation; and consequently} \\ & \text{to find how near the said number } 0.8041 \text{ will approach to the true value of } x \\ & \text{in the said equation. This may be done in the manner following.} \end{aligned}$$

If we suppose x to be $= 0.8041$, we shall have

$$\begin{aligned} & xx = 0.8041^2 = 0.646,576,81, \\ & \text{and } x^3 (= 0.646,577 \times 0.8041) = 0.519,912, \\ & \text{and } x^4 (= 0.519,912 \times 0.8041) = 0.418,061, \\ & \text{and } x^5 (= 0.418,061 \times 0.8041) = 0.336,162, \\ & \text{and } x^6 (= 0.336,162 \times 0.8041) = 0.270,307, \\ & \text{and } x^7 (= 0.270,307 \times 0.8041) = 0.217,353, \\ & \text{and } x^8 (= 0.217,353 \times 0.8041) = 0.174,773, \\ & \text{and consequently } 0.001,132 \times x (= 0.001,132 \times 0.8041) = 0.000,910, \\ & \text{and } 0.029,559 \times xx (= 0.029,559 \times 0.646,577) = 0.019,112, \\ & \text{and } 0.000,565 \times x^3 (= 0.000,565 \times 0.519,912) = 0.000,293, \\ & \text{and } 0.007,425 \times x^4 (= 0.007,425 \times 0.418,061) = 0.003,104, \\ & \text{and } 0.000,141 \times x^5 (= 0.000,141 \times 0.336,162) = 0.000,047, \\ & \text{and } 0.003,707 \times x^6 (= 0.003,707 \times 0.270,307) = 0.001,002, \\ & \text{and } 0.000,070 \times x^7 (= 0.000,070 \times 0.217,353) = 0.000,015, \\ & \text{and } 0.002,316 \times x^8 (= 0.002,316 \times 0.174,773) = 0.000,404, \end{aligned}$$

and

and $-0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3$
 $+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6$
 $+ 0.000,070 \times x^7 + 0.002,316 \times x^8 = -0.000,910 + 0.019,112$
 $+ 0.000,293 + 0.003,104 + 0.000,047 + 0.001,002 + 0.000,015 +$
 $0.000,404 = -0.000,910 + 0.023,977 = 0.023,067$; which is greater
 than 0.022,370, or the absolute term of the equation

$$-0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3$$

$$+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6$$

$+ 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c = 0.022,370$. Therefore
 0.8041 is greater than the true value of x in that equation: but the difference
 between them is not great.

Art. 77. We will therefore, in the next place, suppose the value of x in the
 foregoing equation to be equal to 0.8, and substitute 0.8 instead of x in the
 compound quantity.

$$-0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3$$

$$+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6$$

$+ 0.000,070 \times x^7 + 0.002,316 \times x^8$; (which forms the first, or left-hand,
 side of that equation,) in order to discover how nearly the value of the said
 compound quantity resulting from that substitution will approach to the true
 value of the said compound quantity, or of its equal, the absolute term,
 0.022,370, of the said equation. This may be done in the manner following.

If we suppose x to be = 0.8, we shall have $xx (= 0.8^2) = 0.64$,

and $x^3 (= 0.64 \times 0.8) = 0.512$;

and $x^4 (= 0.512 \times 0.8) = 0.4096$,

and $x^5 (= 0.4096 \times 0.8) = 0.327,68$,

and $x^6 (= 0.327,68 \times 0.8) = 0.262,144$;

and $x^7 (= 0.262,144 \times 0.8) = 0.209,715,2$;

and $x^8 (= 0.209,715,2 \times 0.8) = 0.167,772,16$.

Therefore $0.001,132 \times x$ will be $(= 0.001,132 \times 0.8) = 0.000,905$
 and $0.029,559 \times xx$ will be $(= 0.029,559 \times 0.64) = 0.018,917$;
 and $0.000,565 \times x^3$ will be $(= 0.000,565 \times 0.512) = 0.000,289$;
 and $0.007,425 \times x^4$ will be $(= 0.007,425 \times 0.4096) = 0.003,041$;
 and $0.000,141 \times x^5$ will be $(= 0.000,141 \times 0.327,68) = 0.000,046$;
 and $0.003,707 \times x^6$ will be $(= 0.003,707 \times 0.262,144) = 0.000,971$;
 and $0.000,070 \times x^7$ will be $(= 0.000,070 \times 0.209,715) = 0.000,014$;
 and $0.002,316 \times x^8$ will be $(= 0.002,316 \times 0.167,772) = 0.000,388$;
 and consequently the whole compound quantity

==

$- 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3$
 $+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6$
 $+ 0.000,070 \times x^7 + 0.002,316 \times x^8$ will be $= - 0.000,905 + 0.018,917 + 0.000,289 + 0.003,041 + 0.000,046 + 0.000,971 + 0.000,014 + 0.000,388 = - 0.000,905 + 0.023,666 = 0.022,761$; which is a little greater than 0.022,370, or the absolute term of the equation

$- 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3$
 $+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6$
 $+ 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c = 0.022,370$. Therefore 0.8 will be nearly equal to, but a little greater than, the true value of x in the said equation. Q. E. I.

Art. 78. In order therefore to find a still nearer value of the root of the said equation

$- 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3$
 $+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6$
 $+ 0.000,070 \times x^7 + 0.002,316 \times x^8 = 0.022,370$, by Mr. RAPHSOON'S method of approximation, let us suppose x to be equal to $0.8 - z$, and let us substitute $0.8 - z$ instead of x in the terms of the said equation, omitting all the terms in the values of the several powers of $0.8 - z$ which involve any higher powers of z than the simple power, or z itself. This may be done in the manner following.

Since x is $= 0.8 - z$, we shall have

$$xx (= 0.8 - z)^2 = 0.8^2 - 2 \times 0.8 \times z + \&c) = 0.64 - 1.6 \times z + \&c,$$

$$\text{and } x^3 (= 0.8 - z)^3 = 0.8^3 - 3 \times 0.8^2 \times z + \&c = 0.8^3 - 3 \times 0.64 \times z + \&c) = 0.512 - 1.92 \times z + \&c,$$

$$\text{and } x^4 (= 0.8 - z)^4 = 0.8^4 - 4 \times 0.8^3 \times z + \&c = 0.8^4 - 4 \times 0.512 \times z + \&c) = 0.4096 - 2.048 \times z + \&c,$$

$$\text{and } x^5 (= 0.8 - z)^5 = 0.8^5 - 5 \times 0.8^4 \times z + \&c = 0.8^5 - 5 \times 0.4096 \times z + \&c) = 0.327,68 - 2.0480 \times z + \&c,$$

$$\text{and } x^6 (= 0.8 - z)^6 = 0.8^6 - 6 \times 0.8^5 \times z + \&c = 0.8^6 - 6 \times 0.327,68 \times z + \&c) = 0.262,144 - 1.966,08 \times z + \&c,$$

$$\text{and } x^7 (= 0.8 - z)^7 = 0.8^7 - 7 \times 0.8^6 \times z + \&c = 0.8^7 - 7 \times 0.262,144 \times z + \&c) = 0.209,715,2 - 1.835,008 \times z + \&c,$$

$$\text{and } x^8 (= 0.8 - z)^8 = 0.8^8 - 8 \times 0.8^7 \times z + \&c = 0.8^8 - 8 \times 0.209,715,2 \times z + \&c) = 0.167,772,16 - 1.677,721,6 \times z + \&c,$$

$$\text{and consequently } 0.001,132 \times x (= 0.001,132 \times 0.8 - z) = 0.001,132 \times 0.8 - 0.001,132 \times z,$$

and

$$\text{and } 0.029,559 \times xx (= 0.029,559 \times \overline{0.64 - 1.6 \times z + \&c} = 0.029,559 \times 0.64 - 0.029,559 \times 1.6 \times z + \&c) = 0.018,917,76 - 0.047,294,4 \times z + \&c,$$

$$\text{and } 0.000,565 \times x^3 (= 0.000,565 \times \overline{0.512 - 1.92 \times z + \&c} = 0.000,565 \times 0.512 - 0.000,565 \times 1.92 \times z + \&c) = 0.000,289 - 0.001,084 \times z + \&c,$$

$$\text{and } 0.007,425 \times x^4 (= 0.007,425 \times \overline{0.4096 - 2.048 \times z + \&c} = 0.007,425 \times 0.4096 - 0.007,425 \times 2.048 \times z + \&c) = 0.003,041 - 0.015,205 \times z + \&c,$$

$$\text{and } 0.000,141 \times x^5 (= 0.000,141 \times \overline{0.327,68 - 2.0480 \times z + \&c} = 0.000,141 \times 0.327,68 - 0.000,141 \times 2.0480 \times z + \&c) = 0.000,046 - 0.000,288 \times z + \&c,$$

$$\text{and } 0.003,707 \times x^6 (= 0.003,707 \times \overline{0.262,144 - 1.966,08 \times z + \&c} = 0.003,707 \times 0.262,144 - 0.003,707 \times 1.966,08 \times z + \&c) = 0.009,971 - 0.007,288 \times z + \&c,$$

$$\text{and } 0.000,070 \times x^7 (= 0.000,070 \times \overline{0.209,715 - 1.835,008 \times z + \&c} = 0.000,070 \times 0.209,715 - 0.000,070 \times 1.835,008 \times z + \&c) = 0.000,014 - 0.000,128 \times z + \&c,$$

$$\text{and } 0.002,316 \times x^8 (= 0.002,316 \times \overline{0.167,772 - 1.677,721 \times z + \&c} = 0.002,316 \times 0.167,772 - 0.002,316 \times 1.677,721 \times z + \&c) = 0.000,388 - 0.003,875 \times z + \&c.$$

Therefore the compound quantity

$$- 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3 \\ + 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ + 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c, \text{ will be } =$$

$$\begin{aligned} & - 0.000,905 + 0.001,132 \times z \\ & + 0.018,917 - 0.047,294 \times z + \&c \\ & + 0.000,289 - 0.001,084 \times z + \&c \\ & + 0.003,041 - 0.015,205 \times z + \&c \\ & + 0.000,046 - 0.000,288 \times z + \&c \\ & + 0.009,971 - 0.007,288 \times z + \&c \\ & + 0.000,014 - 0.000,128 \times z + \&c \\ & + 0.000,388 - 0.003,875 \times z + \&c \end{aligned}$$

$$= \left\{ \begin{array}{l} - 0.000,905 + 0.001,132 \times z \\ + 0.023,666 - 0.075,163 \times z + \&c \end{array} \right\} \\ = 0.022,761 - 0.074,031 \times z + \&c.$$

But the said compound quantity is equal to the absolute term 0.022,370, of the equation

$$\begin{aligned} & - 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3 \\ & + 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6 \\ & + 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c = 0.022,370. \end{aligned}$$

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Therefore $0.022,761 - 0.074,031 \times z + \&c.$, will also be equal to the said absolute term $0.022,370$. And consequently $0.022,761$ will be $= 0.022,370 + 0.074,031 \times z$, and $0.074,031 \times z$ will be $(= 0.022,761 - 0.022,370) = 0.000,391$, and z will be $= \frac{0.000,391}{0.074,031} = 0.005,281$. Therefore x , or $0.8 - z$, will be $(= 0.800,000 - 0.005,281) = 0.794,719$; that is, the more accurate value of the root of the said equation will be $0.794,719$. $Q.E.D.$

Art. 80. Having thus obtained the number $0.794,719$, or (dropping the two last figures, 19, as probably not exact,) the number 0.7947 for the value of x in the equation

$$- 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3$$

$$+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6$$

+ $0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c = 0.022,370$, we will now proceed to substitute the said number in the compound quantity which forms the left-hand side of this equation, in order to see how nearly the value of the said compound quantity resulting from this substitution will approach to the true value of the said compound quantity, or to its equal, the absolute term $0.022,370$, and consequently to determine how nearly the said number 0.7947 will approach to the true value of x in the said equation. This substitution may be made in the following manner.

If we suppose x to be equal to 0.7947 , we shall have

$$xx = 0.631,548, \text{ and } x^3 = 0.501,891, \text{ and } x^4 = 0.398,852,$$

$$\text{and } x^5 = 0.316,967, \text{ and } x^6 = 0.251,893, \text{ and } x^7 = 0.200,179,$$

$$\text{and } x^8 = 0.159,082, \text{ and consequently}$$

$$0.001,132 \times x (= 0.001,132 \times 0.7947) = 0.000,899,$$

$$\text{and } 0.029,559 \times xx (= 0.029,559 \times 0.631,548) = 0.018,668,$$

$$\text{and } 0.000,565 \times x^3 (= 0.000,565 \times 0.501,891) = 0.000,282,$$

$$\text{and } 0.007,425 \times x^4 (= 0.007,425 \times 0.398,852) = 0.002,961,$$

$$\text{and } 0.000,141 \times x^5 (= 0.000,141 \times 0.316,967) = 0.000,044,$$

$$\text{and } 0.003,707 \times x^6 (= 0.003,707 \times 0.251,893) = 0.000,933,$$

$$\text{and } 0.000,070 \times x^7 (= 0.000,070 \times 0.200,179) = 0.000,014,$$

$$\text{and } 0.002,316 \times x^8 (= 0.002,316 \times 0.159,082) = 0.000,368.$$

Therefore the compound quantity

$$- 0.001,132 \times x + 0.029,559 \times xx + 0.000,565 \times x^3$$

$$+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6$$

+ $0.000,070 \times x^7 + 0.002,316 \times x^8$ will be $= - 0.000,899 + 0.018,668 + 0.000,282 + 0.002,961 + 0.000,044 + 0.000,933 + 0.000,014 + 0.000,368 = - 0.000,899 + 0.023,270 = + 0.022,371$; which is greater than the absolute term, $0.022,370$, of the equation

$- 0.001,132 \times x + 0.029,559 \times x^2 + 0.000,565 \times x^3$
 $+ 0.007,425 \times x^4 + 0.000,141 \times x^5 + 0.003,707 \times x^6$
 $+ 0.000,070 \times x^7 + 0.002,316 \times x^8 + \&c = 0.022,370$ by only
 an unit in the sixth place of decimal fractions. We may therefore conclude
 that the number 0.7947 is very nearly equal to the true value of x in the said
 equation. Q. E. I.

Art. 81. Now it appears from SHERWIN'S Table of Sines and Tangents,
 that, if the radius of a circle be called 1, the number 0.7947 will be the sine of
 an angle of 52 degrees and 38 minutes, and the co-sine of an angle of 37 de-
 grees and 22 minutes. Therefore the rhumb-angle L A B, or the angle of the
 ship's course, of which x , or 0.7947, is the co-sine, will be an angle of 37 de-
 grees and 22 minutes. Q. E. I.

Art. 82. If the ship sailed in a direction due North-East, the rhumb angle
 L A B would be an angle of 45 degrees. But the angle L A B is only 37°
 22', which is less than 45° by 7°, 38'. Therefore the direction in which the
 ship sails will be to the North of a direction due North-East, and will make an
 angle of 7°, 38', with that direction. This angle of 7°, 38' is less than a
 whole point of the compass, (or a 32d part of the four right angles subtended
 by the whole circumference of a circle, or than $\frac{360^\circ}{32}$, or $11^\circ + \frac{8^\circ}{32}$, or $11^\circ +$
 $\frac{1^\circ}{4}$, or $11^\circ + \frac{60'}{4}$), or $11^\circ, 15'$, by an angle of 3°, 37'. Therefore the direc-
 tion in which the ship sails will be North-East by North, with 3°, 37' to the
 East, or North-East by North, with something less than a third part of ($11^\circ,$
 $15'$, or) a point of the compass towards the East. Q. E. I.

A Determination of the Length of the Arch AL, or the Latitude of the Second Point A.

Art. 83. We may now determine the latitude of the second point A, at
 which the ship arrives after passing along the loxodromick arch B A. This may
 be done in the manner following.

Since r is = 1, and d is = 0.045, and x is = 0.7947, we shall have $\frac{dx}{r}$
 ($= \frac{dx}{1} = dx = d \times 0.7947 = 0.045 \times 0.7947 = 0.035,761,5$, or (ne-
 glecting the last figure 5) = 0.035,761. Therefore $m + \frac{dx}{r}$ will be ($= m +$
 $0.035,761 = 0.708,031 + 0.035,761 = 0.743,792$. But the arch A L, or
 the latitude of the point A, is $= m + \frac{dx}{r}$. Therefore the said arch will be
 $= 0.743,792$, or 743,792 millionth parts of 1, or of the radius of the Earth;
 which may be expressed in degrees and minutes by proceeding as follows.

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The arch BM, or the latitude of the first point B, expressed in millionth parts of 1, or the radius of the Earth, will be to the arch AL, or the latitude of the second point A, expressed in the like parts of the radius of the Earth, in the same proportion as the said first arch BM, expressed in degrees and decimal parts of a degree, is to the said second arch AL, expressed also in degrees and decimal parts of a degree. But the arch BM is = 0.708,031, or 708,031 millionth parts of the radius of the Earth, and contains 40° , 45', or $40^{\circ} + \frac{45^{\circ}}{60}$, or $40^{\circ} + \frac{75^{\circ}}{100}$, or 40.75° . Therefore, if we take a fourth proportional to the three numbers 0.708,031, 0.743,792, and 40.75° , the said fourth proportional will be the value of the arch AL expressed in degrees and decimal parts of a degree. But the said fourth proportional will be ($= \frac{0.743,792}{0.708,031} \times 40.75^{\circ} = 1.050,507 \times 40.75^{\circ} = 42.808^{\circ}$), or $42^{\circ} + \frac{808^{\circ}}{1000}$. Therefore the length of the arch AL, expressed in degrees and decimal parts of a degree, will be $= 42^{\circ} + \frac{808^{\circ}}{1000}$ ($= 42^{\circ} + \frac{60 \times 808'}{1000} = 42^{\circ} + \frac{48,480'}{1000}$) $= 42^{\circ} + 48' + \frac{480'}{1000} = 42^{\circ} + 48' + \frac{60 \times 480''}{1000} = 42^{\circ} + 48' + \frac{28,800''}{1000} = 42^{\circ} + 48' + 28'' + \frac{800''}{1000} = 42^{\circ} + 48' + 28'' + \frac{60 \times 800''' }{1000} = 42^{\circ} + 48' + 28'' + \frac{48,000'''}{1000} = 42^{\circ} + 48' + 28'' + 48'''$, or, very nearly, $42^{\circ} + 48' + 29''$; that is, the latitude of the second point A, (at which the ship arrives after passing over the loxodromick arch BA,) when expressed in degrees and minutes, will be, very nearly, 42° , 48', 29'. Q. E. I.

A Proof of the Truth of the Conclusions that have been obtained in the preceding Articles, by the Application of the foregoing Solution of the Second Case of Dr. HALLEY'S Problem, to the Example given of it in Art. 66.

Art. 84. Having thus obtained the magnitude of the rhumb-angle L A B and the length of the arch AL, or the latitude of the second point A, (at which the ship arrives after passing over the loxodromick arch BA,) and having had the length of the arch BM, or the latitude of the first point B (from which the ship begins her voyage,) given us at first, we may now determine, by the help of Lemma 3d, Coroll. 1, the length of the equatorial arch LM, or c , or the difference of the longitudes of the two points A and B, (which was before supposed to be given and to be = 2° , 15', or 147.375 miles, or 147 miles and something more than a third part of a mile. And, if the value we shall obtain for the said arch LM, or the difference of the longitudes of the points

points A and B, by such determination, shall be found to be nearly equal to the magnitude of it that was given in the foregoing example, to wit, $2^{\circ}, 15'$, or 147.375 miles, the said result will be a confirmation both of the justness of the reasonings adopted in the foregoing solution, and of the exactness with which the several arithmetical operations, contained in the application of that solution to the preceding example, have been performed. This determination of the length of the said equatorial arch LM may be performed in the manner following.

It is shewn in Lemma 3d, Coroll. 1, that, if r be put for the radius of the Earth, and t be put for the tangent of the rhumb-angle LAB in the circle of which r , or the radius of the Earth, is the radius, and a be put for the logarithm of the ratio of the radius r to the tangent of the arch $\frac{PA}{2}$, taken on the axis, or asymptote, of a logarithmick curve, of which r , or the radius of the Earth, is the subtangent; and b be put for the logarithm of the ratio of the radius r to the tangent of the arch $\frac{PB}{2}$, taken on the axis, or asymptote, of the same logarithmick curve; and ϵ be put for the equatorial arch LM, or the difference of the longitudes of the points A and B; we shall have ϵ , or LM, $= \frac{\epsilon}{r} \times a - b$. We must therefore now compute this quantity $\frac{\epsilon}{r} \times a - b$ upon a supposition that the rhumb angle LAB is $= 37^{\circ}, 22'$, and that the arch BM is $= 40^{\circ}, 45'$, and that the arch AL is $= 42^{\circ}, 48', 29''$.

Now, since the angle LAB is $= 37^{\circ}, 22'$, its tangent t , in a circle of which r , or the radius of the Earth, is the radius, will be $= 0.763,636,2 \times r$, or (because r is supposed to be $= 1$.) will be $= (0.763,636,2 \times 1$, or) $0.763,636,2$. And consequently the quantity $\frac{\epsilon}{r} \times a - b$ will be $(= \frac{0.763,636,2}{1} \times a - b)$ $= 0.763,636,2 \times a - b$.

We must next find the values of the logarithms a and b ; which may be done as follows.

The arch AL, or the latitude of the point A, is $= 42^{\circ}, 48', 29''$. Therefore the arch PA, or the complement of the arch AL to the arch PL, or the arch of a quadrant, will be $(= 90^{\circ} - (42^{\circ}, 48', 29'')) = 47^{\circ}, 11', 31''$; and consequently the arch $\frac{PA}{2}$ will be $= 23^{\circ}, 35', 45'', 30'''$; the tangent of which, in a circle of which the radius is called 1, is $= 0.436,801$.

The arch BM, or the latitude of the point B, is $= 40^{\circ}, 45'$. Therefore the arch PB, or the complement of the arch BM to the arch PM, or the arch of a quadrant, will be $= 49^{\circ}, 15'$; and consequently the arch $\frac{PB}{2}$ will be $= 24^{\circ}, 37', 30''$; the tangent of which in a circle of which the radius is called 1, will be $0.458,363$.

Therefore

Therefore the ratio of the radius r to the tangent of the arch $\frac{PA}{2}$, will be equal to the ratio of 1 to 0.436,801; and the ratio of the radius r to the tangent of the arch $\frac{Pn}{2}$ will be equal to the ratio of 1 to 0.458,363. Therefore a will be the logarithm of the ratio of 1 to 0.436,801, in a logarithmick curve of which 1, or the radius of the Earth, is the subtangent; and b will be the logarithm of the ratio of 1 to 0.458,363, in the same logarithmick curve.

Further, the ratio of 1 to 0.436,801 is equal to the ratio of 2.289,372 to 1; and the ratio of 1 to 0.458,363 is equal to the ratio of 2.181,676 to 1. Therefore a will be the logarithm of the ratio of 2.289,372 to 1 in a logarithmick curve, of which 1, or the radius of the Earth, is the subtangent; and b will be the logarithm of the ratio of 2.181,676 to 1 in the same logarithmick curve.

Now, if the radius of the Earth, or the subtangent of the said logarithmick curve, were called 0.434,294,481,903,251,827, &c, and consequently the logarithm of the ratio of 10 to 1 in the same curve were called 1, (as it is in BAIGES's system of logarithms,) the logarithm of the ratio of 2.289,372 to 1 would be = 0.359,708,9, and the logarithm of the ratio of 2.181,676 to 1 would be = 0.338,788,4; as will appear from a table of BAIGES's, or the common, logarithms. Therefore, when the radius of the Earth, or the subtangent of the said logarithmick curve, is called 1, and consequently the logarithm of the ratio of 10 to 1, is called 2.302,585,092,994,045,684, &c, the logarithm of the ratio of 2.289,372 to 1 will be (= 2.302,585, &c $\times$ 0.359,708,9) = 0.828,260, and the logarithm of the ratio of 2.181,676 to 1 will be (= 2.302,585, &c $\times$ 0.338,788,4) = 0.780,089. Therefore a will be = 0.828,260, and b will be = 0.780,089; and consequently $a - b$ will be (= 0.828,260 - 0.780,089) = 0.048,171.

Therefore $\frac{1}{r} \times a - b$, or $0.763,636,2 \times a - b$, will be (= 0.763,636,2 $\times$ 0.048,171) = 0.036,785; that is, e , or the equatorial arch LM, or the difference of the longitudes of the points A and B, will be = 0.036,785, or 36,785 millionth parts of 1, or the radius of the Earth, or 36,785 millionth parts of 4000 miles, or (= $\frac{36,785}{1,000,000} \times 4000$ miles = $\frac{147,140,000}{1,000,000}$ miles = 147.14) = 147 miles + $\frac{14}{100}$ of a mile, or 147 miles and about a seventh part of a mile. Q. E. I.

Art. 85. This quantity, 147.14 miles, differs very little from 147.375 miles, or 2°, 15', which was the length of the equatorial arch LM that was given in the foregoing example. And therefore we may reasonably conclude, both that the reasonings used in the foregoing Solution of this second case of Dr. HALLEY's Problem are just, and that the numerous arithmetical operations, contained in the application of the said solution to the foregoing example, have been rightly performed.

Prepara-

Preparations for the Solution of a Third Case of Dr. HALLEY's Problem.

Art. 86. We have hitherto supposed the two points A and B, (from one of which the ship begins its voyage, and at the other of which it arrives at the end of its voyage,) to be both situated on the same side of the Equator. But it may happen that these points may be on different sides of the Equator, or that the ship may cross the line in its passage from one point to the other. This case therefore remains to be considered, in order to a complete solution of Dr. HALLEY's Problem in all its varieties. Now, in order to obtain the solution of this last case of the Problem, it will be necessary that we should previously give a solution of the following Lemma, or preliminary Problem.

PROPOSITION VI.

LEMMA IV. A PROBLEM.

Art. 87. Let the ship be supposed to sail from the point A (in Fig. 4.) which is situated on one side of the Equator HI, to the point D, which is situated on the other side of it, and in a different meridian from the point A. And let L be the point in which the meridian line PA, which passes through the point A, cuts the equatorial line HI; and N be the point in which the meridian line PD, which passes through the point D, cuts the same line HI: So that the arch AL of the meridian circle PAL shall be the latitude of the point A, and the arch DN of the meridian circle PND shall be the latitude of the point D. And let both the arches AL and DN, or the latitudes of both the points A and D, be supposed to be known; and likewise the rhumb-angle LAD, or the angle of the ship's course; and the length of the rhumb-line AD, or the distance run by the ship in her passage from A to D along the loxodromick curve AD, which we will suppose to cut the line HI, or the circumference of the Equator, in the point V. It is required to determine the length of the arch LN in the circumference of the Equator, or the difference of the longitudes of the points A and D.

SOLUTION.

It appears from Lemma II, and its Corollaries, that, if the ship had continued her course from the point A along the line AV only till she had cut the circumference of the Equator in the point V, the arch LV of the circumference
of

of the Equator would have been equal to the quantity $\frac{t}{r} \times \log. \frac{r}{\text{tangent of } \frac{PA}{2}}$

in a logarithmick curve of which the subtangent is equal to r , or the radius of the Earth. And, if we suppose the opposite pole of the Earth to be denoted by the small letter p , it will follow from the same Lemma and its Corollaries, that the arch VN of the circumference of the Equator will be equal to the quantity $\frac{t}{r} \times \log. \frac{r}{\text{tangent of } \frac{PD}{2}}$ in the same logarithmick curve. Therefore the

equatorial arch LN , which is the sum of the two arches LV and VN , will be equal to the quantity $\frac{t}{r} \times \log. \frac{r}{\text{tangent of } \frac{PA}{2}} + \frac{t}{r} \times \log. \frac{r}{\text{tangent of } \frac{PD}{2}}$, or

to the quantity $\frac{t}{r} \times \left\{ \log. \frac{r}{\text{tangent of } \frac{PA}{2}} + \log. \frac{r}{\text{tangent of } \frac{PD}{2}} \right\}$; that is, the arch

LN , or the difference of the longitudes of the points A and D , will be equal to the product of the multiplication of the fraction $\frac{t}{r}$ into the sum of the logarithm of the ratio of the radius r to the tangent of half the arch PA , or of half the complement of the latitude of the first point A , to the arch of a quadrant, and the logarithm of the ratio of the radius r to the tangent of half the arch PD , or of half the complement of the latitude of the second point D to the arch of a quadrant, in a logarithmick curve of which the subtangent is equal to r , or the radius of the Earth.

Q. E. D.

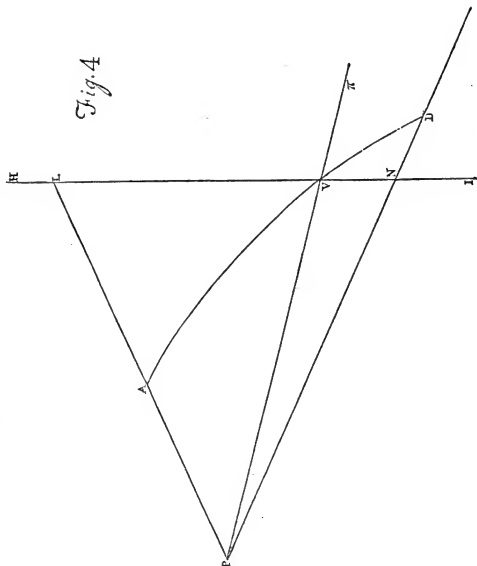
Coroll. 1. To shorten the foregoing expression of the value of the arch LN , let a be put for the logarithm of the ratio of the radius r to the tangent of half the arch PA , and let the small Greek letter ϵ be put for the logarithm of the ratio of the radius r to the tangent of half the arch PD , in a logarithmick curve, of which the subtangent is equal to r , or the radius of the Earth.

And we shall then have $LN = \frac{t}{r} \times a + \epsilon$; or the difference of the longitudes of the two points A and D will be equal to the product of the multiplication of the fraction $\frac{t}{r}$ into the sum of the two logarithms a and ϵ .

Coroll. 2. If LAD , or the angle of the ship's course, be 45 degrees, or half a right angle, the tangent t will be equal to the radius r , and consequently $\frac{t}{r} \times a + \epsilon$ will be $(= \frac{r}{r} \times a + \epsilon) = a + \epsilon$. Therefore in this case the arch LN , or the difference of the longitudes of the two points A and D , will be equal to $a + \epsilon$, or to the sum of the two logarithms a and ϵ .

Fig. 4.

Fig. 4



Art. 88. By the help of this 4th Lemma we may now proceed to solve the third and last case of Dr. HALLEY's Problem, which is as follows.

PROPOSITION VII; A PROBLEM;

BEING

A THIRD CASE OF DOCTOR HALLEY'S PROBLEM.

Let P (in Fig. 4.) be one of the poles of the Earth; and let the other pole, (which is not set down in the figure,) be supposed to be denoted by the small letter p ; let HI be a portion of the circumference of the Equator; A and D two points on the surface of the Earth, on different sides of the Equator HI, to wit, A on the side of the pole P, and D on the side of the opposite pole p , and at either equal or unequal distances from the Equator, but in different meridian circles PAL and PND, which cut the equatorial arch HI in the points L and N. Further, let a ship be supposed to sail from the point A to the point D along the rhumb-line, or loxodromick curve, AVD, which cuts all the meridian-lines, over which it passes, in the same angle, to wit, in an angle equal to LAD, or LAV, and which cuts the equatorial arch HI in the point V. And let the equatorial arch LN, or the difference of the longitudes of the two points A and D, and likewise the length of the rhumb-line, or loxodromick curve, AD, or AVD, (or the number of miles run over by the ship in her passage from A to D,) and the arch AL, or the latitude of the first point A, (from which the ship begins her voyage,) be supposed to be known; but not the arch DN, or the latitude of the second point D, (at which the ship arrives after passing over the loxodromick arch AD,) nor the angle LAD, or LAV, which the line of the ship's course makes with the several meridian-lines over which it passes. It is required to find the said angle LAD, or LAV.

THE SOLUTION.

Art. 89. In the foregoing Problem, or Lemma 4th, the two circular arches AL and DN, or the latitudes of the points A and D, were supposed to be known; as were likewise the rhumb-angle LAD, or LAV, and the length of the

the rhumb-line AD: and the length of the equatorial arch LN, or the difference of the longitudes of the two points A and D, was supposed to be unknown, and was derived from the said former quantities in the solution of that

Problem, and was shewn, in Coroll. 1 of the said Problem, to be $= \frac{t}{r} \times a + \epsilon$, the letters a and ϵ being put for the logarithms of the ratios of the radius r to the tangent of half the arch PA, or half the complement of the latitude of the first point A to the arch of a quadrant, or to an arch of 90 degrees, and to the tangent of half the arch pD, or half the complement of the latitude of the second point D to the arch of a quadrant, or to an arch of 90 degrees, in a logarithmick curve of which the subtangent is equal to r , or the radius of the Earth. But in the present Problem the latter of these two ratios, (which depends on the arch DN, or the latitude of the second point D,) and consequently its logarithm ϵ , is unknown. Nor can the arch DN, or the latitude of the second point D, be derived from the arch AL, or the latitude of the first point A, and the length of the rhumb-line AD, (which are both supposed to be known,) in the manner directed by the first Lemma; because in that Lemma the rhumb-angle LAB was supposed to be known, and consequently its tangent t (in the circle of which r is the radius,) was also known, as well as the length of the rhumb-line AB, and the arch AL, or the latitude of the first point A; whereas in the present Problem the rhumb-angle LAD is supposed to be unknown, and is the very quantity which we are required to

find. Therefore in the equation $LM = \frac{t}{r} \times a + \epsilon$ (obtained in the first Corollary of Lemma 4th,) there will be two unknown quantities, to wit, the tangent t of the unknown angle LAD, and the logarithm ϵ of the ratio of r to the tangent of half the arch pD. We cannot therefore consider the equation $LM = \frac{t}{r} \times a + \epsilon$ as a simple equation containing only one unknown quantity t (in which case we should have $r \times LM = t \times a + \epsilon$, and consequently $t = \frac{r \times LM}{a + \epsilon}$;) but must investigate the relation between the two unknown quantities t and ϵ , (which are mutually dependant on each other,) and must reduce them to one unknown quantity, before we can discover the value of t , or the tangent of the unknown angle LAD. And this investigation is a matter of great nicety and difficulty. as we have found in the solutions of the two former cases of Dr. HALLEY's Problem.

Art. 90. Let x be put for the co-sine of the rhumb-angle LAD, of which t is the tangent, in the circle of which r , or the radius of the Earth, is the radius. Then will the square of the said co-sine be xx , and consequently the square of the sine of the same angle LAD will be $rr - xx$, and the said sine itself will be $= \sqrt{rr - xx}$. But the co-sine of any arc is to its sine as the radius is to the tangent. Therefore x will be to $\sqrt{rr - xx}$ as r
M 2 is

is to t ; and consequently t will be $= \frac{r \times \sqrt{rr - xx}}{x}$. Therefore $\frac{t}{r}$ will be $(= \frac{r \times \sqrt{rr - xx}}{r \times x}) = \frac{\sqrt{rr - xx}}{x}$, and $\frac{t}{r} \times \overline{a + \epsilon}$ will be $= \frac{\sqrt{rr - xx}}{x} \times \overline{a + \epsilon}$. But it has been shewn above in art. 87, Lemma IV, Coroll. 1, that the equatorial arch LN is $= \frac{t}{r} \times \overline{a + \epsilon}$. Therefore the said arch LN, or the difference of the longitudes of the points A and D, will be equal to $\frac{\sqrt{rr - xx}}{x} \times \overline{a + \epsilon}$.

Let ϵ be put for the arch LN, or the difference of the longitudes of the points A and D; which in the present Problem is supposed to be known. And we shall then have $\epsilon = \frac{\sqrt{rr - xx}}{x} \times \overline{a + \epsilon}$; in which equation the quantity ϵ is unknown, as well as the quantity x , but is dependant on it, and may be derived from it, though not without great difficulty.

ART. 91. Let d be put for the length of the loxodromick arch AVD, or the distance run by the ship in her passage from A to D; and l for the length of the arch AL, or the latitude of the first point A.

It is shewn above in Lemma 1, art. 7, page 10, that the loxodromick arch AB, (in fig. 2,) is to the circular arch AQ, (or to the difference of the arches AL and BM, or of the latitudes of the points A and B,) as the radius of a circle is to the co-sine of the rhumb-angle LAB. Therefore, when the arch BM is equal to 0, or the point B co-incides with the point M, or is situated in the Equator, and the arch AQ becomes equal to the whole arch AL, or to the whole latitude of the point A, the loxodromick arch AB will be to the whole arch AL, or to the whole latitude of the point A, as the radius of a circle is to the co-sine of the rhumb-angle LAB. Therefore, in fig. 4, the loxodromick arch AV will be to the whole arch AL, or to the whole latitude of the point A, as the radius of a circle is to the co-sine of the rhumb-angle LAV, or as r is to x ; and, in like manner, the loxodromick arch DV will be to the whole arch DN, or to the whole latitude of the point D, in the proportion of the radius of a circle to the co-sine of the rhumb-angle DV π made by the loxodromick arch, or rhumb-line, VD, with the meridian line PV continued beyond the point V (where it cuts the equatorial arch HI,) to the point π , or, (because the said rhumb-angle DV π is equal to the rhumb-angle LAV, the direction of the ship's course being supposed to cut all the meridian lines, over which it passes, in the same angle,) in the proportion of the radius of a circle to the co-sine of the rhumb-angle LAV, or in the proportion of r to x . We shall therefore have the loxodromick arch VD to the circular arch DN in the same proportion as the loxodromick arch AV to the circular arch AL. Therefore (by EUCLID'S Elements, book 5, prop. 12,) the sum of the two loxodromick arches AV and VD will be to the sum of the two circular

circular arches A L and D N in the same proportion as the loxodromick arch A V is to the circular arch A L, or as r is to x ; that is, the whole loxodromick arch, or rhumb line, A V D will be to A L + D N (the sum of the latitudes of the two points A and D,) as r is to x ; or, (because d is equal to the loxodromick arch A V D, and l is equal to the arch A L,) d will be to $l + D N$ as r is to x . Therefore dx will be $= (l + D N) \times r$, and $l + D N$ will be $= \frac{dx}{r}$, and D N will be $= \frac{dx}{r} - l$.

Art. 92. Now let q be put for the length of the arch $p D N$, or the arch of a whole quadrant of the meridian circle which passes through the point D, and of which r , or the radius of the Earth, is the radius. Then will $p D$, or $p D N - D N$, be $= q - D N$; or (because $D N$ is $= \frac{dx}{r} - l$), $p D$ will be $(= q - (\frac{dx}{r} - l)) = q - \frac{dx}{r} + l$, and consequently $\frac{p D}{2}$ will be $(= \frac{q - \frac{dx}{r} + l}{2}) = \frac{q}{2} - \frac{dx}{2r} + \frac{l}{2}$. Therefore ϵ , or the logarithm of the ratio of the radius r to the tangent of the arch $\frac{p D}{2}$, will be equal to the logarithm of the ratio of the radius r to the tangent of the arch $\frac{q}{2} - \frac{dx}{2r} + \frac{l}{2}$.

Art. 93. Further, because in every circle the radius is a mean proportional between the tangent of any arch less than 90 degrees, or the arch of a quadrant, and the co-tangent of the same arch, or the tangent of its complement to the arch of a quadrant, it follows that the ratio of the radius r to the tangent of the arch $\frac{q}{2} - \frac{dx}{2r} + \frac{l}{2}$ will be equal to the ratio of the tangent of the arch which is the complement of the arch $\frac{q}{2} - \frac{dx}{2r} + \frac{l}{2}$ to the arch of a quadrant, or to q , to the radius r . But the complement of the arch $\frac{q}{2} - \frac{dx}{2r} + \frac{l}{2}$ to the arch q is $(= q - (\frac{q}{2} - \frac{dx}{2r} + \frac{l}{2})) = q - \frac{q}{2} + \frac{dx}{2r} - \frac{l}{2} = \frac{q}{2} + \frac{dx}{2r} - \frac{l}{2}$. Therefore the ratio of the tangent of the arch $\frac{q}{2} + \frac{dx}{2r} - \frac{l}{2}$ to the radius r will be equal to the ratio of the radius r to the tangent of the arch $\frac{q}{2} - \frac{dx}{2r} + \frac{l}{2}$; and consequently the logarithm of the former ratio will be equal to the logarithm of the latter ratio. But the logarithm of the latter ratio is equal to ϵ . Therefore the logarithm of the former ratio will also be equal to ϵ ; or ϵ will be equal to the logarithm of the ratio of the tangent

gent of the arch $\frac{q}{2} + \frac{dx}{2r} - \frac{1}{2}$ to the radius r . Therefore $a + e$ will be $= a + \log.$ of tangent of $\frac{\frac{q}{2} + \frac{dx}{2r} - \frac{1}{2}}{r}$. We must therefore now endeavour to find an expression for the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{dx}{2r} - \frac{1}{2}$ to the radius r , that shall consist of a set of simple terms involving either known quantities, or the powers of the unknown quantity x combined with known quantities. This may be done in the manner following.

Art. 94. The arch pD is less than the arch pN , or than the arch of a whole quadrant of a circle. And consequently the arch $\frac{pD}{2}$ must be less than an arch of 45 degrees, or half the arch of a quadrant. Therefore the complement of the arch $\frac{pD}{2}$ to an arch of 90 degrees, must be greater than an arch of 45 degrees. But it has been shewn (in art. 92,) that the arch $\frac{pD}{2}$ is equal to $\frac{q}{2} - \frac{dx}{2r} + \frac{1}{2}$, the complement of which to the arch q , or an arch of 90 degrees, is $\frac{q}{2} + \frac{dx}{2r} - \frac{1}{2}$. Therefore $\frac{q}{2} + \frac{dx}{2r} - \frac{1}{2}$ is also the complement of the arch $\frac{pD}{2}$ to an arch of 90 degrees. Therefore the arch $\frac{q}{2} + \frac{dx}{2r} - \frac{1}{2}$ must be greater than an arch of 45 degrees.

But, when an arch is greater than 45 degrees, but less than 90 degrees, or the arch of a quadrant, Mr. JAMES GREGORY has given us a series that will express the logarithm of the ratio of its tangent to the radius of the circle, (which logarithm he calls its *artificial, or logarithmick, tangent,*) in terms involving the powers of the excess of twice the said arch above the arch of a quadrant, and the powers of the radius of the circle; which series is as follows. If the radius of the circle be called R , and an arch of the circle that is less than an arch of 90 degrees, but greater than an arch of 45 degrees, be called A , and an arch of 90 degrees in the said circle be called Q , and an arch that is equal to $2A - Q$ be called E , the logarithm of the ratio of the tangent of the arch A to the radius R , in a logarithmick curve of which the subtangent is equal to the radius R , will be equal to the infinite series $E + \frac{E^2}{6R} + \frac{E^3}{24R^2} + \frac{61E^4}{5040R^3} + \frac{277E^5}{72576R^4} + \&c.$ Therefore, if, in the circle of which r , or the radius of the Earth, is the radius, and in which q is put for the arch of a quadrant, the arch $\frac{q}{2} + \frac{dx}{2r} - \frac{1}{2}$ (which has been shewn to be greater

greater than an arch of 45 degrees in the said circle, but at the same time is less than the arch of a whole quadrant, being the complement of the arch $\frac{q}{2}$ to q , or the arch of a quadrant,) be doubled, and from the double of it, which will be $q + \frac{dx}{r} = l$, we subtract the quadrantal arch q , and substitute the remaining arch $\frac{dx}{r} = l$ instead of E , and r instead of R , in the terms of the foregoing series $x + \frac{x^3}{6rr} + \frac{x^5}{24r^3} + \frac{61x^7}{5040r^5} + \frac{2771x^9}{72576r^7} + \&c$, *ad infinitum*, the complicated series thence arising, to wit, the series $\frac{dx}{r} = l + \frac{\left(\frac{dx}{r} - l\right)^3}{6rr} + \frac{\left(\frac{dx}{r} - l\right)^5}{24r^3} + \frac{61 \times \left(\frac{dx}{r} - l\right)^7}{5040r^5} + \frac{277 \times \left(\frac{dx}{r} - l\right)^9}{72576r^7} + \&c$, *ad infinitum*, will be equal to the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{dx}{2r} - \frac{l}{2}$ to the radius r , in a logarithmick curve of which the subtangent is equal to the radius r .

But it has been shewn in the preceeding art. 93, that the logarithm of this ratio is equal to ζ .

Therefore ζ will be equal to the said infinite series $\frac{dx}{r} = l + \frac{\left(\frac{dx}{r} - l\right)^3}{6rr} + \frac{\left(\frac{dx}{r} - l\right)^5}{24r^3} + \frac{61 \times \left(\frac{dx}{r} - l\right)^7}{5040r^5} + \frac{277 \times \left(\frac{dx}{r} - l\right)^9}{72576r^7} + \&c$. And consequently the equatorial arch LN , or c , (which is equal to $\frac{\sqrt{rr - xx}}{x} \times a + \zeta$,) will be $= \frac{\sqrt{rr - xx}}{x} \times$ the sum of the known logarithm a and the said complicated infinite series $\frac{dx}{r} = l + \frac{\left(\frac{dx}{r} - l\right)^3}{6rr} + \frac{\left(\frac{dx}{r} - l\right)^5}{24r^3} + \frac{61 \times \left(\frac{dx}{r} - l\right)^7}{5040r^5} + \frac{277 \times \left(\frac{dx}{r} - l\right)^9}{72576r^7} + \&c$. This equation must now be reduced into a proper form for a resolution of it; which may be done in the manner following.

Art. 95. The second term, $\frac{\left(\frac{dx}{r} - l\right)^3}{6rr}$, of the foregoing series is $=$

$$\frac{dx^3}{r^3}$$

$$\frac{d^3x^3}{r^3} - \frac{3d^2x^2l}{r^3} + \frac{3dx^2l^2}{r} - l^3 = \frac{d^3x^3}{r^3} - \frac{3rd^2x^2l}{r^3} + \frac{3r^2dx^2l^2}{r^3} - \frac{r^3l^3}{r^3} =$$

$$\frac{d^3x^3 - 3rd^2x^2l + 3r^2l^2dx - r^3l^3}{6rr} =$$

And the third term, $\frac{\left(\frac{dx}{r} - l\right)^3}{34r^4}$ is = $\frac{d^3x^3}{r^3} - \frac{3d^2x^2l}{r^3} + \frac{3rd^2x^2l^2}{r^3} - \frac{3rd^2x^2l^3}{r^3} + \frac{3dx^2l^3}{r} - l^3$

$$= \frac{\frac{d^3x^3}{r^3} - \frac{3rd^2x^2l}{r^3} + \frac{3rd^2l^2dx^2}{r^3} - \frac{3rd^2l^3dx^2}{r^3} + \frac{3r^2l^3dx}{r^3} - \frac{r^3l^3}{r^3}}{24r^4} =$$

$$\frac{d^3x^3 - 3rd^2x^2l + 3r^2l^2dx - r^3l^3}{24r^4}$$

And the fourth term, $\frac{61 \times \left(\frac{dx}{r} - l\right)^7}{5040r^6}$ is = $\frac{61}{5040r^6} \times \left[\frac{dx}{r} - l\right]^7 = \frac{61}{5040r^6} \times$ the compound quantity $\frac{d^7x^7}{r^7} - \frac{7d^6x^6l}{r^6} + \frac{21d^5x^5l^2}{r^5} - \frac{35d^4x^4l^3}{r^4} + \frac{21d^3x^3l^4}{r^3} - \frac{7d^2x^2l^5}{r^2} + \frac{d^2x^2l^5}{r^2}$

$$+ \frac{7dx^2l^6}{r} - l^7 = \frac{61}{5040r^6} \times \text{the compound quantity } \frac{d^7x^7}{r^7} - \frac{7rd^6x^6l}{r^6} + \frac{21r^2d^5x^5l^2}{r^5} - \frac{35r^3d^4x^4l^3}{r^4} + \frac{21r^4d^3x^3l^4}{r^3} - \frac{7r^5d^2x^2l^5}{r^2} + \frac{21r^6dx^2l^6}{r} - \frac{r^7l^7}{r^7}$$

$$= \frac{61d^7x^7 - 427rd^6x^6l + 1281r^2d^5x^5l^2 - 2135r^3d^4x^4l^3 + 2135r^4d^3x^3l^4 - 1281r^5d^2x^2l^5 + 427r^6dx^2l^6 - 61r^7l^7}{5040r^{12}}$$

And the fifth term, $\frac{277 \times \left(\frac{dx}{r} - l\right)^9}{72576r^8}$ is = $\frac{277}{72576r^8} \times$ the compound quantity $\frac{d^9x^9}{r^9} - \frac{9d^8x^8l}{r^8} + \frac{36d^7x^7l^2}{r^7} - \frac{84d^6x^6l^3}{r^6} + \frac{126d^5x^5l^4}{r^5} - \frac{126d^4x^4l^5}{r^4} + \frac{84d^3x^3l^6}{r^3} - \frac{36d^2x^2l^7}{r^2} + \frac{9dx^2l^8}{r} - l^9$

$$= \frac{277}{72576r^8} \times \text{the compound quantity } \frac{d^9x^9}{r^9} - \frac{9rd^8x^8l}{r^8} + \frac{36r^2d^7x^7l^2}{r^7} - \frac{84r^3d^6x^6l^3}{r^6} + \frac{126r^4d^5x^5l^4}{r^5} - \frac{126r^5d^4x^4l^5}{r^4} + \frac{84r^6d^3x^3l^6}{r^3} - \frac{36r^7d^2x^2l^7}{r^2} + \frac{9r^8dx^2l^8}{r} - \frac{r^9l^9}{r^9}$$

$$= \frac{277d^9x^9 - 2493rd^8x^8l + 9972r^2d^7x^7l^2 - 23268r^3d^6x^6l^3 + 34902r^4d^5x^5l^4 - 34902r^5d^4x^4l^5 + 23268r^6d^3x^3l^6 - 2493r^7d^2x^2l^7 + 277r^8dx^2l^8 - 277r^9l^9}{72576r^{16}}$$

Therefore the said complicated series $\frac{dx}{r} - l + \frac{\left(\frac{dx}{r} - l\right)^3}{6rr} + \frac{\left(\frac{dx}{r} - l\right)^5}{34r^4} + \frac{61 \times \left(\frac{dx}{r} - l\right)^7}{5040r^6} + \frac{277 \times \left(\frac{dx}{r} - l\right)^9}{72576r^8} + \&c$ *ad infinitum* will be equal to the series $\frac{dx}{r}$

$$\begin{aligned}
 & \frac{dx}{r} = l + \frac{d^1x^1 - 3rd^2x^2 + 3r^2d^3x^3 - r^3d^4}{(1)^3} \\
 & + \frac{d^2x^2 - 5rd^3x^3 + 10r^2d^4x^4 - 10r^3d^5x^5 + 5r^4d^6x^6 - r^5d^7}{24r^9} \\
 & + \frac{61d^7x^7 - 427rd^8x^8 + 1281r^2d^9x^9 - 2135r^3d^{10}x^{10} + 2135r^4d^{11}x^{11} - 1281r^5d^{12}x^{12} +}{427r^6d^{13}x^{13} - 61r^7d^{14}} \\
 & + \frac{277d^{15}x^{15} - 2403rd^{16}x^{16} + 9972r^2d^{17}x^{17} - 23268r^3d^{18}x^{18} + 34902r^4d^{19}x^{19} - 34902r^5d^{20}x^{20} +}{5040r^{21}} \\
 & + 23268r^6d^{21}x^{21} - 9972r^7d^{22}x^{22} + 2403r^8d^{23}x^{23} - 277r^9d^{24}x^{24} \\
 & + \&c., \text{ ad infinitum,} = \\
 & - \text{ the series } l + \frac{r^{3/2}}{6r^3} + \frac{r^{5/2}}{24r^9} + \frac{61r^{7/2}}{5040r^{15}} + \frac{277r^{9/2}}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & + \text{ the series } \frac{dx}{r} + \frac{3r^{1/2}dx}{6r^3} + \frac{5r^{3/2}dx}{24r^9} + \frac{61r^{5/2}dx}{5040r^{15}} + \frac{427r^{7/2}dx}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & - \text{ the series } \frac{3rd^2x^2}{6r^3} + \frac{10r^2d^3x^3}{24r^9} + \frac{1281r^3d^4x^4}{5040r^{15}} + \frac{9972r^4d^5x^5}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & + \text{ the series } \frac{d^2x^2}{6r^3} + \frac{10r^2d^3x^3}{24r^9} + \frac{2135r^4d^4x^4}{5040r^{15}} + \frac{23,268r^6d^5x^5}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & - \text{ the series } \frac{5rd^4x^4}{24r^9} + \frac{2135r^3d^5x^5}{5040r^{15}} + \frac{34,902r^4d^6x^6}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & + \text{ the series } \frac{d^3x^3}{24r^9} + \frac{1281r^2d^4x^4}{5040r^{15}} + \frac{34,902r^4d^5x^5}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & - \text{ the series } \frac{427rd^6x^6}{5040r^{15}} + \frac{23,268r^3d^7x^7}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & + \text{ the series } \frac{61d^7x^7}{5040r^{15}} + \frac{9972r^2d^8x^8}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & - \text{ the series } \frac{2403rd^9x^9}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & + \text{ the series } \frac{277d^{10}x^{10}}{72,576r^{21}} + \&c. \text{ ad infinitum,} - \&c. = \\
 & - \text{ the series } l + \frac{r^3}{6r^3} + \frac{r^5}{24r^9} + \frac{61r^7}{5040r^{15}} + \frac{277r^9}{72,576r^{21}} + \&c. \text{ ad infinitum,} \\
 & + \text{ the series } \frac{dx}{r} + \frac{r^2dx}{2r^3} + \frac{5r^4dx}{24r^9} + \frac{61r^6dx}{720r^{15}} + \frac{277r^8dx}{8064r^{21}} + \&c. \text{ ad infinitum,} \\
 & - \text{ the series } \frac{ld^2x^2}{2r^4} + \frac{5r^2d^3x^3}{12r^9} + \frac{61r^4d^4x^4}{240r^{15}} + \frac{277r^6d^5x^5}{2016r^{21}} + \&c. \text{ ad infinitum,} \\
 & + \text{ the series } \frac{d^2x^2}{6r^3} + \frac{5r^2d^3x^3}{12r^9} + \frac{61r^4d^4x^4}{144r^{15}} + \frac{277r^6d^5x^5}{864r^{21}} + \&c. \text{ ad infinitum,} \\
 & - \text{ the series } \frac{5ld^4x^4}{24r^9} + \frac{61r^2d^5x^5}{144r^{15}} + \frac{277r^4d^6x^6}{576r^{21}} + \&c. \text{ ad infinitum,} \\
 & + \text{ the series } \frac{d^3x^3}{24r^9} + \frac{61r^2d^4x^4}{240r^{15}} + \frac{277r^4d^5x^5}{576r^{21}} + \&c. \text{ ad infinitum,} \\
 & - \text{ the series } \frac{61ld^6x^6}{720r^{15}} + \frac{277r^2d^7x^7}{864r^{21}} + \&c. \text{ ad infinitum,}
 \end{aligned}$$

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- + the series $\frac{61d^2x^2}{5040r^{15}} + \frac{277^2d^2x^2}{2016r^{15}} + \&c$ *ad infinitum*,
 - the series $\frac{277ld^2x^2}{8064r^{15}} + \&c$ *ad infinitum*,
 + the series $\frac{277d^2x^2}{72,576r^{15}} + \&c$ *ad infinitum*,
 - $\&c$ *ad infinitum*.

Art. 96. Now let the capital letter B be put = the infinite series $l + \frac{r}{6rr}$
 + $\frac{r}{24r^4} + \frac{61l^2}{5040r^8} + \frac{277l^2}{72,576r^8} + \&c$; and let the capital letter C be put = the
 infinite series $d + \frac{r^2}{2r^3} + \frac{5l^2d}{24r^4} + \frac{61l^2d}{720r^8} + \frac{277l^2d}{8064r^8} + \&c$; and let D be = the
 infinite series $\frac{ld^2}{2r^3} + \frac{5l^2ld^2}{12r^4} + \frac{61l^2ld^2}{240r^8} + \frac{277l^2ld^2}{2016r^8} + \&c$; and E be = the infinite
 series $\frac{d^2}{6r^3} + \frac{5l^2d^2}{12r^4} + \frac{61l^2d^2}{144r^8} + \frac{277l^2d^2}{864r^8} + \&c$; and F be = the infinite series
 $\frac{5l^2d^2}{24r^4} + \frac{61l^2ld^2}{114r^8} + \frac{277l^2ld^2}{576r^8} + \&c$; and G be = the infinite series $\frac{d^3}{24r^4} + \frac{61l^2ld^3}{240r^8}$
 + $\frac{277l^2ld^3}{576r^8} + \&c$; and H be = the infinite series $\frac{61ld^3}{720r^8} + \frac{277ld^3}{8064r^8} + \&c$; and
 K be = the infinite series $\frac{61d^3}{5040r^8} + \frac{277l^2d^3}{2016r^8} + \&c$; and L be = the infinite
 series $\frac{277ld^3}{8064r^8} + \&c$; and M be = the infinite series $\frac{277d^3}{72,576r^8} + \&c$.

And the foregoing complicated series $\frac{dx}{r} - l + \frac{\left(\frac{dx}{r} - l\right)^2}{6rr} + \frac{\left(\frac{dx}{r} - l\right)^3}{24r^4} +$
 $\frac{61 \times \left(\frac{dx}{r} - l\right)^4}{5040r^8} + \frac{277 \times \left(\frac{dx}{r} - l\right)^5}{72,576r^8} + \&c$ *ad infinitum* will be equal to the series
 - B + $\frac{C}{r} - \frac{Dx^2}{r^3} + \frac{Ex^2}{r^3} - \frac{Gx^4}{r^5} + \frac{Hx^4}{r^5} - \frac{Ix^4}{r^5} + \frac{Kx^2}{r^3} - \frac{Lx^3}{r^3} + \frac{Mx^3}{r^3} - \&c$ *ad*
infinitum.

Art. 97. But, by art. 94, the complicated series $\frac{dx}{r} - l + \frac{\left(\frac{dx}{r} - l\right)^2}{6rr} +$
 $\frac{\left(\frac{dx}{r} - l\right)^3}{24r^4} + \frac{61 \times \left(\frac{dx}{r} - l\right)^4}{5040r^8} + \frac{277 \times \left(\frac{dx}{r} - l\right)^5}{72,576r^8} + \&c$ *ad infinitum* is equal to C, or
 the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{dx}{2r} - \frac{l}{2}$ to the
 radius r in a logarithmick curve of which the subtangent is equal to the radius r .
 Therefore

Therefore the series $-B + \frac{cx}{r} - \frac{dx^3}{r^3} + \frac{fx^5}{r^5} - \frac{gx^7}{r^7} + \frac{hx^9}{r^9} - \frac{ix^{11}}{r^{11}} + \frac{kx^{13}}{r^{13}} - \frac{lx^{15}}{r^{15}} + \frac{mx^{17}}{r^{17}} - \&c$ *ad infinitum* will also be equal to the said logarithm, or to ϕ . Therefore $a + \phi$ will be equal to the sum of a and the said series $-B + \frac{cx}{r} - \frac{dx^3}{r^3} + \frac{fx^5}{r^5} - \frac{gx^7}{r^7} + \frac{hx^9}{r^9} - \frac{ix^{11}}{r^{11}} + \frac{kx^{13}}{r^{13}} - \frac{lx^{15}}{r^{15}} + \frac{mx^{17}}{r^{17}} - \&c$ *ad infinitum*, or will be equal to the series $a - B + \frac{cx}{r} - \frac{dx^3}{r^3} + \frac{fx^5}{r^5} - \frac{gx^7}{r^7} + \frac{hx^9}{r^9} - \frac{ix^{11}}{r^{11}} + \frac{kx^{13}}{r^{13}} - \frac{lx^{15}}{r^{15}} + \frac{mx^{17}}{r^{17}} - \&c$ *ad infinitum*.

Art. 98. But, because l is the excess of the arch $q + l$, or twice the arch $\frac{q}{2} + \frac{l}{2}$, above q , or the arch of a quadrant, it follows from art. 94, that $l + \frac{l}{6rr} + \frac{l^3}{24r^3} + \frac{61l^5}{5040r^5} + \frac{277l^7}{72,576r^7} + \&c$ *ad infinitum* will be equal to the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{l}{2}$ to the radius r in a logarithmick curve of which the subtangent is equal to r . Therefore the quantity B , (which is equal to the said series $l + \frac{l}{6rr} + \frac{l^3}{24r^3} + \frac{61l^5}{5040r^5} + \frac{277l^7}{72,576r^7} + \&c$ *ad infinitum*,) will be equal to the logarithm of the said ratio of the tangent of the arch $\frac{q}{2} + \frac{l}{2}$ to the radius r in the said logarithmick curve.

But the ratio of the tangent of the arch $\frac{q}{2} + \frac{l}{2}$, or $\frac{q+l}{2}$, to the radius r is equal to the ratio of the radius r to the tangent of the arch $\frac{q-l}{2}$, which is the complement of the arch $\frac{q+l}{2}$ to q , or the arch of a quadrant; and consequently the logarithm of the former ratio in a logarithmick curve of which r is the subtangent is equal to the logarithm of the latter ratio in the same logarithmick curve.

Therefore B (which has just now been shewn to be equal to the logarithm of the ratio of the tangent of the arch $\frac{q}{2} + \frac{l}{2}$, or $\frac{q+l}{2}$, to the radius r in the logarithmick curve of which r is the subtangent,) will also be equal to the logarithm of the ratio of the radius r to the tangent of the arch $\frac{q-l}{2}$ in the same logarithmick curve.

But this arch $\frac{r-L}{2}$ is equal to $\frac{rL-AL}{2}$, or $\frac{rA}{2}$.

Therefore B will be equal to the logarithm of the ratio of the radius r to the tangent of the arch $\frac{rA}{2}$ in a logarithmick curve of which r is the subtangent.

But a is equal to this logarithm, as appears from Coroll. 1 to Lemma 3d.

Therefore B will be equal to a . And consequently the series $a - B + \frac{cx}{r} - \frac{dx^3}{r^3} + \frac{fx^5}{r^5} - \frac{gx^7}{r^7} + \frac{hx^9}{r^9} - \frac{ix^{11}}{r^{11}} + \frac{kx^{13}}{r^{13}} - \frac{lx^{15}}{r^{15}} + \frac{mx^{17}}{r^{17}} - \frac{nx^{19}}{r^{19}} + \frac{ox^{21}}{r^{21}} - \frac{px^{23}}{r^{23}} + \frac{qx^{25}}{r^{25}} - \frac{rx^{27}}{r^{27}} + \frac{sx^{29}}{r^{29}} - \frac{tx^{31}}{r^{31}} + \frac{ux^{33}}{r^{33}} - \frac{vx^{35}}{r^{35}} + \frac{wx^{37}}{r^{37}} - \frac{yx^{39}}{r^{39}} + \frac{zx^{41}}{r^{41}} - \frac{ax^{43}}{r^{43}} + \frac{bx^{45}}{r^{45}} - \frac{cx^{47}}{r^{47}} + \frac{dx^{49}}{r^{49}} - \frac{ex^{51}}{r^{51}} + \frac{fx^{53}}{r^{53}} - \frac{gx^{55}}{r^{55}} + \frac{hx^{57}}{r^{57}} - \frac{ix^{59}}{r^{59}} + \frac{kx^{61}}{r^{61}} - \frac{lx^{63}}{r^{63}} + \frac{mx^{65}}{r^{65}} - \frac{nx^{67}}{r^{67}} + \frac{ox^{69}}{r^{69}} - \frac{px^{71}}{r^{71}} + \frac{qx^{73}}{r^{73}} - \frac{rx^{75}}{r^{75}} + \frac{sx^{77}}{r^{77}} - \frac{tx^{79}}{r^{79}} + \frac{ux^{81}}{r^{81}} - \frac{vx^{83}}{r^{83}} + \frac{wx^{85}}{r^{85}} - \frac{yx^{87}}{r^{87}} + \frac{zx^{89}}{r^{89}} - \frac{ax^{91}}{r^{91}} + \frac{bx^{93}}{r^{93}} - \frac{cx^{95}}{r^{95}} + \frac{dx^{97}}{r^{97}} - \frac{ex^{99}}{r^{99}} + \frac{fx^{101}}{r^{101}} - \frac{gx^{103}}{r^{103}} + \frac{hx^{105}}{r^{105}} - \frac{ix^{107}}{r^{107}} + \frac{kx^{109}}{r^{109}} - \frac{lx^{111}}{r^{111}} + \frac{mx^{113}}{r^{113}} - \frac{nx^{115}}{r^{115}} + \frac{ox^{117}}{r^{117}} - \frac{px^{119}}{r^{119}} + \frac{qx^{121}}{r^{121}} - \frac{rx^{123}}{r^{123}} + \frac{sx^{125}}{r^{125}} - \frac{tx^{127}}{r^{127}} + \frac{ux^{129}}{r^{129}} - \frac{vx^{131}}{r^{131}} + \frac{wx^{133}}{r^{133}} - \frac{yx^{135}}{r^{135}} + \frac{zx^{137}}{r^{137}} - \frac{ax^{139}}{r^{139}} + \frac{bx^{141}}{r^{141}} - \frac{cx^{143}}{r^{143}} + \frac{dx^{145}}{r^{145}} - \frac{ex^{147}}{r^{147}} + \frac{fx^{149}}{r^{149}} - \frac{gx^{151}}{r^{151}} + \frac{hx^{153}}{r^{153}} - \frac{ix^{155}}{r^{155}} + \frac{kx^{157}}{r^{157}} - \frac{lx^{159}}{r^{159}} + \frac{mx^{161}}{r^{161}} - \frac{nx^{163}}{r^{163}} + \frac{ox^{165}}{r^{165}} - \frac{px^{167}}{r^{167}} + \frac{qx^{169}}{r^{169}} - \frac{rx^{171}}{r^{171}} + \frac{sx^{173}}{r^{173}} - \frac{tx^{175}}{r^{175}} + \frac{ux^{177}}{r^{177}} - \frac{vx^{179}}{r^{179}} + \frac{wx^{181}}{r^{181}} - \frac{yx^{183}}{r^{183}} + \frac{zx^{185}}{r^{185}} - \frac{ax^{187}}{r^{187}} + \frac{bx^{189}}{r^{189}} - \frac{cx^{191}}{r^{191}} + \frac{dx^{193}}{r^{193}} - \frac{ex^{195}}{r^{195}} + \frac{fx^{197}}{r^{197}} - \frac{gx^{199}}{r^{199}} + \frac{hx^{201}}{r^{201}} - \frac{ix^{203}}{r^{203}} + \frac{kx^{205}}{r^{205}} - \frac{lx^{207}}{r^{207}} + \frac{mx^{209}}{r^{209}} - \frac{nx^{211}}{r^{211}} + \frac{ox^{213}}{r^{213}} - \frac{px^{215}}{r^{215}} + \frac{qx^{217}}{r^{217}} - \frac{rx^{219}}{r^{219}} + \frac{sx^{221}}{r^{221}} - \frac{tx^{223}}{r^{223}} + \frac{ux^{225}}{r^{225}} - \frac{vx^{227}}{r^{227}} + \frac{wx^{229}}{r^{229}} - \frac{yx^{231}}{r^{231}} + \frac{zx^{233}}{r^{233}} - \frac{ax^{235}}{r^{235}} + \frac{bx^{237}}{r^{237}} - \frac{cx^{239}}{r^{239}} + \frac{dx^{241}}{r^{241}} - \frac{ex^{243}}{r^{243}} + \frac{fx^{245}}{r^{245}} - \frac{gx^{247}}{r^{247}} + \frac{hx^{249}}{r^{249}} - \frac{ix^{251}}{r^{251}} + \frac{kx^{253}}{r^{253}} - \frac{lx^{255}}{r^{255}} + \frac{mx^{257}}{r^{257}} - \frac{nx^{259}}{r^{259}} + \frac{ox^{261}}{r^{261}} - \frac{px^{263}}{r^{263}} + \frac{qx^{265}}{r^{265}} - \frac{rx^{267}}{r^{267}} + \frac{sx^{269}}{r^{269}} - \frac{tx^{271}}{r^{271}} + \frac{ux^{273}}{r^{273}} - \frac{vx^{275}}{r^{275}} + \frac{wx^{277}}{r^{277}} - \frac{yx^{279}}{r^{279}} + \frac{zx^{281}}{r^{281}} - \frac{ax^{283}}{r^{283}} + \frac{bx^{285}}{r^{285}} - \frac{cx^{287}}{r^{287}} + \frac{dx^{289}}{r^{289}} - \frac{ex^{291}}{r^{291}} + \frac{fx^{293}}{r^{293}} - \frac{gx^{295}}{r^{295}} + \frac{hx^{297}}{r^{297}} - \frac{ix^{299}}{r^{299}} + \frac{kx^{301}}{r^{301}} - \frac{lx^{303}}{r^{303}} + \frac{mx^{305}}{r^{305}} - \frac{nx^{307}}{r^{307}} + \frac{ox^{309}}{r^{309}} - \frac{px^{311}}{r^{311}} + \frac{qx^{313}}{r^{313}} - \frac{rx^{315}}{r^{315}} + \frac{sx^{317}}{r^{317}} - \frac{tx^{319}}{r^{319}} + \frac{ux^{321}}{r^{321}} - \frac{vx^{323}}{r^{323}} + \frac{wx^{325}}{r^{325}} - \frac{yx^{327}}{r^{327}} + \frac{zx^{329}}{r^{329}} - \frac{ax^{331}}{r^{331}} + \frac{bx^{333}}{r^{333}} - \frac{cx^{335}}{r^{335}} + \frac{dx^{337}}{r^{337}} - \frac{ex^{339}}{r^{339}} + \frac{fx^{341}}{r^{341}} - \frac{gx^{343}}{r^{343}} + \frac{hx^{345}}{r^{345}} - \frac{ix^{347}}{r^{347}} + \frac{kx^{349}}{r^{349}} - \frac{lx^{351}}{r^{351}} + \frac{mx^{353}}{r^{353}} - \frac{nx^{355}}{r^{355}} + \frac{ox^{357}}{r^{357}} - \frac{px^{359}}{r^{359}} + \frac{qx^{361}}{r^{361}} - \frac{rx^{363}}{r^{363}} + \frac{sx^{365}}{r^{365}} - \frac{tx^{367}}{r^{367}} + \frac{ux^{369}}{r^{369}} - \frac{vx^{371}}{r^{371}} + \frac{wx^{373}}{r^{373}} - \frac{yx^{375}}{r^{375}} + \frac{zx^{377}}{r^{377}} - \frac{ax^{379}}{r^{379}} + \frac{bx^{381}}{r^{381}} - \frac{cx^{383}}{r^{383}} + \frac{dx^{385}}{r^{385}} - \frac{ex^{387}}{r^{387}} + \frac{fx^{389}}{r^{389}} - \frac{gx^{391}}{r^{391}} + \frac{hx^{393}}{r^{393}} - \frac{ix^{395}}{r^{395}} + \frac{kx^{397}}{r^{397}} - \frac{lx^{399}}{r^{399}} + \frac{mx^{401}}{r^{401}} - \frac{nx^{403}}{r^{403}} + \frac{ox^{405}}{r^{405}} - \frac{px^{407}}{r^{407}} + \frac{qx^{409}}{r^{409}} - \frac{rx^{411}}{r^{411}} + \frac{sx^{413}}{r^{413}} - \frac{tx^{415}}{r^{415}} + \frac{ux^{417}}{r^{417}} - \frac{vx^{419}}{r^{419}} + \frac{wx^{421}}{r^{421}} - \frac{yx^{423}}{r^{423}} + \frac{zx^{425}}{r^{425}} - \frac{ax^{427}}{r^{427}} + \frac{bx^{429}}{r^{429}} - \frac{cx^{431}}{r^{431}} + \frac{dx^{433}}{r^{433}} - \frac{ex^{435}}{r^{435}} + \frac{fx^{437}}{r^{437}} - \frac{gx^{439}}{r^{439}} + \frac{hx^{441}}{r^{441}} - \frac{ix^{443}}{r^{443}} + \frac{kx^{445}}{r^{445}} - \frac{lx^{447}}{r^{447}} + \frac{mx^{449}}{r^{449}} - \frac{nx^{451}}{r^{451}} + \frac{ox^{453}}{r^{453}} - \frac{px^{455}}{r^{455}} + \frac{qx^{457}}{r^{457}} - \frac{rx^{459}}{r^{459}} + \frac{sx^{461}}{r^{461}} - \frac{tx^{463}}{r^{463}} + \frac{ux^{465}}{r^{465}} - \frac{vx^{467}}{r^{467}} + \frac{wx^{469}}{r^{469}} - \frac{yx^{471}}{r^{471}} + \frac{zx^{473}}{r^{473}} - \frac{ax^{475}}{r^{475}} + \frac{bx^{477}}{r^{477}} - \frac{cx^{479}}{r^{479}} + \frac{dx^{481}}{r^{481}} - \frac{ex^{483}}{r^{483}} + \frac{fx^{485}}{r^{485}} - \frac{gx^{487}}{r^{487}} + \frac{hx^{489}}{r^{489}} - \frac{ix^{491}}{r^{491}} + \frac{kx^{493}}{r^{493}} - \frac{lx^{495}}{r^{495}} + \frac{mx^{497}}{r^{497}} - \frac{nx^{499}}{r^{499}} + \frac{ox^{501}}{r^{501}} - \frac{px^{503}}{r^{503}} + \frac{qx^{505}}{r^{505}} - \frac{rx^{507}}{r^{507}} + \frac{sx^{509}}{r^{509}} - \frac{tx^{511}}{r^{511}} + \frac{ux^{513}}{r^{513}} - \frac{vx^{515}}{r^{515}} + \frac{wx^{517}}{r^{517}} - \frac{yx^{519}}{r^{519}} + \frac{zx^{521}}{r^{521}} - \frac{ax^{523}}{r^{523}} + \frac{bx^{525}}{r^{525}} - \frac{cx^{527}}{r^{527}} + \frac{dx^{529}}{r^{529}} - \frac{ex^{531}}{r^{531}} + \frac{fx^{533}}{r^{533}} - \frac{gx^{535}}{r^{535}} + \frac{hx^{537}}{r^{537}} - \frac{ix^{539}}{r^{539}} + \frac{kx^{541}}{r^{541}} - \frac{lx^{543}}{r^{543}} + \frac{mx^{545}}{r^{545}} - \frac{nx^{547}}{r^{547}} + \frac{ox^{549}}{r^{549}} - \frac{px^{551}}{r^{551}} + \frac{qx^{553}}{r^{553}} - \frac{rx^{555}}{r^{555}} + \frac{sx^{557}}{r^{557}} - \frac{tx^{559}}{r^{559}} + \frac{ux^{561}}{r^{561}} - \frac{vx^{563}}{r^{563}} + \frac{wx^{565}}{r^{565}} - \frac{yx^{567}}{r^{567}} + \frac{zx^{569}}{r^{569}} - \frac{ax^{571}}{r^{571}} + \frac{bx^{573}}{r^{573}} - \frac{cx^{575}}{r^{575}} + \frac{dx^{577}}{r^{577}} - \frac{ex^{579}}{r^{579}} + \frac{fx^{581}}{r^{581}} - \frac{gx^{583}}{r^{583}} + \frac{hx^{585}}{r^{585}} - \frac{ix^{587}}{r^{587}} + \frac{kx^{589}}{r^{589}} - \frac{lx^{591}}{r^{591}} + \frac{mx^{593}}{r^{593}} - \frac{nx^{595}}{r^{595}} + \frac{ox^{597}}{r^{597}} - \frac{px^{599}}{r^{599}} + \frac{qx^{601}}{r^{601}} - \frac{rx^{603}}{r^{603}} + \frac{sx^{605}}{r^{605}} - \frac{tx^{607}}{r^{607}} + \frac{ux^{609}}{r^{609}} - \frac{vx^{611}}{r^{611}} + \frac{wx^{613}}{r^{613}} - \frac{yx^{615}}{r^{615}} + \frac{zx^{617}}{r^{617}} - \frac{ax^{619}}{r^{619}} + \frac{bx^{621}}{r^{621}} - \frac{cx^{623}}{r^{623}} + \frac{dx^{625}}{r^{625}} - \frac{ex^{627}}{r^{627}} + \frac{fx^{629}}{r^{629}} - \frac{gx^{631}}{r^{631}} + \frac{hx^{633}}{r^{633}} - \frac{ix^{635}}{r^{635}} + \frac{kx^{637}}{r^{637}} - \frac{lx^{639}}{r^{639}} + \frac{mx^{641}}{r^{641}} - \frac{nx^{643}}{r^{643}} + \frac{ox^{645}}{r^{645}} - \frac{px^{647}}{r^{647}} + \frac{qx^{649}}{r^{649}} - \frac{rx^{651}}{r^{651}} + \frac{sx^{653}}{r^{653}} - \frac{tx^{655}}{r^{655}} + \frac{ux^{657}}{r^{657}} - \frac{vx^{659}}{r^{659}} + \frac{wx^{661}}{r^{661}} - \frac{yx^{663}}{r^{663}} + \frac{zx^{665}}{r^{665}} - \frac{ax^{667}}{r^{667}} + \frac{bx^{669}}{r^{669}} - \frac{cx^{671}}{r^{671}} + \frac{dx^{673}}{r^{673}} - \frac{ex^{675}}{r^{675}} + \frac{fx^{677}}{r^{677}} - \frac{gx^{679}}{r^{679}} + \frac{hx^{681}}{r^{681}} - \frac{ix^{683}}{r^{683}} + \frac{kx^{685}}{r^{685}} - \frac{lx^{687}}{r^{687}} + \frac{mx^{689}}{r^{689}} - \frac{nx^{691}}{r^{691}} + \frac{ox^{693}}{r^{693}} - \frac{px^{695}}{r^{695}} + \frac{qx^{697}}{r^{697}} - \frac{rx^{699}}{r^{699}} + \frac{sx^{701}}{r^{701}} - \frac{tx^{703}}{r^{703}} + \frac{ux^{705}}{r^{705}} - \frac{vx^{707}}{r^{707}} + \frac{wx^{709}}{r^{709}} - \frac{yx^{711}}{r^{711}} + \frac{zx^{713}}{r^{713}} - \frac{ax^{715}}{r^{715}} + \frac{bx^{717}}{r^{717}} - \frac{cx^{719}}{r^{719}} + \frac{dx^{721}}{r^{721}} - \frac{ex^{723}}{r^{723}} + \frac{fx^{725}}{r^{725}} - \frac{gx^{727}}{r^{727}} + \frac{hx^{729}}{r^{729}} - \frac{ix^{731}}{r^{731}} + \frac{kx^{733}}{r^{733}} - \frac{lx^{735}}{r^{735}} + \frac{mx^{737}}{r^{737}} - \frac{nx^{739}}{r^{739}} + \frac{ox^{741}}{r^{741}} - \frac{px^{743}}{r^{743}} + \frac{qx^{745}}{r^{745}} - \frac{rx^{747}}{r^{747}} + \frac{sx^{749}}{r^{749}} - \frac{tx^{751}}{r^{751}} + \frac{ux^{753}}{r^{753}} - \frac{vx^{755}}{r^{755}} + \frac{wx^{757}}{r^{757}} - \frac{yx^{759}}{r^{759}} + \frac{zx^{761}}{r^{761}} - \frac{ax^{763}}{r^{763}} + \frac{bx^{765}}{r^{765}} - \frac{cx^{767}}{r^{767}} + \frac{dx^{769}}{r^{769}} - \frac{ex^{771}}{r^{771}} + \frac{fx^{773}}{r^{773}} - \frac{gx^{775}}{r^{775}} + \frac{hx^{777}}{r^{777}} - \frac{ix^{779}}{r^{779}} + \frac{kx^{781}}{r^{781}} - \frac{lx^{783}}{r^{783}} + \frac{mx^{785}}{r^{785}} - \frac{nx^{787}}{r^{787}} + \frac{ox^{789}}{r^{789}} - \frac{px^{791}}{r^{791}} + \frac{qx^{793}}{r^{793}} - \frac{rx^{795}}{r^{795}} + \frac{sx^{797}}{r^{797}} - \frac{tx^{799}}{r^{799}} + \frac{ux^{801}}{r^{801}} - \frac{vx^{803}}{r^{803}} + \frac{wx^{805}}{r^{805}} - \frac{yx^{807}}{r^{807}} + \frac{zx^{809}}{r^{809}} - \frac{ax^{811}}{r^{811}} + \frac{bx^{813}}{r^{813}} - \frac{cx^{815}}{r^{815}} + \frac{dx^{817}}{r^{817}} - \frac{ex^{819}}{r^{819}} + \frac{fx^{821}}{r^{821}} - \frac{gx^{823}}{r^{823}} + \frac{hx^{825}}{r^{825}} - \frac{ix^{827}}{r^{827}} + \frac{kx^{829}}{r^{829}} - \frac{lx^{831}}{r^{831}} + \frac{mx^{833}}{r^{833}} - \frac{nx^{835}}{r^{835}} + \frac{ox^{837}}{r^{837}} - \frac{px^{839}}{r^{839}} + \frac{qx^{841}}{r^{841}} - \frac{rx^{843}}{r^{843}} + \frac{sx^{845}}{r^{845}} - \frac{tx^{847}}{r^{847}} + \frac{ux^{849}}{r^{849}} - \frac{vx^{851}}{r^{851}} + \frac{wx^{853}}{r^{853}} - \frac{yx^{855}}{r^{855}} + \frac{zx^{857}}{r^{857}} - \frac{ax^{859}}{r^{859}} + \frac{bx^{861}}{r^{861}} - \frac{cx^{863}}{r^{863}} + \frac{dx^{865}}{r^{865}} - \frac{ex^{867}}{r^{867}} + \frac{fx^{869}}{r^{869}} - \frac{gx^{871}}{r^{871}} + \frac{hx^{873}}{r^{873}} - \frac{ix^{875}}{r^{875}} + \frac{kx^{877}}{r^{877}} - \frac{lx^{879}}{r^{879}} + \frac{mx^{881}}{r^{881}} - \frac{nx^{883}}{r^{883}} + \frac{ox^{885}}{r^{885}} - \frac{px^{887}}{r^{887}} + \frac{qx^{889}}{r^{889}} - \frac{rx^{891}}{r^{891}} + \frac{sx^{893}}{r^{893}} - \frac{tx^{895}}{r^{895}} + \frac{ux^{897}}{r^{897}} - \frac{vx^{899}}{r^{899}} + \frac{wx^{901}}{r^{901}} - \frac{yx^{903}}{r^{903}} + \frac{zx^{905}}{r^{905}} - \frac{ax^{907}}{r^{907}} + \frac{bx^{909}}{r^{909}} - \frac{cx^{911}}{r^{911}} + \frac{dx^{913}}{r^{913}} - \frac{ex^{915}}{r^{915}} + \frac{fx^{917}}{r^{917}} - \frac{gx^{919}}{r^{919}} + \frac{hx^{921}}{r^{921}} - \frac{ix^{923}}{r^{923}} + \frac{kx^{925}}{r^{925}} - \frac{lx^{927}}{r^{927}} + \frac{mx^{929}}{r^{929}} - \frac{nx^{931}}{r^{931}} + \frac{ox^{933}}{r^{933}} - \frac{px^{935}}{r^{935}} + \frac{qx^{937}}{r^{937}} - \frac{rx^{939}}{r^{939}} + \frac{sx^{941}}{r^{941}} - \frac{tx^{943}}{r^{943}} + \frac{ux^{945}}{r^{945}} - \frac{vx^{947}}{r^{947}} + \frac{wx^{949}}{r^{949}} - \frac{yx^{951}}{r^{951}} + \frac{zx^{953}}{r^{953}} - \frac{ax^{955}}{r^{955}} + \frac{bx^{957}}{r^{957}} - \frac{cx^{959}}{r^{959}} + \frac{dx^{961}}{r^{961}} - \frac{ex^{963}}{r^{963}} + \frac{fx^{965}}{r^{965}} - \frac{gx^{967}}{r^{967}} + \frac{hx^{969}}{r^{969}} - \frac{ix^{971}}{r^{971}} + \frac{kx^{973}}{r^{973}} - \frac{lx^{975}}{r^{975}} + \frac{mx^{977}}{r^{977}} - \frac{nx^{979}}{r^{979}} + \frac{ox^{981}}{r^{981}} - \frac{px^{983}}{r^{983}} + \frac{qx^{985}}{r^{985}} - \frac{rx^{987}}{r^{987}} + \frac{sx^{989}}{r^{989}} - \frac{tx^{991}}{r^{991}} + \frac{ux^{993}}{r^{993}} - \frac{vx^{995}}{r^{995}} + \frac{wx^{997}}{r^{997}} - \frac{yx^{999}}{r^{999}} + \frac{zx^{1001}}{r^{1001}} - \frac{ax^{1003}}{r^{1003}} + \frac{bx^{1005}}{r^{1005}} - \frac{cx^{1007}}{r^{1007}} + \frac{dx^{1009}}{r^{1009}} - \frac{ex^{1011}}{r^{1011}} + \frac{fx^{1013}}{r^{1013}} - \frac{gx^{1015}}{r^{1015}} + \frac{hx^{1017}}{r^{1017}} - \frac{ix^{1019}}{r^{1019}} + \frac{kx^{1021}}{r^{1021}} - \frac{lx^{1023}}{r^{1023}} + \frac{mx^{1025}}{r^{1025}} - \frac{nx^{1027}}{r^{1027}} + \frac{ox^{1029}}{r^{1029}} - \frac{px^{1031}}{r^{1031}} + \frac{qx^{1033}}{r^{1033}} - \frac{rx^{1035}}{r^{1035}} + \frac{sx^{1037}}{r^{1037}} - \frac{tx^{1039}}{r^{1039}} + \frac{ux^{1041}}{r^{1041}} - \frac{vx^{1043}}{r^{1043}} + \frac{wx^{1045}}{r^{1045}} - \frac{yx^{1047}}{r^{1047}} + \frac{zx^{1049}}{r^{1049}} - \frac{ax^{1051}}{r^{1051}} + \frac{bx^{1053}}{r^{1053}} - \frac{cx^{1055}}{r^{1055}} + \frac{dx^{1057}}{r^{1057}} - \frac{ex^{1059}}{r^{1059}} + \frac{fx^{1061}}{r^{1061}} - \frac{gx^{1063}}{r^{1063}} + \frac{hx^{1065}}{r^{1065}} - \frac{ix^{1067}}{r^{1067}} + \frac{kx^{1069}}{r^{1069}} - \frac{lx^{1071}}{r^{1071}} + \frac{mx^{1073}}{r^{1073}} - \frac{nx^{1075}}{r^{1075}} + \frac{ox^{1077}}{r^{1077}} - \frac{px^{1079}}{r^{1079}} + \frac{qx^{1081}}{r^{1081}} - \frac{rx^{1083}}{r^{1083}} + \frac{sx^{1085}}{r^{1085}} - \frac{tx^{1087}}{r^{1087}} + \frac{ux^{1089}}{r^{1089}} - \frac{vx^{1091}}{r^{1091}} + \frac{wx^{1093}}{r^{1093}} - \frac{yx^{1095}}{r^{1095}} + \frac{zx^{1097}}{r^{1097}} - \frac{ax^{1099}}{r^{1099}} + \frac{bx^{1101}}{r^{1101}} - \frac{cx^{1103}}{r^{1103}} + \frac{dx^{1105}}{r^{1105}} - \frac{ex^{1107}}{r^{1107}} + \frac{fx^{1109}}{r^{1109}} - \frac{gx^{1111}}{r^{1111}} + \frac{hx^{1113}}{r^{1113}} - \frac{ix^{1115}}{r^{1115}} + \frac{kx^{1117}}{r^{1117}} - \frac{lx^{1119}}{r^{1119}} + \frac{mx^{1121}}{r^{1121}} - \frac{nx^{1123}}{r^{1123}} + \frac{ox^{1125}}{r^{1125}} - \frac{px^{1127}}{r^{1127}} + \frac{qx^{1129}}{r^{1129}} - \frac{rx^{1131}}{r^{1131}} + \frac{sx^{1133}}{r^{1133}} - \frac{tx^{1135}}{r^{1135}} + \frac{ux^{1137}}{r^{1137}} - \frac{vx^{1139}}{r^{1139}} + \frac{wx^{1141}}{r^{1141}} - \frac{yx^{1143}}{r^{1143}} + \frac{zx^{1145}}{r^{1145}} - \frac{ax^{1147}}{r^{1147}} + \frac{bx^{1149}}{r^{1149}} - \frac{cx^{1151}}{r^{1151}} + \frac{dx^{1153}}{r^{1153}} - \frac{ex^{1155}}{r^{1155}} + \frac{fx^{1157}}{r^{1157}} - \frac{gx^{1159}}{r^{1159}} + \frac{hx^{1161}}{r^{1161}} - \frac{ix^{1163}}{r^{1163}} + \frac{kx^{1165}}{r^{1165}} - \frac{lx^{1167}}{r^{1167}} + \frac{mx^{1169}}{r^{1169}} - \frac{nx^{1171}}{r^{1171}} + \frac{ox^{1173}}{r^{1173}} - \frac{px^{1175}}{r^{1175}} + \frac{qx^{1177}}{r^{1177}} - \frac{rx^{1179}}{r^{1179}} + \frac{sx^{1181}}{r^{1181}} - \frac{tx^{1183}}{r^{1183}} + \frac{ux^{1185}}{r^{1185}} - \frac{vx^{1187}}{r^{1187}} + \frac{wx^{1189}}{r^{1189}} - \frac{yx^{1191}}{r^{1191}} + \frac{zx^{1193}}{r^{1193}} - \frac{ax^{1195}}{r^{1195}} + \frac{bx^{1197}}{r^{1197}} - \frac{cx^{1199}}{r^{1199}} + \frac{dx^{1201}}{r^{1201}} - \frac{ex^{1203}}{r^{1203}} + \frac{fx^{1205}}{r^{1205}} - \frac{gx^{1207}}{r^{1207}} + \frac{hx^{1209}}{r^{1209}} - \frac{ix^{1211}}{r^{1211}} + \frac{kx^{1213}}{r^{1213}} - \frac{lx^{1215}}{r^{1215}} + \frac{mx^{1217}}{r^{1217}} - \frac{nx^{1219}}{r^{1219}} + \frac{ox^{1221}}{r^{1221}} - \frac{px^{1223}}{r^{1223}} + \frac{qx^{1225}}{r^{1225}} - \frac{rx^{1227}}{r^{1227}} + \frac{sx^{1229}}{r^{1229}} - \frac{tx^{1231}}{r^{1231}} + \frac{ux^{1233}}{r^{1233}} - \frac{vx^{1235}}{r^{1235}} + \frac{wx^{1237}}{r^{1237}} - \frac{yx^{1239}}{r^{1239}} + \frac{zx^{1241}}{r^{1241}} - \frac{ax^{1243}}{r^{1243}} + \frac{bx^{1245}}{r^{1245}} - \frac{cx^{1247}}{r^{1247}} + \frac{dx^{1249}}{r^{1249}} - \frac{ex^{1251}}{r^{1251}} + \frac{fx^{1253}}{r^{1253}} - \frac{gx^{1255}}{r^{1255}} + \frac{hx^{1257}}{r^{1257}} - \frac{ix^{1259}}{r^{1259}} + \frac{kx^{1261}}{r^{1261}} - \frac{lx^{1263}}{r^{1263}} + \frac{mx^{1265}}{r^{1265}} - \frac{nx^{1267}}{r^{1267}} + \frac{ox^{1269}}{r^{1269}} - \frac{px^{1271}}{r^{1271}} + \frac{qx^{1273}}{r^{1273}} - \frac{rx^{1275}}{r^{1275}} + \frac{sx^{1277}}{r^{1277}} - \frac{tx^{1279}}{r^{1279}} + \frac{ux^{1281}}{r^{1281}} - \frac{vx^{1283}}{r^{1283}} + \frac{wx^{1285}}{r^{1285}} - \frac{yx^{1287}}{r^{1287}} + \frac{zx^{1289}}{r^{1289}} - \frac{ax^{1291}}{r^{1291}} + \frac{bx^{1293}}{r^{1293}} - \frac{cx^{1295}}{r^{1295}} + \frac{dx^{1297}}{r^{1297}} - \frac{ex^{1299}}{r^{1299}} + \frac{fx^{1301}}{r^{1301}} - \frac{gx^{1303}}{r^{1303}} + \frac{hx^{1305}}{r^{1305}} - \frac{ix^{1307}}{r^{1307}} + \frac{kx^{1309}}{r^{1309}} - \frac{lx^{1311}}{r^{1311}} + \frac{mx^{1313}}{r^{1313}} - \frac{nx^{1315}}{r^{1315}} + \frac{ox^{1317}}{r^{1317}} - \frac{px^{1319}}{r^{1319}} + \frac{qx^{1321}}{r^{1321}} - \frac{rx^{1323}}{r^{1323}} + \frac{sx^{1325}}{r^{1325}} - \frac{tx^{1327}}{r^{1327}} + \frac{ux^{1329}}{r^{1329}} - \frac{vx^{1331}}{r^{1331}} + \frac{wx^{1333}}{r^{1333}} - \frac{yx^{1335}}{r^{1335}} + \frac{zx^{1337}}{r^{1337}} - \frac{ax^{1339}}{r^{1339}} + \frac{bx^{1341}}{r^{1341}} - \frac{cx^{1343}}{r^{1343}} + \frac{dx^{1345}}{r^{1345}} - \frac{ex^{1347}}{r^{1347}} + \frac{fx^{1349}}{r^{1349}} - \frac{gx^{1351}}{r^{1351}} + \frac{hx^{1353}}{r^{1353}} - \frac{ix^{1355}}{r^{1355}} + \frac{kx^{1357}}{r^{1357}} - \frac{lx^{1359}}{r^{1359}} + \frac{mx^{1361}}{r^{1361}} - \frac{nx^{1363}}{r^{1363}} + \frac{ox^{1365}}{r^{1365}} - \frac{px^{1367}}{r^{1367}} + \frac{qx^{1369}}{r^{1369}} - \frac{rx^{1371}}{r^{1371}} + \frac{sx^{1373}}{r^{1373}} - \frac{tx^{1375}}{r^{1375}} + \frac{ux^{1377}}{r^{1377}} - \frac{vx^{1379}}{r^{1379}} + \frac{wx^{1381}}{r^{1381}} - \frac{yx^{1383}}{r^{1383}} + \frac{zx^{1385}}{r^{1$

$$\left\{ \begin{aligned} & Cx - \frac{Dx^6}{r} + \frac{Fx^3}{r^3} - \frac{Gx^4}{r^3} + \frac{Hx^5}{r^3} - \frac{Ix^6}{r^3} + \frac{Kx^7}{r^3} - \frac{Lx^8}{r^3} + \frac{Mx^9}{r^3} - && \&c \\ & \quad - \frac{Cx^3}{2r^3} + \frac{Dx^4}{2r^3} - \frac{Fx^5}{2r^3} + \frac{Gx^6}{2r^3} - \frac{Hx^7}{2r^3} + \frac{Ix^8}{2r^3} - \frac{Kx^9}{2r^3} + && \&c \\ & \quad \quad - \frac{Cx^5}{8r^4} + \frac{Dx^6}{8r^4} - \frac{Fx^7}{8r^4} + \frac{Gx^8}{8r^4} - \frac{Hx^9}{8r^4} + && \&c \\ & \quad \quad \quad - \frac{Cx^7}{16r^5} + \frac{Dx^8}{16r^5} - \frac{Fx^9}{16r^5} + && \&c \\ & \quad \quad \quad \quad - \frac{5Cx^9}{128r^6} + && \&c \\ & \quad \quad \quad \quad \quad - && \&c; \end{aligned} \right.$$

and consequently (by dividing all the terms of this last equation by x ,) we shall have the complicated series

$$\left\{ \begin{aligned} & C - \frac{Dx}{r} + \frac{Fx^3}{r^3} - \frac{Gx^4}{r^3} + \frac{Hx^5}{r^3} - \frac{Ix^6}{r^3} + \frac{Kx^7}{r^3} - \frac{Lx^8}{r^3} + \frac{Mx^9}{r^3} - && \&c \\ & \quad - \frac{Cx^3}{2r^3} + \frac{Dx^4}{2r^3} - \frac{Fx^5}{2r^3} + \frac{Gx^6}{2r^3} - \frac{Hx^7}{2r^3} + \frac{Ix^8}{2r^3} - \frac{Kx^9}{2r^3} + && \&c \\ & \quad \quad - \frac{Cx^5}{8r^4} + \frac{Dx^6}{8r^4} - \frac{Fx^7}{8r^4} + \frac{Gx^8}{8r^4} - \frac{Hx^9}{8r^4} + && \&c \\ & \quad \quad \quad - \frac{Cx^7}{16r^5} + \frac{Dx^8}{16r^5} - \frac{Fx^9}{16r^5} + && \&c \\ & \quad \quad \quad \quad - \frac{5Cx^9}{128r^6} + && \&c \\ & \quad \quad \quad \quad \quad - && \&c \end{aligned} \right.$$

$= c$: in which equation there is only one unknown quantity, to wit, x , or the co-sine of the rhumb-angle LAD. Therefore by resolving this equation we shall find the value of the said co-sine of the rhumb-angle LAD, and consequently shall, by the help of a Table of Sines and Tangents, discover the magnitude of the said angle LAD itself. Q. E. I.

Art. 100. This equation will be true only when the several infinite serieses denoted by the capital letters C, D, F, G, H, I, K, L, M, &c, are converging serieses. But this will happen only when l , or the latitude of the point A, or of the first place of the ship, from which it begins its voyage, is less than r , or the radius of the Earth, and when d , or the loxodromick arch AD, or the distance run by the ship in going from A to D, is also less than r , or the radius of the Earth; because the quantities l and d occur in the numerators of the terms of the said infinite serieses, and the quantity r occurs in their denominators. Therefore the said equation will be true only when l and d , or the said latitude of the first point A and the distance run by the ship in going from A to D, are, each of them, less than r , or the radius of the Earth.

Art. 101. In order to simplify the notation of the foregoing equation, let 1 be substituted in its terms instead of r , or the radius of the Earth; or let the radius

radius of the Earth be called 1. And then we shall have $r^2 = 1$, and $r^3 = 1$, and $r^4 = 1$, and every following power of r in like manner = 1; and consequently the foregoing equation will become as follows, to wit,

$$\begin{aligned}
 & c - dx + fx^2 - gx^3 + hx^4 - lx^5 + kx^6 - lx^7 + mx^8 - \&c \\
 & - \frac{cx^2}{2} + \frac{dx^3}{3} - \frac{fx^4}{2} + \frac{gx^5}{2} - \frac{hx^6}{2} + \frac{lx^7}{3} - \frac{kx^8}{2} + \&c \\
 & - \frac{cx^6}{8} + \frac{dx^7}{6} - \frac{fx^8}{8} + \frac{gx^9}{8} - \frac{hx^8}{8} + \&c \\
 & - \frac{cx^9}{16} + \frac{dx^7}{16} - \frac{fx^9}{16} + \&c \\
 & - \frac{5cx^9}{128} + \&c \\
 & = c.
 \end{aligned}$$

Art. 102. It remains that we make an application of the foregoing solution to some particular example, as we did of the two solutions of the two foregoing cases of this Problem in the former part of this Discourse. The first of these examples (which is given above in pages 33, 34, 35, 36, &c, - - - 45) was taken from Mr. ROBERTSON'S Treatise on Navigation, page 560; and the other example (which is given above in pages 60, 61, 62, 63, &c, - - - 75,) was taken from Monsieur BOUGUER'S French Treatise on Navigation, page 353. But neither of those eminent authors has given us an example of this third case of Dr. HALLEY'S Problem. And therefore I have been obliged to frame one myself. And, in endeavouring to do this, I have found that there is no small danger of assigning to the loxodromick arch AVD , or the distance run by the ship in her passage from A to D (which is denoted in the foregoing general solution by the small letter d), and to the equatorial arch LN (which is denoted in the foregoing general solution by the small letter c), magnitudes which may afterwards prove to be incompatible with each other, and consequently make the solution of the Problem impossible. This misfortune I at first fell into in my endeavours to form an example to the foregoing solution. But I have since taken proper measures to avoid it, and have framed an example in which the magnitudes assigned to the loxodromick arch AVD and the equatorial arch LN , or to the quantities d and c , (which are both supposed to be given in this Problem,) are compatible with each other, and the solution of the Problem is consequently possible. This example is as follows.

An

*An Example of the foregoing Solution of the Third Case of
Dr. HALLEY's Problem.*

Art. 103. Let the point P, in Fig. 4, denote the North Pole of the Earth, and HI denote a part of the circumference of the Equator. And let the point A, (which lies between the North Pole P and the Equator HI,) be only five degrees to the North of the Equator; or let the arch AL, (which is the latitude of the point A,) be an arch of five degrees. Let the ship sail, in a direction partly tending to the South and partly to the West, along the loxodromick curve, or rhumb-line, AD, which cuts all the meridian-lines, over which it passes, in the same angle LAD, which is supposed to be unknown, and of which π is the co-sine in a circle of which r , or 1, or the radius of the Earth, is the radius. And let the said rhumb-line AD be supposed to cross the equatorial arch HI, and to cut it in the point V, and to be continued to the point D, which lies on the South side of the Equator, or between the South Pole, (which I suppose to be denoted by the small letter p , though, from want of room, it is not set down in the figure,) and the Equator. And let the length of the said rhumb-line, AVD, or the distance run by the ship in passing from A to D, be 564 miles. And, lastly, let the length of the equatorial arch LN, or the difference of the longitudes of the points A and D, be 324.904 miles. It is required to find, from these three quantities so given, (to wit, the first latitude AL, or five degrees, the rhumb line AVD, or 564 miles, and the equatorial arch LN, or 324.904 miles,) the angle LAD, of the ship's course, by means of the foregoing solution.

Art. 104. A degree of a great circle of the Earth is equal to about 69 miles and a half, or 69.5 miles. Therefore the arch AL, or the latitude of the first point A, (being equal to 5 degrees,) will be equal to 5×69.5 miles, or 347.5 miles. But the radius of the Earth is = 4000 miles. Therefore AL will be = $347.5 \times$ the 4000th part of the radius of the Earth, or will be $(= 347.5 \times \frac{1}{4000} = \frac{347.5}{4000}) = 0.086,875$ of the radius of the Earth; or (because the radius of the Earth is supposed to be called 1,) AL will be = $0.086,875 \times 1$, or 0.086,875. But l is = AL. Therefore l will be = 0.086,875.

In the next place the loxodromick arch AD, or the distance run by the ship in passing from A to D, is supposed to be 564 miles, which is = $\frac{564}{4000}$ of the radius of the Earth, or 0.141 of the radius of the Earth; or (because the radius

dus of the Earth is called 1,) the loxodromick arch AD will be $= 0.141 \times 1$, or 0.141. But d is $=$ the loxodromick arch AD. Therefore d will be $= 0.141$.

Thirdly, the small letter e , (which is equal to the equatorial arch LN) will be $= 324.904$ miles, or $\frac{324.904}{4000}$ of the radius of the Earth; or 0.081,201 of the radius of the Earth, or (because the radius of the Earth is called 1,) will be $= 0.081,201 \times 1$, or 0.081,201.

Therefore the general equation set down above in art. 101, to wit, the equation

$$\left\{ \begin{aligned} C - Dx + Fx^2 - Gx^3 + Hx^4 - Ix^5 + Kx^6 - Lx^7 + Mx^8 - \&c, \\ - \frac{Cx^3}{2} + \frac{Dx^3}{2} - \frac{Fx^4}{2} + \frac{Gx^5}{2} - \frac{Hx^6}{2} + \frac{Ix^7}{2} - \frac{Kx^8}{2} + \&c, \\ - \frac{Cx^4}{8} + \frac{Dx^5}{8} - \frac{Fx^6}{8} + \frac{Gx^7}{8} - \frac{Hx^8}{8} + \&c, \\ - \frac{Cx^5}{16} + \frac{Dx^6}{16} - \frac{Fx^7}{16} + \&c, \\ - \frac{5Cx^8}{128} + \&c, \end{aligned} \right.$$

$= e$, will, in this case, become

$$\left\{ \begin{aligned} C - Dx + Fx^2 - Gx^3 + Hx^4 - Ix^5 + Kx^6 - Lx^7 + Mx^8 - \&c, \\ - \frac{Cx^3}{2} + \frac{Dx^3}{2} - \frac{Fx^4}{2} + \frac{Gx^5}{2} - \frac{Hx^6}{2} + \frac{Ix^7}{2} - \frac{Kx^8}{2} + \&c, \\ - \frac{Cx^4}{8} + \frac{Dx^5}{8} - \frac{Fx^6}{8} + \frac{Gx^7}{8} - \frac{Hx^8}{8} + \&c, \\ - \frac{Cx^5}{16} + \frac{Dx^6}{16} - \frac{Fx^7}{16} + \&c, \\ - \frac{5Cx^8}{128} + \&c, \end{aligned} \right.$$

$= 0.081,201$.

Art. 105. We must next find the value of the capital letter C; which may be done as follows.

The capital letter C is equal to the infinite series $d + \frac{Pd}{2r^3} + \frac{5Pd}{24r^4} + \frac{61Pd}{720r^5} + \&c$, and therefore (as r is $= 1$, and consequently r^3, r^4, r^5, r^6 , &c, are also, each of them, $= 1$.) will be equal to the infinite series $d + \frac{Pd}{2} + \frac{5Pd}{24} + \frac{61Pd}{720} + \frac{277Pd}{8064} + \&c$, $= d \times$ the infinite series $1 + \frac{P}{2} + \frac{5P}{24} + \frac{61P}{720} + \frac{277P}{8064} + \&c$. But l is $= 0.086,875$. Therefore P will be $(=$
0.086,875

$0.086,875 \times 0.086,875 = 0.007,547$, and i^4 will be $(= 0.007,547 \times 0.007,547) = 0.000,056,9$, and i^5 will be $(= i^4 \times i^1 = 0.000,056,9 \times 0.007,547) = 0.000,000,429$, and i^6 will be $(= i^5 \times i^1 = 0.000,000,429 \times 0.007,547) = 0.000,000,003$; and consequently the series $1 + \frac{p}{2} + \frac{5i^2}{24} + \frac{61i^4}{720} + \frac{277i^6}{8064} + \&c$ will be $= 1 + \frac{0.007,547}{2} + \frac{5 \times 0.000,056,9}{24} + \frac{61 \times 0.000,000,429}{720} + \frac{277 \times 0.000,000,003}{8064} + \&c = 1 + 0.003,773 + \frac{0.000,284,5}{24} + \frac{0.000,026,169}{720} + \frac{0.000,000,831}{8064} + \&c = 1 + 0.003,773 + 0.000,011,8 + 0.000,000,036 + 0.000,000,000, \&c + \&c$, or (if we neglect the figures that are below the sixth place of decimal fractions,) will be $= 1 + 0.003,773 + 0.000,011 + 0.000,000, \&c + 0.000,000, \&c + \&c = 1.003,784$. Therefore $d \times$ the infinite series $1 + \frac{p}{2} + \frac{5i^2}{24} + \frac{61i^4}{720} + \frac{277i^6}{8064} + \&c$ will be $= d \times 1.003,784, \&c = 0.141 \times 1.003,784, \&c = 0.141,533, \&c$. Therefore the capital letter C will be $= 0.141,533, \&c$.

ART. 106. Therefore the equation set down above in art. 104, will become as follows; to wit,

$$\left\{ \begin{array}{l} 0.141,533 - Dx + Fx^2 - Gx^3 + Hx^4 - Ix^5 + Kx^6 - Lx^7 + Mx^8 - \&c \\ - \frac{Cx^2}{2} + \frac{Dx^3}{2} - \frac{Fx^4}{2} + \frac{Gx^5}{2} - \frac{Hx^6}{2} + \frac{Ix^7}{2} - \frac{Kx^8}{2} + \&c \\ - \frac{Cx^4}{8} + \frac{Dx^5}{8} - \frac{Fx^6}{8} + \frac{Gx^7}{8} - \frac{Hx^8}{8} + \&c \\ - \frac{Cx^6}{16} + \frac{Dx^7}{16} - \frac{Fx^8}{16} + \&c \\ - \frac{5Cx^8}{128} + \&c \\ - \&c \end{array} \right.$$

$= 0.081,201.$

Now let all the terms of this equation that are marked with the sign $-$ be added to both sides of the equation. And we shall have

$$\left\{ \begin{array}{l} 0.141,533 + Fx^2 + Hx^4 + Kx^6 + Mx^8 + \&c \\ + \frac{Dx^3}{2} + \frac{Gx^5}{2} + \frac{Ix^7}{2} + \&c \\ + \frac{Dx^5}{8} + \frac{Gx^7}{8} + \&c \\ + \frac{Dx^7}{16} + \&c \\ + \&c \end{array} \right\} =$$

$$\left\{ \begin{aligned} &0.081,201 + Dx + Gx^3 + Ix^5 + Lx^7 + \&c \\ &\quad + \frac{Cx^3}{2} + \frac{Fx^4}{2} + \frac{Hx^6}{2} + \frac{Kx^8}{2} + \&c \\ &\quad \quad + \frac{Cx^5}{8} + \frac{Fx^6}{8} + \frac{Hx^8}{8} + \&c \\ &\quad \quad \quad + \frac{Cx^7}{16} + \frac{Fx^8}{16} + \&c \\ &\quad \quad \quad \quad + \frac{5Cx^9}{128} + \&c \\ &\quad \quad \quad \quad \quad + \&c; \end{aligned} \right.$$

and consequently (subtracting 0.081,201 from both sides,) we shall have

$$\left\{ \begin{aligned} &0.060,332 + Fx^3 + Hx^5 + Kx^7 + Mx^9 + \&c \\ &\quad + \frac{Dx^3}{2} + \frac{Gx^5}{2} + \frac{Ix^7}{2} + \&c \\ &\quad \quad + \frac{Dx^5}{8} + \frac{Gx^7}{8} + \&c \\ &\quad \quad \quad + \frac{Dx^7}{16} + \&c \\ &\quad \quad \quad \quad + \&c \end{aligned} \right\} =$$

$$\left\{ \begin{aligned} &Dx + Gx^3 + Ix^5 + Lx^7 + \&c \\ &\quad + \frac{Cx^3}{2} + \frac{Fx^4}{2} + \frac{Hx^6}{2} + \frac{Kx^8}{2} + \&c \\ &\quad \quad + \frac{Cx^5}{8} + \frac{Fx^6}{8} + \frac{Hx^8}{8} + \&c \\ &\quad \quad \quad + \frac{Cx^7}{16} + \frac{Fx^8}{16} + \&c \\ &\quad \quad \quad \quad + \frac{5Cx^9}{128} + \&c; \end{aligned} \right.$$

and lastly, (subtracting all the terms on the left-hand side of the equation, except the first term 0.060,332, from both sides of the equation,) we shall have the equation

$$\left\{ \begin{aligned} &Dx - Fx^3 + Gx^3 - Hx^4 + Ix^5 - Kx^6 + Lx^7 - Mx^9 + \&c \\ &\quad + \frac{Cx^3}{2} - \frac{Dx^3}{2} + \frac{Fx^4}{2} - \frac{Gx^5}{2} + \frac{Hx^6}{2} - \frac{Ix^7}{2} + \frac{Kx^8}{2} - \&c \\ &\quad \quad + \frac{Cx^5}{8} - \frac{Dx^5}{8} + \frac{Fx^6}{8} - \frac{Gx^7}{8} + \frac{Hx^8}{8} - \&c \\ &\quad \quad \quad + \frac{Cx^7}{16} - \frac{Dx^7}{16} + \frac{Fx^8}{16} - \&c \\ &\quad \quad \quad \quad + \frac{5Cx^9}{128} + \&c \\ &\quad \quad \quad \quad \quad + \&c \end{aligned} \right\} \\ = 0.060,332. \quad \text{Art. 107.}$$

Art. 107. We will now proceed to find the values of the capital letters, D, F, G, H, I, K, L, and M; which are equal to the several infinite serieses set forth above in art. 96, which consist of terms that involve the powers of the known quantities l and d , together with certain numeral co-efficients. This may be done in the manner following.

The capital letter D is equal to the infinite series $\frac{ld^3}{2r^3} + \frac{5l^2d^2}{12r^4} + \frac{61l^3d}{240r^5} + \frac{277l^4d^2}{2016r^6} + \&c$, or (because r is = 1,) to the infinite series $\frac{ld^3}{2} + \frac{5l^2d^2}{12} + \frac{61l^3d}{240} + \frac{277l^4d^2}{2016} + \&c$, or to $d^3 \times$ the infinite series $\frac{1}{2} + \frac{5l}{12} + \frac{61l^2}{240} + \frac{277l^3}{2016} + \&c$. But l is = 0.086,875, and l^2 is = 0.007,547, and l^3 is = 0.000,057, and l^4 is = 0.000,000, &c. Therefore the infinite series $\frac{1}{2} + \frac{5l}{12} + \frac{61l^2}{240} + \frac{277l^3}{2016} + \&c$ will be (= the infinite series $\frac{1}{2} + \frac{5 \times 0.007,547}{12} + \frac{61 \times 0.000,057}{240} + \frac{277 \times 0.000,000}{2016} + \&c$, = the infinite series $\frac{1}{2} + \frac{0.037,735}{12} + \frac{0.003,477}{240} + 0.000,000 + \&c$ = the infinite series 0.500,000 + 0.003,145 + 0.000,014 + 0.000,000 + &c) = 0.503,159; and consequently $d^3 \times$ the infinite series $\frac{1}{2} + \frac{5l}{12} + \frac{61l^2}{240} + \frac{277l^3}{2016} + \&c$ will be = $d^3 \times 0.503,159$ = $d^3 \times 0.086,875 \times 0.503,159$, &c = $d^3 \times 0.043,712$ = $0.141^3 \times 0.043,712$ = $0.141 \times 0.006,163$ = 0.000,869. Therefore the capital letter D will be = 0.000,869. Q. E. I.

The capital letter F is = the infinite series $\frac{d^3}{6r^3} + \frac{5l^2d^2}{12r^4} + \frac{61l^3d}{144r^5} + \frac{277l^4d^2}{864r^6} + \&c$, = the infinite series $\frac{d^3}{6} + \frac{5l^2d^2}{12} + \frac{61l^3d}{144} + \frac{277l^4d^2}{864} + \&c$, = $d^3 \times$ the infinite series $\frac{1}{6} + \frac{5l^2}{12} + \frac{61l^3}{144} + \frac{277l^4}{864} + \&c$, = $d^3 \times$ the infinite series $\frac{1}{6} + \frac{5 \times 0.007,547}{12} + \frac{61 \times 0.000,057}{144} + \frac{277 \times 0.000,000}{864} + \&c$, = $d^3 \times$ the infinite series $\frac{1}{6} + \frac{0.037,735}{12} + \frac{0.003,477}{144} + 0.000,000 + \&c$, = $d^3 \times$ the infinite series 0.166,666 + 0.003,145 + 0.000,024 + 0.000,000 + &c, = $d^3 \times 0.169,835$, &c = $0.141^3 \times 0.169,835$, &c = $0.141 \times 0.023,946$ = 0.141 $\times$ 0.003,376 = 0.000,476. Q. E. I.

The capital letter G is = the infinite series $\frac{5ld^4}{24r^4} + \frac{61l^2d^3}{144r^5} + \frac{277l^3d^2}{576r^6} + \&c$ =

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the

the infinite series $\frac{61d^4}{24} + \frac{61^3d^4}{144} + \frac{277^3d^4}{576} + \&c = d^4 \times$ the infinite series $\frac{5}{24} + \frac{61}{144} + \frac{277^3}{576} + \&c = d^4 \times$ the infinite series $\frac{5}{24} + \frac{61 \times 0.007,547}{144} + \frac{277 \times 0.000,057}{576} + \&c = d^4 \times$ the infinite series $\frac{5}{24} + \frac{0.460,367}{144} + \frac{0.015,789}{576} + \&c = d^4 \times$ the infinite series $0.208,333 + 0.003,156 + 0.000,027 + \&c = d^4 \times$ the infinite series $0.211,556 + \&c = d^4 \times 0.086,875 \times 0.211,556 = d^4 \times 0.018,379 = \frac{0.141}{1} \times 0.018,379 = \frac{0.141}{1} \times 0.002,591 = \frac{0.141}{1} \times 0.000,365 = 0.141 \times 0.000,051 = 0.000,007. \quad Q. E. I.$

The capital letter I is equal to the infinite series $\frac{d^5}{24} + \frac{61^3d^5}{240} + \frac{277^3d^5}{576} + \&c = d^5 \times$ the infinite series $\frac{1}{24} + \frac{61^3}{240} + \frac{277^3}{576} + \&c = d^5 \times$ the infinite series $\frac{1}{24} + \frac{61 \times 0.007,547}{240} + \frac{277 \times 0.000,057}{576} + \&c = d^5 \times$ the infinite series $\frac{1}{24} + \frac{0.460,367}{240} + \frac{0.015,789}{576} + \&c = d^5 \times$ the infinite series $0.041,666 + 0.001,918 + 0.000,027 + \&c = d^5 \times 0.043,611 = \frac{0.141}{1} \times 0.043,611 = 0.141 \times 0.006,149 = 0.141 \times 0.000,867 = \frac{0.141}{1} \times 0.000,122 = 0.141 \times 0.000,017 = 0.000,002. \quad Q. E. I.$

The capital letter I is equal to the infinite series $\frac{61d^6}{720} + \frac{277^3d^6}{804} + \&c =$ the infinite series $\frac{61d^6}{720} + \frac{277^3d^6}{804} + \&c = d^6 \times$ the infinite series $\frac{61}{720} + \frac{277^3}{804} + \&c = d^6 \times$ the infinite series $\frac{61}{720} + \frac{277 \times 0.007,547}{804} + \&c = d^6 \times$ the infinite series $\frac{61}{720} + \frac{2.000,519}{804} + \&c = d^6 \times$ the infinite series $0.084,722 + 0.002,419 + \&c = d^6 \times 0.087,141 = d^6 \times 0.086,875 \times 0.087,141 = d^6 \times 0.007,571 = \frac{0.141}{1} \times 0.007,571 = \frac{0.141}{1} \times 0.001,067 = \frac{0.141}{1} \times 0.000,150 = \frac{0.141}{1} \times 0.000,021 = 0.141 \times 0.000,003 = 0.141 \times 0.000,000 = 0.000,000; \text{ that is, the capital letter I will be equal to a quantity which is less than } 0.000,001, \text{ and which therefore need not be taken into consideration in a calculation which is intended to be carried to only fix places of decimal fractions.} \quad Q. E. I.$

The capital letter K is equal to the infinite series $\frac{61d^7}{5040} + \frac{277^3d^7}{2016} + \&c =$ the infinite series $\frac{61d^7}{5040} + \frac{277^3d^7}{2016} + \&c = d^7 \times$ the infinite series $\frac{61}{5040} +$

+ $\frac{277^9}{2016} + \&c = d^7 \times$ the infinite series $\frac{61}{5040} + \frac{277 \times 0.007,547}{2016} + \&c = d^7 \times$ the infinite series $\frac{61}{5040} + \frac{2.090,519}{2016} + \&c = d^7 \times$ the infinite series $0.012,103 + 0.001,037 + \&c = d^7 \times 0.013,140$ &c = $\frac{0.141}{1} \times 0.013,140 = 0.141^6 \times 0.001,852 = 0.141^5 \times 0.000,261 = 0.141^4 \times 0.000,036 = 0.141^3 \times 0.000,005 = 0.141^2 \times 0.000,000 = 0.000,000$; that is, the capital letter K will be equal to a quantity which is less than 0.000,001, and which therefore need not be taken into consideration in a calculation which is intended to be carried to only six places of decimal fractions. Q. E. I.

The capital letter L is equal to the infinite series $\frac{277^8 d^8}{8064} + \&c =$ the infinite series $\frac{277^8 d^8}{8064} + \&c = d^8 \times$ the infinite series $\frac{277}{8064} + \&c = d^8 \times$ the infinite series $0.034,350 + \&c = d^8 \times 0.086,875 \times 0.034,350 = d^8 \times 0.003,004 = 0.141^6 \times 0.003,004 = 0.141^5 \times 0.000,423 = 0.141^4 \times 0.000,059 = 0.141^3 \times 0.000,008 = 0.141^2 \times 0.000,001 = 0.141 \times 0.000,000 = 0.000,000$; that is, the capital letter L will be equal to a quantity that is less than 0.000,001, and which therefore need not be taken into consideration in a calculation which is intended to be carried to only six places of decimal fractions. Q. E. I.

And the capital letter M is equal to the infinite series $\frac{277^9 d^9}{72,576} + \&c =$ the infinite series $\frac{277^9 d^9}{72,576} + \&c = d^9 \times$ the infinite series $\frac{277}{72,576} + \&c = d^9 \times$ the infinite series $0.003,816 + \&c = 0.141^6 \times 0.003,816 = 0.141^5 \times 0.000,538 = 0.141^4 \times 0.000,075 = 0.141^3 \times 0.000,010 = 0.141^2 \times 0.000,001 = 0.141 \times 0.000,000 = 0.000,000$; that is, the capital letter M will be equal to a quantity that is less than 0.000,001, and which therefore need not be taken into consideration in a calculation which is intended to be carried to only six places of decimal fractions. Q. E. I.

Art. 108. We must now reduce the several compound co-efficients of x^3 , x^4 , x^5 , x^6 , x^7 , and x^8 , to wit, the compound quantities — $F + \frac{c}{2} + G - \frac{1}{2} \frac{D}{2} - H + \frac{F}{2} + \frac{c}{8}$, + $I - \frac{G}{2} - \frac{H}{8} - K + \frac{H}{2} + \frac{F}{8} + \frac{c}{16}$, + $L - \frac{I}{2} - \frac{G}{8} - \frac{D}{16}$, and — $M + \frac{K}{2} + \frac{H}{8} + \frac{F}{16} + \frac{5c}{32}$, into simple quantities: which may be done as follows.

Since c is = 0.141,533, and F is = 0.000,476, we shall have — F +

$$+ \frac{c}{2} (= - 0.000,476 + \frac{0.141,533}{2} = - 0.000,476 + 0.070,766) = + 0.070,290.$$

And, since p is = 0.000,869, and g is = 0.000,007, we shall have $+ g - \frac{p}{2} (= + 0.000,007 - \frac{0.000,869}{2} = + 0.000,007 - 0.000,434) = - 0.000,427.$

And, since c is = 0.141,533, and F is = 0.000,476, and H is = 0.000,002, we shall have $- H + \frac{F}{2} + \frac{c}{8} (= - 0.000,002 + \frac{0.000,476}{2} + \frac{0.141,533}{8} = - 0.000,002 + 0.000,238 + 0.017,691 = - 0.000,002 + 0.017,929) = + 0.017,927.$

And, since p is = 0.000,869, and g is = 0.000,007, and I is = 0.000,000, we shall have $+ I - \frac{g}{2} - \frac{p}{8} (= + 0.000,000 - \frac{0.000,007}{2} - \frac{0.000,869}{8} = + 0.000,000 - 0.000,003 - 0.000,108) = - 0.000,111.$

And, since c is = 0.141,533, and F is = 0.000,476, and H is = 0.000,002, and K is = 0.000,000, we shall have $- K + \frac{H}{2} + \frac{F}{8} + \frac{c}{16} (= - 0.000,000 + \frac{0.000,002}{2} + \frac{0.000,476}{8} + \frac{0.141,533}{16} = - 0.000,000 + 0.000,001 + 0.000,059 + 0.008,845) = + 0.008,905.$

And, since p is = 0.000,869, and g is = 0.000,007, and I is = 0.000,000, and L is also = 0.000,000, we shall have $+ L - \frac{I}{2} - \frac{g}{8} - \frac{p}{16} (= + 0.000,000 - \frac{0.000,000}{2} - \frac{0.000,007}{8} - \frac{0.000,869}{16} = + 0.000,000 - 0.000,000 - 0.000,000 - 0.000,054) = - 0.000,054.$

And, lastly, since c is = 0.141,533, and F is = 0.000,476, and H is = 0.000,002, and K is = 0.000,000, and M is also = 0.000,000, we shall have $- M + \frac{K}{2} + \frac{H}{8} + \frac{F}{16} + \frac{5c}{128} (= - 0.000,000 + \frac{0.000,000}{2} + \frac{0.000,002}{8} + \frac{0.000,476}{16} + \frac{5 \times 0.141,533}{128} = - 0.000,000 + 0.000,000 + 0.000,000 + 0.000,029 + \frac{0.707,665}{128} = + 0.000,029 + 0.005,528) = + 0.005,557.$

Therefore the equation, set down above in art. 106, to wit, the equation

DX

$$\begin{aligned}
 & \left\{ \begin{aligned}
 & Dx = Fx^3 + Gx^2 + Hx^4 + Lx^5 - Kx^6 + Lx^7 - Mx^8 + \&c, \\
 & + \frac{Cx^3}{2} - \frac{Dx^2}{2} + \frac{Fx^4}{2} - \frac{Gx^3}{2} + \frac{Hx^5}{3} - \frac{Ix^7}{3} + \frac{Kx^8}{2} - \&c, \\
 & + \frac{Cx^4}{8} - \frac{Dx^3}{8} + \frac{Fx^6}{8} - \frac{Gx^7}{8} + \frac{Hx^8}{8} - \&c, \\
 & + \frac{Cx^6}{16} - \frac{Dx^7}{16} + \frac{Fx^8}{16} - \&c, \\
 & + \frac{5Cx^9}{128} - \&c,
 \end{aligned} \right.
 \end{aligned}$$

= 0.060,332, will become as follows, to wit,

$$\begin{aligned}
 & 0.000,869 \times x + 0.070,290 \times x^2 - 0.000,427 \times x^3 \\
 & + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\
 & - 0.000,054 \times x^7 + 0.005,557 \times x^8 = 0.060,332; \text{ which equation} \\
 & \text{we must therefore now endeavour to resolve, in order to discover the magni-} \\
 & \text{titude of the rhumb-angle LAD, of which } x, \text{ or the root of the said equa-} \\
 & \text{tion, denotes the co-line.}
 \end{aligned}$$

Art. 109. In this equation

$$\begin{aligned}
 & 0.000,869 \times x + 0.070,290 \times x^2 - 0.000,427 \times x^3 \\
 & + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\
 & - 0.000,054 \times x^7 + 0.005,557 \times x^8 = 0.060,332, \text{ it is remarkable} \\
 & \text{that } 0.070,290, \text{ the co-efficient of } x^2, \text{ is greater than } 0.000,869, \text{ the co-effi-} \\
 & \text{cient of } x; \text{ and that } 0.017,927, \text{ the co-efficient of } x^4, \text{ is greater than } 0.000,427, \\
 & \text{the co-efficient of } x^3; \text{ and that } 0.008,905, \text{ the co-efficient of } x^6, \text{ is greater} \\
 & \text{than } 0.000,111, \text{ the co-efficient of } x^5; \text{ and that } 0.005,557, \text{ the co-efficient of } \\
 & x^8, \text{ is greater than } 0.000,054, \text{ the co-efficient of } x^7. \text{ But the co-efficients of } x, \\
 & x^2, x^3, \text{ and } x^4, \text{ (to wit, } 0.000,869, \text{ and } 0.000,427, \text{ and } 0.000,111, \text{ and} \\
 & 0.000,054,) \text{ are every one less than the next before it, or decrease when the} \\
 & \text{powers of } x \text{ increase; and, in like manner, the co-efficients of } x^5, x^6, x^7, \text{ and} \\
 & x^8, \text{ or of the even powers of } x, \text{ (to wit, } 0.070,290, \text{ and } 0.017,927, \text{ and} \\
 & 0.008,905, \text{ and } 0.005,557,) \text{ are every one less than the next before it, or} \\
 & \text{decrease when the powers of } x \text{ increase; as was the case with the co-efficients} \\
 & \text{of } x, x^2, x^3, \text{ and } x^4, \text{ or of the odd powers of } x.
 \end{aligned}$$

Art. 110. We must now endeavour to resolve the final equation obtained in art. 108, to wit, the equation $0.000,869 \times x + 0.070,290 \times x^2$

$$\begin{aligned}
 & - 0.000,427 \times x^3 + 0.017,927 \times x^4 - 0.000,111 \times x^5 \\
 & + 0.008,905 \times x^6 - 0.000,054 \times x^7 + 0.005,557 \times x^8 - \&c \\
 & = 0.060,332; \text{ which may be most easily done by Mr. RAPHSOON's method of} \\
 & \text{approximation. I shall therefore now endeavour to find a first near value of } x, \\
 & \text{that shall differ from its true value by only a small part of the said true value,} \\
 & \text{in order to make the said first near value of } x \text{ the basis of a further approxima-} \\
 & \text{tion to its true value, according to the directions of Mr. RAPHSOON's method.}
 \end{aligned}$$

Art. 111.

ART. 111. Now, in order to obtain this first near value of x , I begin by observing that, since r , or 1, is the radius of a great circle of the Earth, and x is the sine of a certain angle in the said circle, (to wit, the sine of the angle which is the complement of the rhumb angle LAD to a right angle.) x must be less than 1; and, consequently, that xx must be less than x , and x^3 than xx , and x^4 than x^3 , and every following power of x than that which immediately precedes it. And hence we may conclude that the two first terms on the left-hand side of the foregoing equation, to wit, the terms $0.000,869 \times x$, and $0.070,290 \times xx$, will be greater than the two next terms, or than any other two terms on the same side of the equation, and (because those following terms are marked alternately with the signs $+$ and $-$, and consequently tend to counterbalance each other, or to diminish each other's magnitudes,) may be reasonably supposed to be pretty nearly equal to the whole infinite series that forms the left-hand side of the said equation. We will therefore suppose these two terms, taken together, to be equal to the whole of the said infinite series, and consequently to its equal, the absolute term, $0.060,332$, of the said equation. And then we shall have the following quadratrix equation to resolve, to wit, $0.000,869 \times x + 0.070,290 \times xx = 0.060,332$. This equation may be resolved as follows.

In the first place, let both sides of it be divided by the co-efficient of xx , to wit, $0.070,290$. And we shall have $xx + \frac{0.000,869}{0.070,290} \times x = \frac{0.060,332}{0.070,290}$, or $xx + 0.012,363 \times x = 0.858,319$.

Secondly, add to both the sides of this equation the square of half the co-efficient of x , or of half $0.012,363$, or the square of $0.006,181$; which square is $0.000,038,204,761$. And we shall have $xx + 0.012,363 \times x + 0.006,181 = 0.858,319 + 0.000,038$, &c $= 0.858,367$. Therefore (extracting the square-roots of both sides,) we shall have $x + 0.006,181 = 0.926$, &c, and consequently $x (= 0.926 - 0.006) = 0.92$. Therefore the first near value of x in the foregoing high equation, will be 0.92 ; of which number it seems reasonable to suppose the first figure $.9$ to be exact.

ART. 112. We will therefore now substitute 0.9 , instead of x , in the compound quantity which forms the left-hand side of the foregoing infinite equation $0.000,869 \times x + 0.070,290 \times xx - 0.000,417 \times x^3 + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 - 0.000,054 \times x^7 + 0.005,557 \times x^8 - \&c, = 0.060,332$, in order to discover, whether the value of the said compound quantity resulting from such substitution, will be greater, or less, than the absolute term, $0.060,332$, of the said equation, and whether its difference from the said absolute term will be great or small.

Now, if we suppose x to be $= 0.9$, we shall have

xx

$$xx (= 0.9^2) = 0.81,$$

$$\text{and } x^3 (= 0.81 \times 0.9) = 0.729,$$

$$\text{and } x^4 (= 0.729 \times 0.9) = 0.656,1,$$

$$\text{and } x^5 (= 0.656,1 \times 0.9) = 0.590,49,$$

$$\text{and } x^6 (= 0.590,49 \times 0.9) = 0.531,441,$$

$$\text{and } x^7 (= 0.531,441 \times 0.9) = 0.478,296,9$$

$$\text{and } x^8 (= 0.478,296,9 \times 0.9) = 0.430,467,21, \text{ and consequently}$$

$$0.000,869 \times x (= 0.000,869 \times 0.9) = 0.000,782,$$

$$\text{and } 0.070,290 \times xx (= 0.070,290 \times 0.81) = 0.056,935,$$

$$\text{and } 0.000,427 \times x^3 (= 0.000,427 \times 0.729) = 0.000,311,$$

$$\text{and } 0.017,927 \times x^4 (= 0.017,927 \times 0.656,1) = 0.011,762,$$

$$\text{and } 0.000,111 \times x^5 (= 0.000,111 \times 0.590,49) = 0.000,065,$$

$$\text{and } 0.008,905 \times x^6 (= 0.008,905 \times 0.531,441) = 0.004,732,$$

$$\text{and } 0.000,054 \times x^7 (= 0.000,054 \times 0.478,296,9) = 0.000,026,$$

$$\text{and } 0.005,557 \times x^8 (= 0.005,557 \times 0.430,467,21) = 0.002,392.$$

Therefore the whole compound quantity

$$0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^3 \\ + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\ - 0.000,054 \times x^7 + 0.005,557 \times x^8 \text{ will be } =$$

$$\left\{ \begin{array}{rcl} & 0.000,782 & \\ + & .056,935 & - 0.000,311 \\ + & .011,762 & - 0.000,065 \\ + & .004,732 & - 0.000,026 \\ + & .002,392 & - \&c \end{array} \right\}$$

$$= 0.076,603 - 0.000,402$$

$\approx 0.076,201$; which is greater than $0.060,332$, or the absolute term of the equation

$$0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^3 \\ + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\ - 0.000,054 \times x^7 + 0.005,557 \times x^8 - \&c = 0.060,332. \text{ Therefore } 0.9 \\ \text{must be greater than the true value of } x \text{ in the same equation.} \quad Q. E. I.$$

Art. 113. Since it appears that x is less than 0.9 , let us, in the next place, suppose it to be equal to 0.8 , and substitute 0.8 , instead of it, in the terms of the compound quantity

$$0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^3 \\ + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\ - 0.000,054 \times x^7 + 0.005,557 \times x^8, \text{ in order to discover, whether the} \\ \text{Vol. IV.} \quad \quad \quad \text{P} \quad \quad \quad \text{value}$$

value of the said compound quantity resulting from such substitution will be greater, or less, than 0.060,332, or the absolute term of the equation which we are endeavouring to resolve, and consequently whether 0.8 is greater, or less, than the true value of x in that equation.

Now, if x be supposed to be = 0.8, we shall have
 $xx = 0.64$, and $x^3 = 0.512$, and $x^4 = 0.4096$, and $x^5 = 0.327,68$, and $x^6 = 0.262,144$, and $x^7 = 0.209,715,2$, and $x^8 = 0.167,772,16$, and consequently

$0.000,869 \times x (= 0.000,869 \times 0.8) = 0.000,695$,
 and $0.070,290 \times xx (= 0.070,290 \times 0.64) = 0.044,985$,
 and $0.000,427 \times x^3 (= 0.000,427 \times 0.512) = 0.000,218$,
 and $0.017,927 \times x^4 (= 0.017,927 \times 0.409,6) = 0.007,343$,
 and $0.000,111 \times x^5 (= 0.000,111 \times 0.327,68) = 0.000,036$,
 and $0.008,905 \times x^6 (= 0.008,905 \times 0.262,144) = 0.002,334$,
 and $0.000,054 \times x^7 (= 0.000,054 \times 0.209,715,2) = 0.000,011$,
 and $0.005,557 \times x^8 (= 0.005,557 \times 0.167,772,16) = 0.000,932$.

Therefore the whole compound quantity

$$0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^3 \\ + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\ - 0.000,054 \times x^7 + 0.005,557 \times x^8 \text{ will be } =$$

$$\left\{ \begin{array}{rcl} 0.000,695 & & \\ + 0.044,985 & - & 0.000,218 \\ + 0.007,343 & - & 0.000,036 \\ + 0.002,334 & - & 0.000,011 \\ + 0.000,932 & - & \&c \end{array} \right\}$$

= 0.056,289 - 0.000,265 = 0.056,024; which is less than 0.060,332, or the absolute term of the equation

$$0.000,869 \times x + 0.070,290 \times x^2 - 0.000,427 \times x^3 \\ + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\ - 0.000,054 \times x^7 + 0.005,557 \times x^8 - \&c = 0.060,332. \text{ Therefore} \\ 0.8 \text{ must be less than the true value of } x \text{ in that equation. } \quad \text{Q. E. I.}$$

ART. 114. We have now found that the true value of x is less than 0.9, but greater than 0.8, or is of an intermediate magnitude between 0.8 and 0.9. And, as the quantity 0.056,024, which results from the supposition that x is = 0.8, differs much less from 0.060,332, or the absolute term of the equation

$$0.000,869 \times x + 0.070,290 \times x^2 - 0.000,427 \times x^3 \\ + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\ - 0.000,054 \times x^7 + 0.005,557 \times x^8 - \&c = 0.060,332, \text{ than} \\ 0.076,201, \text{ which results from the supposition that } x \text{ is } = 0.9, \text{ (the difference of}$$

of 0.056,024 and 0.060,332, being only 0.004,308, whereas the difference between 0.076,201 and 0.060,332 is 0.015,869, which is more than triple of 0.004,308,) it seems reasonable to conclude that the difference between 0.8 and the true value of x in the said equation, will be less than the difference between 0.9 and the said true value. It therefore will be more convenient to suppose x to be $= 0.8 + z$, and to substitute $0.8 + z$, instead of x , in the terms of the foregoing equation, in order to discover a near value of the difference z , by which the quantity 0.8 falls short of the true value of x , than to suppose x to be $= 0.9 - z$, and to substitute $0.9 - z$, instead of x , in the terms of the said equation, in order to discover a near value of the difference z by which the quantity 0.9 exceeds the said true value. We will therefore now suppose x to be $= 0.8 + z$, and will proceed to substitute $0.8 + z$ instead of x in the terms of the foregoing equation

$$0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^3$$

$$+ 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.003,905 \times x^6$$

$- 0.000,054 \times x^7 + 0.005,557 \times x^8 - \&c = 0.060,332$, but with an omission of all the terms that involve any higher power of z than its simple power, or z itself, in order to obtain a near value of the difference z by which x exceeds 0.8, and consequently a more exact near value of x than 0.8, or that which we had before. This may be done in the manner following.

If we suppose x to be $= 0.8 + z$, we shall have

$$xx (= 0.8 + z)^2 = 0.8^2 + 2 \times 0.8 \times z + \&c = 0.64 + 1.6 \times z + \&c,$$

$$\text{and } x^3 (= 0.8 + z)^3 = 0.8^3 + 3 \times 0.8^2 \times z + \&c = 0.8^3 + 3 \times 0.64 \times z + \&c = 0.512 + 1.92 \times z + \&c,$$

$$\text{and } x^4 (= 0.8 + z)^4 = 0.8^4 + 4 \times 0.8^3 \times z + \&c = 0.8^4 + 4 \times 0.512 \times z + \&c = 0.4096 + 2.048 \times z + \&c,$$

$$\text{and } x^5 (= 0.8 + z)^5 = 0.8^5 + 5 \times 0.8^4 \times z + \&c = 0.8^5 + 5 \times 0.4096 \times z + \&c = 0.327,68 + 2.0480 \times z + \&c,$$

$$\text{and } x^6 (= 0.8 + z)^6 = 0.8^6 + 6 \times 0.8^5 \times z + \&c = 0.8^6 + 6 \times 0.327,68 \times z + \&c = 0.262,144 + 1.966,08 \times z + \&c,$$

$$\text{and } x^7 (= 0.8 + z)^7 = 0.8^7 + 7 \times 0.8^6 \times z + \&c = 0.8^7 + 7 \times 0.262,144 \times z + \&c = 0.209,715,2 + 1.835,008 \times z + \&c,$$

$$\text{and } x^8 (= 0.8 + z)^8 = 0.8^8 + 8 \times 0.8^7 \times z + \&c = 0.8^8 + 8 \times 0.209,715,2 \times z + \&c = 0.167,772,16 + 1.677,721,6 \times z + \&c,$$

And consequently

$$0.000,869 \times x (= 0.000,869 \times \overline{0.8 + z} = 0.000,869 \times 0.8 + 0.000,869 \times z) = 0.000,695 + 0.000,869 \times z,$$

$$\text{and } 0.070,290 \times xx (= 0.070,290 \times \overline{0.64 + 1.6 \times z + \&c} = 0.070,290 \times 0.64 + 0.070,290 \times 1.6 \times z + \&c) = 0.044,985 + 0.112,464 \times z + \&c,$$

$$\text{and } 0.000,427 \times x^2 (= 0.000,427 \times \overline{0.512 + 1.92 \times z + \&c} = 0.000,427 \times 0.512 + 0.000,427 \times 1.92 \times z + \&c) = 0.000,218 + 0.000,819 \times z + \&c,$$

$$\text{and } 0.017,927 \times x^3 (= 0.017,927 \times \overline{0.4096 + 2.048 \times z + \&c} = 0.017,927 \times 0.4096 + 0.017,927 \times 2.048 \times z + \&c) = 0.007,343 + 0.036,714 \times z + \&c,$$

$$\text{and } 0.000,111 \times x^4 (= 0.000,111 \times \overline{0.327,68 + 2.0480 \times z + \&c} = 0.000,111 \times 0.327,68 + 0.000,111 \times 2.0480 \times z + \&c) = 0.000,036 + 0.000,227 \times z + \&c,$$

$$\text{and } 0.008,905 \times x^5 (= 0.008,905 \times \overline{0.262,144 + 1.966,08 \times z + \&c} = 0.008,905 \times 0.262,144 + 0.008,905 \times 1.966,08 \times z + \&c) = 0.002,334 + 0.017,507 \times z + \&c,$$

$$\text{and } 0.000,054 \times x^6 (= 0.000,054 \times \overline{0.209,715,2 + 1.835,008 \times z + \&c} = 0.000,054 \times 0.209,715,2 + 0.000,054 \times 1.835,008 \times z + \&c) = 0.000,011 + 0.000,099 \times z + \&c,$$

$$\text{and } 0.005,557 \times x^7 (= 0.005,557 \times \overline{0.167,772,16 + 1.677,721,6 \times z + \&c} = 0.005,557 \times 0.167,772,16 + 0.005,557 \times 1.677,721,6 \times z + \&c) = 0.000,932 + 0.009,323 \times z + \&c.$$

Therefore the compound quantity

$$\begin{aligned} & 0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^2 \\ & + 0.017,927 \times x^3 - 0.000,111 \times x^4 + 0.008,905 \times x^5 \\ & - 0.000,054 \times x^6 + 0.005,557 \times x^7 \text{ will be} \end{aligned}$$

$$\begin{aligned} & = \left\{ \begin{array}{l} 0.000,695 + 0.000,869 \times z \\ + 0.044,985 + 0.112,464 \times z + \&c \\ - 0.000,218 - 0.000,819 \times z - \&c \\ + 0.007,343 + 0.036,714 \times z + \&c \\ - 0.000,036 - 0.000,227 \times z - \&c \\ + 0.002,334 + 0.017,507 \times z + \&c \\ - 0.000,011 - 0.000,099 \times z - \&c \\ + 0.000,932 + 0.009,323 \times z + \&c \end{array} \right\} \\ & = \left\{ \begin{array}{l} 0.056,289 + 0.176,877 \times z + \&c \\ - 0.000,265 - 0.001,145 \times z - \&c \end{array} \right\} \\ & = 0.056,024 + 0.175,732 \times z + \&c. \end{aligned}$$

But

But the said compound quantity is equal to the absolute term 0.060,332.

Therefore the quantity $0.056,024 + 0.175,732 \times z + \&c$, will also be equal to 0.060,332. And consequently $0.175,732 \times z + \&c$, will be $(= 0.060,332 - 0.056,024) = 0.004,308$, and $z + \&c$ will be $(= \frac{0.004,308}{0.175,732}) = 0.024$, and z will be $= 0.024 - \&c$, or will be somewhat less than 0.024. Therefore x , or $0.8 + z$, will be $= 0.8 + 0.024 - \&c = 0.824 - \&c$; that is, the next near value of x in the equation

$0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^3$
 $+ 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6$
 $- 0.000,054 \times x^7 + 0.005,557 \times x^8 - \&c = 0.060,332$, will be $= 0.824 - \&c$. or somewhat less than 0.824, and may therefore be supposed to be, very nearly, $= 0.82$. Q. E. I.

ART. 115. We will now try the degree of exactness of the number 0.82, just now found for the last near value of x , or the root of the equation

$0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^3$
 $+ 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6$
 $- 0.000,054 \times x^7 + 0.005,557 \times x^8 - \&c = 0.060,332$, by substituting the said number instead of x in the terms of the compound quantity which forms the left-hand side of the said equation; by which means we shall discover how nearly the result of the said substitution will approach to the absolute term, 0.060,332, of the said equation, or to the true value of the said compound quantity in the said equation, and consequently shall be able to form a judgement of the degree of exactness with which the said number 0.82 will approach to the true value of x in the said equation. This substitution may be made as follows.

If we suppose x to be $= 0.82$, we shall have

$xx = 0.672,4$, and $x^3 = 0.551,368$, and $x^4 = 0.452,121$,
 and $x^5 = 0.370,739$, and $x^6 = 0.304,006$, and $x^7 = 0.249,285$,
 and $x^8 = 0.204,414$, and consequently

$0.000,869 \times x (= 0.000,869 \times 0.82) = 0.000,712$,
 and $0.070,290 \times xx (= 0.070,290 \times 0.6724) = 0.047,263$,
 and $0.000,427 \times x^3 (= 0.000,427 \times 0.551,368) = 0.000,235$,
 and $0.017,927 \times x^4 (= 0.017,927 \times 0.452,121) = 0.008,105$,
 and $0.000,111 \times x^5 (= 0.000,111 \times 0.370,739) = 0.000,041$,
 and $0.008,905 \times x^6 (= 0.008,905 \times 0.304,006) = 0.002,707$,
 and $0.000,054 \times x^7 (= 0.000,054 \times 0.249,285) = 0.000,013$,
 and $0.005,557 \times x^8 (= 0.005,557 \times 0.204,414) = 0.001,136$.

Therefore

Therefore the compound quantity

$$\begin{aligned} & 0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^3 \\ & + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\ & - 0.000,054 \times x^7 + 0.005,557 \times x^8 \text{ will be } = \end{aligned}$$

$$\left\{ \begin{array}{rcl} & 0.000,712 & \\ + & .047,293 & - 0.000,235 \\ + & .008,105 & - 0.000,041 \\ + & .002,707 & - 0.000,013 \\ + & .001,136 & \end{array} \right\}$$

$= 0.059,923 - 0.000,289 = 0.059,634$; which is very nearly equal to, but somewhat less than, the absolute term, $0.060,332$, the difference being only $0.000,698$, or about the 86th part of the said absolute term $0.060,332$. Therefore 0.82 will be very nearly equal to, but somewhat less than, the true value of x in the said equation

$$\begin{aligned} & 0.000,869 \times x + 0.070,290 \times xx - 0.000,427 \times x^3 \\ & + 0.017,927 \times x^4 - 0.000,111 \times x^5 + 0.008,905 \times x^6 \\ & - 0.000,054 \times x^7 + 0.005,557 \times x^8 - \&c = 0.060,332. \end{aligned}$$

Q. E. I.

Art. 116. Having thus found the value of x , or the root of the foregoing equation, to be pretty nearly $= 0.82$, we will now proceed to apply this number to the determination of the magnitude of the rhumb-angle LAD . Now it appears from the Tables of Sines and Tangents, that, if the radius of a circle be called 1 , the co-sine of an arch of $34^\circ, 54'$ will be $= 0.820,151,9$, and the co-sine of an arch of $34^\circ, 55'$, will be $= 0.819,985,4$. Therefore the angle LAD , of which we have found the co-sine x to be $= 0.82$, will be very nearly equal to $34^\circ, 55'$. So that the said rhumb-angle, or angle of the ship's course, which we have been so long in search of, may now be concluded to be very nearly equal to 34 degrees and 55 minutes. Q. E. I.

And, because $\frac{d}{r}$ is $= \frac{564}{4000} = 0.141$, we shall have $\frac{dx}{r}$ ($= 0.141 \times x$ $= 0.141 \times 0.82$) $= 0.115,62$, and consequently $\frac{dx}{r} - 1$ ($= 0.115,62 - 1$ $= 0.115,620 - 0.086,875$) $= 0.028,745$; that is, the length of the arch DN of the meridian circle PND , or the latitude of the second point D , will be $= 0.028,745$; which may be converted into degrees and minutes in the following manner.

Since $0.086,875$ is the length of the arch AL , or the latitude of the first point A , expressed in decimal parts of 1 , or the radius of the Earth, and $0.028,745$ is the length of the arch DN , or the latitude of the second point D , expressed in decimal parts of the same radius, and 5° , is the length of the former arch expressed in degrees, it follows that the length of the latter arch DN ,

DN, expressed likewise in degrees and in decimal parts of a degree, will be a fourth proportional to the three numbers 0.086,875, 0.028,745, and 5°, and consequently will be $= \frac{5^\circ \times 0.028,745}{0.086,875} = \frac{0.143,715^\circ}{0.086,875} = 1.654,388^\circ$, or 1 degree and $\frac{654,388}{1,000,000}$ of a degree. But $\frac{654,388}{1,000,000}$ of a degree are equal to $\frac{60 \times 654,388}{1,000,000}$ of a minute, or to $\frac{38,263,280}{1,000,000}$ of a minute, or 38 minutes and $\frac{263,280}{1,000,000}$ of a minute, or 38 minutes and $\frac{26,328}{100,000}$ of a minute, or 38 minutes and $\frac{60 \times 26,328}{100,000}$ of a second, or 38 minutes and $\frac{1,579,680}{100,000}$ of a second, or 38 minutes and $\frac{157,968}{10,000}$ of a second, or 38 minutes and 15 seconds, and more than $\frac{3}{4}$ of a second, or 38 minutes and nearly 16 seconds. Therefore the arch DN, or the latitude of the second point D, will be equal to 1 degree, 38 minutes, and nearly 16 seconds. Q. E. I.

A Proof of the Truth of the Conclusions that have been obtained in the preceding Articles, by the Application of the foregoing Solution of the Third Case of Dr. HALLEY's Problem, to the Example given of it in Art. 103.

ART. 117. Having thus obtained the magnitude of the rhumb-angle LAD, and the length of the arch DN, or the latitude of the second point D, (at which the ship arrives after passing over the loxodromick arch AVD,) and having had the length of the arch AL, or the latitude of the first point A (from which the ship begins her voyage,) given us at first, we may now, from these three things assumed as known, determine, by the help of Lemma 4th, Coroll. 1, the length of the equatorial arch LN, or the difference of the longitudes of the two points A and D, which was before supposed to be given and to be = 324.904 miles, or, very nearly, 325 miles. And, if the value we shall obtain for the said arch LN, or the difference of the longitudes of the points A and D, by such determination, shall be found to be nearly equal to the magnitude of it that was given in the foregoing example, to wit, 324.904 miles, or 325 miles, the said result will be a confirmation both of the justness of the reasonings adopted in the foregoing solution, and of the exactness with which the several arithmetical operations, contained in the application of that solution to the preceding example, have

have been performed. This determination of the length of the said equatorial arch LN may be performed in the manner following.

It is shewn in Lemma 4th, Coroll. 1, that, if r be put for the radius of the Earth, and t be put for the tangent of the rhumb-angle LAD in the circle of which r , or the radius of the Earth, is the radius, and a be put for the logarithm of the ratio of the radius r to the tangent of the arch $\frac{PA}{2}$, taken on the axis, or asymptote, of a logarithmick curve, of which r , or the radius of the Earth, is the subtangent; and ϵ be put for the logarithm of the ratio of the radius r to the tangent of the arch $\frac{PN}{2}$, taken on the axis, or asymptote, of the same logarithmick curve; and e be put for the equatorial arch LN , or the difference of the longitudes of the points A and D ; we shall have e , or LN , $= \frac{t}{r} \times \overline{a + \epsilon}$. We must therefore now compute this quantity $\frac{t}{r} \times \overline{a + \epsilon}$ upon a supposition that the rhumb angle LAD is $= 34^\circ, 55'$, and that the arch AL is $= 5$ degrees, and that the arch DN is $= 1^\circ, 38', 16''$.

Now, since the angle LAD is $= 34^\circ, 55'$, its tangent t , in a circle of which r , or the radius of the Earth, is the radius, will be $= 0.698,042,2 \times r$, or (because r is supposed to be $= 1$,) will be $= 0.698,042,2 \times 1$, or $0.698,042,2$. And consequently the quantity $\frac{t}{r} \times \overline{a + \epsilon}$ will be $(= \frac{0.698,042,2}{1} \times \overline{a + \epsilon})$ or $0.698,042,2 \times \overline{a + \epsilon}$.

We must next find the values of the logarithms a and ϵ ; which may be done as follows.

The arch AL , or the latitude of the point A , is 5 degrees. Therefore the arch PA , or the complement of the arch AL to the arch PL , or to the arch of a quadrant, or an arch of 90 degrees, will be $(= 90^\circ - 5) = 85^\circ$; and consequently the arch $\frac{PA}{2}$ will be $(= \frac{85^\circ}{2}) = 42^\circ, 30'$; the tangent of which, in a circle of which the radius is called 1 , is $= 0.916,331,2$. Therefore a is the logarithm of the ratio of 1 to $0.916,331,2$.

The arch DN , or the latitude of the second point D , is $= 1^\circ, 38', 16''$. Therefore the arch pD , or the complement of the arch DN to the arch pN , or to the arch of a quadrant, or to an arch of 90 degrees, will be $(= 90^\circ - [1^\circ, 38', 16''] = 88^\circ, 21', 44'')$; and consequently the arch $\frac{pD}{2}$ will be $= 44^\circ, 10', 52''$, or, nearly, $44^\circ, 11'$; the tangent of which, in a circle of which the radius is called 1 , is $0.971,891,7$. Therefore ϵ is the logarithm of the ratio of 1 to $0.971,891,7$.

Further,

Further, the ratio of 1 to 0.916,331,2 is equal to the ratio of 1.091,308,5 to 1; and the ratio of 1 to 0.971,891,7 is equal to the ratio of 1.028,921,2 to 1. Therefore a will be equal to the logarithm of the ratio of 1.091,308,5 to 1, and e will be equal to the logarithm of the ratio of 1.028,921,2 to 1, taken on the axis, or asymptote, of a logarithmick curve, of which r , or 1, or the radius of the Earth, is the subtangent.

Now, if the radius of the Earth, or the subtangent of the said logarithmick curve, were called 0.434,294,481,903,251,827, &c, and consequently the logarithm of the ratio of 10 to 1 in the same curve were called 1, (as it is in BRIGGS's, or the common, system of logarithms, of which there are tables published,) the logarithm of the ratio of 1.091,308,5 to 1 would be = 0.037,947,5, and the logarithm of the ratio of 1.028,921,2 to 1 would be = 0.012,582,1; as will appear from a table of BRIGGS's, or the common, logarithms. Therefore, when the radius of the Earth, or the subtangent of the said logarithmick curve, is called 1, and consequently the logarithm of the ratio of 10 to 1, is called 2.302,585,092,994,045,684, &c, the logarithm of the ratio of 1.091,308,5 to 1 will be (= 2.302,585, &c $\times$ 0.037,947,5) = 0.087,377, and the logarithm of the ratio of 1.028,921,2 to 1 will be (= 2.302,585, &c $\times$ 0.012,582,1) = 0.028,510. Therefore the logarithm a will be = 0.087,377, and the logarithm e will be = 0.028,510; and consequently $a + e$ will be (= 0.087,377 + 0.028,510) = 0.115,887.

Therefore $\frac{r}{r} \times a + e$, or 0.698,042,2 $\times a + e$, will be (= 0.689,042,2 $\times$ 0.115,887) = 0.080,894; that is, e , or the equatorial arch LN, or the difference of the longitudes of the points A and D, will be = 0.080,894, or 80,894 millionth parts of 1, or the radius of the Earth, or = $\frac{80,894}{1000,000} \times 4000$ miles = $\frac{323,576,000}{1000,000}$ miles = $\frac{323,576}{1000}$ miles = 323.576, or 323 miles + $\frac{576}{1000}$ of a mile, or 323 miles and something more than half a mile.

Q. E. I.

Art. 118. This quantity, 323.576 miles, differs but little from 324.904 miles, which was the length of the equatorial arch LN that was given in the foregoing example. And therefore we may reasonably conclude, both that the reasonings used in the foregoing solution of this third case of Dr. HALLER's Problem are just, and that the numerous arithmetical operations, contained in the application of the said solution to the foregoing example, have been rightly performed.

Art. 119. The solutions which have been given above of the foregoing three cases.

Q

cases of Dr. HALLEY's Problem *, with the illustration of each solution by a suitable example, seem to contain a full and satisfactory answer to the inquiries of that learned writer. And therefore I shall here take my leave of this subject, and conclude the present Discourse.

JULY 4, 1795.

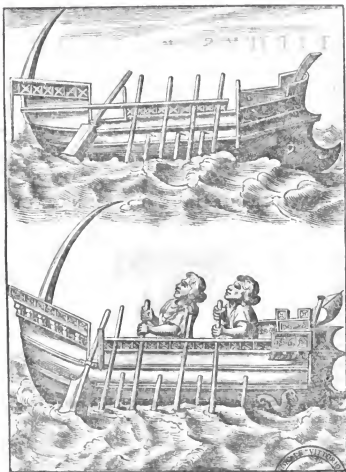
* I have called this Problem *Dr. Halley's Problem*, because I believed it to have been first proposed to the learned world by Dr. Halley; which he himself also, from the manner in which he mentions it, seems to have thought to have been the case. But the learned Dr. Andrew Mackay, of Aberdeen, L. L. D. (who has lately been honoured with a degree of Doctor of Laws by the University of Aberdeen,) has informed me by a letter dated from Aberdeen, May 13th, 1795, that Dr. Halley was not the first proposer of this Problem, but that it had been proposed before in the year 1639, in page 181 of a book published in duodecimo on the subject of Navigation, and intitled, *A learned Treatise on Globes, &c.*, written first in Latin by Mr. Robert Hues, and by him so published; afterwards illustrated with Notes by *Johannes Isaacus Pontanus*; and now lately made into English for the benefit of the unlearned, by John Chilmead, M. A. of Christ-Church in Oxford. London, 1639. And the words in which it is proposed in this book, are (as Dr. Mackay informs me,) these that follow, to wit, "*The difference of Longitude and the distance being given, how to find the Rumb and the difference of Latitude.*" After which the Proposer adds these words: *There is not any thing in all this art more difficult and hard to be found than the Rumb out of the distance and difference of Longitude given. Neither can it be done upon the Globe without long and tedious practise, and many repetitions and measurements.*

WILLEBRORDI SNELLII,
À ROYEN, R. F.
TIPHYS BATAVUS,
SIVE
HISTIODROMICE,
DE
NAVIVM CURSIBUS, ET RE NAVALI.



ALTER ERAT TUNC TIPHYS, ET ALTERA QUÆ VEHAŖ ARGO
DELECTOS HEROAS. ERUNT ETIAM ALTERA BELLA.

Edita primò Lugduni Batavorum, ex Officinâ Elzevirianâ, Anno Domini 1624; nunc
verò demùm impressâ in Officinâ JONÆ DAVIS, Londini, Anno Domini 1795.



P R Æ F A T I O

AD ILLUSTRISSIMOS POTENTISSIMOSQUE

HOLLANDIÆ ET WESTFRISIÆ

O R D I N E S.

ILLUSTRISSIMI ORDINES,

TYRUS illa, quæ olim partu clara, urbibus genitis, Lepti, Uticâ, & illâ Romani imperii æmulâ, terrarum orbis avidâ, Carthagine; etiâ Gadibus extrâ orbem conditis; ut ait Plinius: Tyrus, inquam, illa, quæ mercaturis & mercium exagogy faciundis omnia mediterranei maris littora complevit, et navigandi studio reliquas Phœniciæ maritimas urbes longè antecelluit: quæque sola, superatis terrarum angustiiis, per mare Erythræum elapsa, Asiæ et Africæ littora lustrare, et externa maria tentare, naturâ propemodùm reluctante et invitâ, ausa est: cui uni Rex Solomon, Isthmi illius dominus, classem suam auro, argento, aromatis, lapidibus pretiosis et gemmis reportandis, committendam censuit: Ea, inquam, Tyrus, universam Phœniciam novarum rerum, et artium investigandarum studio inflammavit. Huic enim Grammatica, in literarum inventionem; Arithmetica, ob numerorum usum; Astronomia, in navigiorum cursu regundo; Geographia, in terræ marisque delineatione; originem suam debent: ut reliquarum rerum plurimas, et humanæ vitæ opportunas, inventiones nunc missas faciam. Tyrii enim, ob mercimonia et commercia, quæ cum omnibus gentibus exercebant, ingeniis utilitate et emolumento ad abdita perferutandum excitatis, quam plurima in lucem protraxerunt. Quotiescunque igitur istud cum animo meo reputo, permulta in hæc nostrâ patriâ illis haud dissimilia adverto; quod tanto evidentius erit, si superiorum seculorum tenebras

tenebras cum hæc luce conferamus. Cum enim jam pridem Arabicarum et Indicarum mercium monopolion Alexandrini in Africâ et Asiâ possiderent, Veneti in Europâ; et (ut omnium rerum suæ sunt periodi, suus ortus et interitus) cum tandem aliquando Columbus in Americam expeditione susceptâ penetravisset: Lusitani quoque, vitatis longo circuitu terrarum excurrentium promontoriis, in Indian Orientalem, et insulas aromatiferas navigationem instituerunt. Inde paulatim accitâ Alexandrinorum portoria; et Venetorum uberrimus ille vestigalium proventus non parùm est imminutus. Omnemque hunc aded censum Lusitanæ reges in suum fiscum redegerunt. Atque isto successu et occasione totam propemodùm Europam multos annos vestigalem habuit Hispania: donec tandem ipse Novus Orbis (postquam jugi aded diri depudiisset) satis ita volentibus in vestris penetralibus Amstelodami, atque alibi, horrea aromataria et piperaria propemodùm ignaris vobis exstruxisset. Quâ opportunitate excitati, navigandi studio clauibus vestris maria omnia complevistis. Inde aded factum, ut homines nostri, quibus antea nihil esset inerti otio antiquius, jam omnia maria sollicitare, et deficiente terrâ in ipso Oceano errare gaudeant, terras ignotas quærentes. Atque hinc quoque tantus vestris urbibus splendor accessit, atque illarum artium cultura singularis et eximia emicuit, quarum præconio Ploenices tantoperè celebrantur. VOS enim, quamvis maximo et difficillimo bello, ab hoste potentissimo, terrâ marique distineremini, Academiam tamen Lugduni in Batavis erexistis, viris celeberrimis et doctrinâ excellenti undique evocatis, quorum operâ liberalia ingenia excolerentur; et profectò spem vestram eventus non est frustratus. Habetis enim ingenia ad omnem doctrinam percipiendam excitata: homines, inquam, non minùs gnavos, quàm ingeniosos, qui omnia sua studia, omnem operam ad Patriæ salutem et celebritatem conferre summâ contentione nituntur: quibus omnibus, ut in doctrinâ et ingenio priores partes non invitus tribuam; ita affectu, et animo in patriam propenso me cuiquam concessurum negabo; et, si non assequor quantum quidem optem, at saltèm, quantum sedulitate et industriâ consequi potero, id omne Deo et Patriæ sacrum lubens offeram. Utque hujus affectûs aliquod pignus extaret; non quidem illud vestrâ Amplitudine dignum, nisi ipsi non indignum esse voluissetis; cum rei maritimæ curam aded anxie et sollicitè geratis, ut etiam ultrò præmiis propositis omnes ad eam juvandam promovendamque inviteris. Vestrâ igitur autoritate, *Ordines Illustrissimi*, permotus, existimavi, me operam non inutilem aut instructuosam collocaturum, si navigandi leges accuratâ et non fallaci demonstratione instruerem; quod ipsum quoque videbam viros in hoc docto pulvere versatissimos semper omni studio esse conatos. Postquam enim

humanum

humanum genus Deo docente navigare didicisset, et cum ipse pontus littoribus careret; Noachus cum universi mundi deposito solis fluctibus vehebatur. Postea hominum industriâ etiam vela excogitata sunt, quæ ventos procellâsque reciperent, quorum impetu navis impulsâ curreret. Sed eadem frenum ei quoque iniecit, et clavo gubernatorem admovit. Ut adeo vasta moles exiguo gubernaculo regeretur, et pro vectoris arbitrio motus suos componeret; atque cursus suos ita inflecteret, ut etiam contra ventorum flutis obniteretur, et hoc pacto terrarum orbem ultrò citroque portaret. Eam adeo ob causam jam olim omnes tam sollicitè huic uni rei incubuerunt, ut cursuum ratio ipsis certa conflaret, et eos regundi scientia quædam extaret, cujus præceptis et ductu navem salvam in portu collocarent. Multò autem maximè proavorum nostrorum memoriâ, postquam immani et audacissimo ausu longinquæ illæ navigationes trans mare Oceanum tentari occæptæ. Tunc enim homines docti, et qui suprà cæteros saperent, id unice operam dederunt, ut certâ aliquâ ratione et viâ navigandi artem constituerent, quæ in eo tota maximè est sita, ut longitudinem et latitudinem cujusque loci habeant exploratam. Quarum altera ex observationibus cœlestibus haud admodum difficultè quotidie eruitur. Ad longitudinis autem investigationem nihil usquam extabat auxilii, aut præsidii. Tandem Magnes repertus, et ex ipsis terræ visceribus est erutus; saxum impositum id quidem, nullo splendore, opacum, asperum, scabrum, colore ferrugineo; sed cuius gemmæ contra comparandum. Hic est ille magnes, qui nostri misertus, tanquam officiosissimus *ἑοδὴγος*, miseros mortales, etiam vitæ suæ prodigos, per totum terrarum orbem salvos et sospites deduceret; et in ipsis tenebris, errantibus illis facem alluceret; ac noctū, interdū, tanquam nomenclator aliquis, omnium cursuum plagas referret, et fidi monitoris officium faceret. Atque illinc hominibus sagacibus cogitatio injecta, horum cursuum affectiones accuratius scrutandi. Unde illud continuò notatum, longitudinis et latitudinis investigandæ rationem perbellam hinc posse derivari; et ex cursûs plagâ, atque æstimatione ejus quantitate, istas quotidie ad abacum venire. In scrutandis igitur legibus ejus lineæ, quæ magneticæ acûs ductu à navibus describitur, istque accuratius explicandis, certatim à doctis pariter et indoctis integrum jam sæculum laboratur. Cum longitudinis inveniendæ via in medio mari nulla extet alia, præter eam unam, quæ acûs magneticæ indicio eruitur. Et prosequi omnes oræ maritimæ hanc in eadem jam diù tuditârunt; neque id privatim solum, sed etiam publicè. Sensit ea res et Vestrâ, *Illustriissimi Ordines*, quoque curam. Quoties enim liberalitatem vestram senserunt ii, qui loxodromiarum usum faci-

liorem

liorem feliciorémque proferre sunt conati? five id in tabulis planis, seu in gibbis sphaerarum segmentis explicarent: ut nullus quidquam hâc in parte rem maximam juvisse existimatus sit, qui non reapse sit expertus, VOS id omni sollicitudine et curâ eniti, ut ea quoque pars tam absoluta sit, quàm per naturam ei liceat. Quod adeò non indignum vestrà Majestate existimatis; ut etiàm ultrò omnium studia provocetis. Nec immeritò: cum tot hominum fortunæ rudentibus quotidie committantur, et in vallo illo Oceano natent. Ut enim navigandi industriâ nulli gentium concedimus, ita navigiorum multitudine et agilitate reliquis facillè pares sumus. Cùmque multi, laudando de patriâ benè merendi studio, aures vestras crebrò fatigent, et longitudinis rationes in mari explicas vobis exhibere gestiant: non existimavi, me quoque, aut patriæ, aut harum rerum cupidis deesse debere. Quamobrem ista omnia ab ipso fundamento solidis rationibus arcessere institui, ut inde quantum res ipsa in mari ferat, aut quàm propè ad verum accedat, definitè constaret. Dedi itaque operam, ut, summâ cum diligentia annotaris difficultatibus, quas vitari aut interpolare necesse sit, præcepta certa, brevia, factione expedita et parabilia, in medium proferrem, et omnia demonstrationibus firmata, usu observata, ordine et viâ constituerem. Quæ ut omnibus bonis et veritatis studiosis grata fore mihi est persuasissimum; ita hunc meum laborem vobis quoque, *Ordines Illustrissimi*, probari unicè optem,

Vestra Amplitudini

Additissimus,

WILLEBRORDUS SNELLIUS,

à Royen, R. F.

BENEVOLO

BENEVOLO LECTORI.

CUM supremi numinis providentia primam illam rerum omnium massam, sub quâ totius naturæ moles latebat indigesta, suis locis disparavisset; et hæc inferiora, sive vi magneticâ divinitus insitâ, seu ratione Isorropicâ libratis circâ id mundi punctum elementis, dissociavisset: Aer quidem ille spirabilis suprâ nos, et inde purius elementum, suas sibi sedes delegerunt. Ne autem illius molis nucleus, et reliquis gravior terræ globus aqueo humore proximè ambitus et circumdatus, terrenis animantibus esset inhabitabilis; ideò Divina illa mens, mundi architectatrix, Terram hanc diffregit; et montes quidem attolli, ac valles jussit subsidere: ut in his cavitatibus, tanquam sinubus, circumfluum et innaturum alioquin, humorem exciperet; aut in terræ visceribus concretum egestumque, in amplissimum æquor evolveret. Atque, ita positis diversorum elementorum limitibus, terrâ eminente, et aquâ depressâ, in eadem perfecti orbis specie, humanum genus alios alio disparavit. Inde necessitas, an libido, terras elemento adeò dissociabili disseptas lustrandi, mortales incessit animos. A quorum cupiditate et ingenio, ut neque aer, nec ipsius terræ intima penetralia tuta esse potuerunt: ita quoque non marini æquoris vehementia, non infani fluctus, nec infida vada, neque ventorum frementium impetus eorum conatus inhibere potuerunt; quo minùs regiones adeò immenso æquore disjunctas, mercibus ex alio Orbe, et ad luxuriam petitis, adirent; vel cælum ipsum petitori, si lucri odor ullus istinc adspiraret. Primis quidem sæculis, et sub ipsis mundi incunabulis, quamdiù mortales in locis mediterraneis ætatem agebant, ubi avitis laribus et fundis vitam parcè ac duritèr tolerabant,

De origine
artis Naviga-
torie.

— Non turbida ponti
Æquora lædebant naves ad faxa viròsque.
Sed temerè incaustum mare, fluctibus sæpe coortis,
Sævibat, levitèrque minas ponebat inaneis.
Nec poterat quenquam rapidi * pellacia ponti
Subdola pellicere in fraudem ridentibus undis.
IMPROBA navigii ratio tum cæca jacebat.

* Fortè le-
gendum soni,
placidi.

Phœnices primùm mercaturis et mercibus suis avaritiam, et magnificentiam, et inexplēbiles cupiditates omnium rerum importaverunt in Græciam, inquit Cicero: *ἀναγνώσκεις γόστια Αἰγύπτου καὶ καὶ Ἀσσυρίαν*. Et sanè ipsa gens Phœnicum in magnâ gloriâ literarum inventionis et fiderum, navaliumque et bellicarum artium, Phœnio

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R

fuit.

fuit. Et Herodoto quoque in suarum historiarum limine, Phœnices, inquit, qui à mari quod rubrum vocatur profecti, positis sedibus in eâ regione, quam nunc quoque incolunt, longinquis continuò navigationibus incubuerunt: faciendisque Ægyptiarum et Assyriarum mercium vecturis, cum in alias plagas, tum etiam Argos pervenerunt. Argos enim eâ tempestate omnia Græciæ civitatibus antecellebat. Huc porò appullos Phœnicias mercimonia exposuisse Persæ memorant: et quinto sextiôve quàm appulissent die, cunctis ferè diven-
ditis, foeminas ad mare venisse, cum alias multas, tum regis quoque filiam, cui nomen esset idem quod Græci tradunt, Io, filiam Ioachu. Dúmque hæ navi assistentes mercarentur quæ maximi expeterent, in eas unanimi consensu Phœnices impetum fecisse; et, plerisque earum fugâ clapsis, Io cum aliis aliquot cor-
reptâ, in Ægyptum retrò vela convertisse. Scilicet tu adhuc

Ultimus immenso restabas, Nîle, labori.

Verum navigandi artis gloriam, et fabricæ rationem antiquissimus Prometheus apud Æschylum sibi vendicat,

Θαλασσοπλοῦχτα δ' ὅτις ἄλλῃ ἐστ' ἰσὺ
Λυσίτηρ ἔρι ναυίων ὄχηματα.

Et profectò qui naves ad oras aliquas ignotas appulissent, et præter illorum littora vel remis, vel expallis velis novo et ante eum diem ipsis invisò vehiculo per maria discurrent, Barbarorum animos non minùs percelluerunt, quàm Argo pastoris illius animum apud Arctium, qui, cum navem nunquam ante vidisset, ut prociò divinum, et novum Argonautarum vehiculum stupens et perterritus conspexit. Ea quidem res adeò grata illis temporibus accidit, ut quicumque commodiorem velociorémque aliquam navigii speciem excogitavisset, vel expedi-
tionem in oras nulli priorum aditas instituisset, eorum industriam ad errorem fabulæ traduxerint; ut divinitùs naves hominibus datas, et Deorum auspiciis structas et fabricatas asseverare non dubitaverint,

— ipsamque secandis

Argois navibus * jactent sudasse Minervam.

* Fortè le-
gendum foret,
utribus, vel
ratibus.

Fuga Phryxi et Hellæ Arieti vectori χρονομάλλω imputatur, qui ob hoc benefi-
cium princeps inter Zodiaci signa retulget, et nobis ver aperit. Hæc omnes poetarum fabulæ personant, omnis scena fervet, et ab eo casu etiàm angustissimo inter Asiam et Europam freto nomen est impositum. Atqui dum Phryxus cum Helle sorore novæcalas insidias fugeret, ἰνὸχρῆς πλοῖον ἢ τὸ παρασήμενον χροῖς ἦν, navem nactus est, cujus insigne, et inde quoque nomen, Aries erat. Ille enim in prorâ vel pictus vel fictus stabat. Verùm quia non tantùm à Grammaticis, sed etiam à Philologis doctissimis de hoc navis insigni seu παρασήμενον multa prodita exstant, quæ nimis jocularia et perquam absurda sunt, operæ pretium erit hæc enucleatè explicantem Josephum Scaligerum audire. Παρασήμενον (inquit) erat in prorâ: et ἴνευ παρασήμενον, ibi statuebatur Deus, Heros, bestia, aut aliud quid, unde nomen habebat nav-
vis. In xxvii cap. Actorum Apostolicorum; ἡ πλοῖον παρασήμενον Διοσκύρου, ni-
mirum τὸ παρασήμενον τὸ πλοῖον ἐν τῇ πρώτῃ ἔχε τοὺς Διοσκύρους. Glossa juvenalis: Pe-
gasus. Trierarchi filius, ex cujus Liburnæ παρασά nomen accipit. Videtur accipere

De Insigni,
ἴνευ παρασήμενον,
navis in prorâ
posita.

accipere parafiam pro parafemo, si locus mendo non vacat, et Liburna ea habuit *παράσημον* Pegasum. At Tutela statuebatur in puppe, quæ differebat à parafemo dupliciter; tum quod illius locus semper erat in prorâ, hujus autem in puppe; tum quod tutela semper erat Deus, aut Dea: *παράσημον* erat Deus, aut Heros, aut animal, aut aliud quid. Denique navis à parafemo, non a tutelâ nomen habebat. Ovidius:

De Tutelâ
navis, in puppe
positâ.

Est mihi, sitque precor, flavæ tutela Minervæ
Navis, & à pictâ Casside nomen habet.

Nimirum Cassis erat τὸ παράσημον, et ab eâ Navis vocabatur Cassis. Minerva autem, cujus erat tutela, destituta erat in puppi, aut in aplustri. In primo Iliados: εὐταὶ γὰρ νῦν ἀπεκόψεν ἄκρα κόρυμβα. Scholion: ἐπεὶ ἐπὶ τῶν ἀκραγολίων ἦσαν ἀγάλματα καὶ εἰκότες τῶν Θεῶν. Hefychius: Πάτακται. Θεὸς, φρίκας ἔχων κατὰ τὰς πρύμνας τῶν νηῶν. Persius:

Ingentes de puppe Dei ———

Glossa vetus: Navium tutelam dicit, quam in puppibus habent, vel pingunt. Plutarchus: πλοῦταινοί τῆτι ναυκλῆρα τῶν νηῶν, καὶ τὰ κυβερνήτη, καὶ τὰς νῆας τὸ παράσημον. Nam ἀπὸ τοῦ παρασημον nomen habebat, ut illa, in quâ Paulus Apostolus navigabat, vocabatur Castores. Scio equidem de Tutelâ navium multos scripsisse. Sed tùm eos video non bene hæc distinguere, tùm etiam Grammaticos τὸ παράσημον cum Tutelâ confundere. Græci παράσημον expriment. Tutelam autem non habent, quo expriment.

Fucrit ergò Phryxæus Arias, non Aries, sed Navigii genus. De Tauro fabula haud est abîmilis:

Sic et Europæ niveum doloso
Credidit Tauro latus, et scatentem
Belluis Pontum, mediâsque fraudes
Palluit audax.

Contrâ Lycophron in obscuro suo poemate cecinit, Europam à Piratis in manus iraditam Aëtero Crætæ regi.

——— αἰχμάλωτον ἤμενρεσαν πόρει
Ἐν ταυρομόρφῳ τράμπιδος τυπώματι
Σαραπίαν δικάϊον ἐν ἀνάλτορον
Δαμαρῆα Κρήτης Ἀγρίῳ ἐρεχλάτῃ.

Id est, vertente Scaligero:

——— extulère captivam bovem
Paronis intrâ taurifrontis alveum
Ad sancta templa conjugem Saraptiam
Diētæa Cretes Stellioni principi.

Rapta igitur Europa Agenoris Phœniciæ Regis filia à Cretensibus navi, cujus παράσημον insigne Taurus erat; idque, quemadmodum ex Perfarum historiis Herodotus interpretatur, tanquam ἀνίστατον, hostimentum injuriarum à Phœnicibus

cibus prius illatarum. Ita naves vel à figurâ, vel à παρασκευή in fabulas et miracula vertuntur. Triptolemi beneficium in humanum genus singulare fuit, qui, glandibus repudiatis, frugibus velsi docuit, et iussu Cereris agriculturam intulit.

— Geminos dei fertilis Angues
Curribus admovit, frænsque coercuit oras:
Et medium cæli, terræque per æra vecta est.
Atque levem currum Tritonida misit in arcem
Triptolemo, partimque rudi data semina iussit
Spargere humo, partim post tempora longa recultæ.
Jam super Europen sublimis et Alida terram
Vectus erat juvenis, Scythicæque advertitur oras.

Et quæ sequuntur. Atqui Philochorus autor est, Τριπτόλεμον μακρὴν πλοῦν παραλάβοντα τῆς πόλεως τὸν εἶτον διαδόναι. ὑποτίθεται δὲ πτερωτὸν ὄντι εἶναι τὴν ναύα, ἔχον δὲ τι καὶ τὸ σχῆμα τῆς Τριπτόλεμου, longâ navi ad civitates maritimas appulsum, frumentum ἀστρίβουσαι; navem autem visam esse anguem alatum, quia hanc speciem utriusque suâ figurâ referret. Credo, rostra navium animalium quorundam rostra similitudine quâdam expressisse, ut inde etiâ rostra dicta sint; atque ita veteres rei gestæ veritatem fabulæ coloribus involvisse, et posteris per manus tradidisse. Haud alio argumento Dædali et Icarî fabula conflata est:

De fabulâ
Dædali et Icarî.
ii.

Dædalus (ut fama est) fugiens Minoiâ regna
Præpetibus pennis ausus se credere cælo,
Insuper per iter gelidas enavit ad Arcetos,
Chalcidicæque levis tandem astitit arce.

Sed iste quoque ἀποκρίναι μὲνεται ἐν ἱκάδῃ τῇ ὑπ' αὐτοῦ, πλοῦν λαβόμενον καὶ διαφυγόντων τὸν Μῖνον, διὰ τὸ ἀνθρώπον γινέσθαι τῇ δίακονί, fabulosè ἀνελαιψίτῃ narratur cum filio Icaro, quod, navem nactus, Minoem effugerit, cum persequentes eum non invenirent. Ego existimem illum non magnâ nave fugam sollicitasse, ne vel levissimam fugæ suspicionem Tyranno daret; sed singulos minoribus cymbulis corio undique contextis se commississe: quemadmodum barbari etiamnum hodiè propè Fretum Davis in Americâ his illigati ad umbilicum usque, et coriaceo invidētῃ parte corporis eminente texti, μονάκῃ remo uno, duabus palmulis instructo, navigant.

Non hi carinas quippe è pinu texere
Fecere morem, non abiete (ut usus est)
Curvant faselos: Sed rei ad miraculum
Navigia junctis semper aptant pellibus,
Coriόque vastum sæpe percurrunt salum:

De equo Pegaso et Bellerophonte.

Et Dædalum quidem ita insularum aut continentis littora propius legisse: Icarum verò longius provectum undarum impetu subversum periisse. Et Pegasus quoque et Bellerophon huc accedat: isto nempe equo ligneo per vias cæruleas vectus; ita enim Conon narratione suâ quadragesimâ apud Photium prodidit: ἡ τριταρετὴ ἐστὶν ἡ περὶ Ἀνδρομέδας ἱστορίᾳ ἢ ὡς ὁ Ἑλλάνων μύθος. Duos fratres

tres fuisse Cephea et Phineas : ac Cephei quidem regnum, Phœniciæ postea non De Perseo et Andromedâ.
 men induisse, cum ad eum diem Joppe à maritimâ civitate appellatum fuisset : habuissetque imperii terminos à mari nostro, ad eam usque Arabiam, quæ ad Erythræum mare pertinet. Fuisse item Cepheo insigni pulchritudine filiam quam proci ambirent, Phœnix quidam, et ipse Cephei frater Phineus. At Cepheum, postquam varia de utroque secum ipse considerasset, Phœnici tandem despondere filiam statuisse, ac per filiæ raptum consentientem suam voluntatem obtegere. Abreptam igitur fuisse ex desertâ quâdam insulâ Andromedam, in quam soleret Veneri sacrificia oblatura secedere. Ita cum Phœnicis hujus rapientis navi (quæ Cetus dicta est, sive à piscis illius similitudine, sive casu aliquo) Andromeda tanquam insciente patre aveheretur, miserè eam ejulasse, opemque cum lachrymis invocasse. Ad ea Perseum, Danaës filium, sorte fortunâ præternavigantem, cursum inhibuisse : primoque mox puellæ adspectu, pietate simul et amore captum : tum navigium illud Ceti nomine expugnâsse, et vectores, tantum non in lapidem præ metu versos, occidisse. Atque hunc illum esse Græcæ fabulæ Cetum, hos homines, Gorgonis viso capite in saxa mutatos. Et Palæphatus libro primo mirabilium historiarum, (qui solus è quinque hodiè superest,) eandem fabulam ad historiæ veritatem paulo aliter revocat. Βελλεροφόντης (φει) το φέργιος ἀνὴρ, τὸ γένος Κορίνθιος, καλὸς ἡ ἀγαθὸς, δι, πλοῦτος κατασκευάσας μηχανήν, διήκειτο τὰ παραθαλάσσια χώρα. ὄνομα τῇ τῷ πλοῦτι Πύργος, ὅς καὶ τὸν ἑκάστη του πλοῦτος ὄνομα ἔχει. Bellerophon, inquit, erat Phryx, domo, seu genere, Corinthius, vir formosus et strenuus, qui, fabricatâ longâ navi, à locis maritimis, prædas agebat : navi nomen erat Περγος ; quemadmodum etiam nunc dierum singula naves nomen aliquod sortiuntur. Ita quoque Troicis temporibus Tyrrheni piraticam exercebant, myoparonibus et triremibus suis mare infestantes, quarum triremium una nomine Scylla, ut reliquis vincebat celeritate, ita suprà omnes mercatoris navigiis insidiabatur : eam triremem, ut ait Palæphatus, Ulysses et celeritate navis suæ et secundo vento usus, feliciter effugit : καὶ καὶ τοὺς τεύχερα ταχῆα, τὸ τε ὄνομα Σκύλλα· αὕτη ἡ τρέρης τὰ λοιπὰ του πλοῦτος συλλαμβάνουσα πολλὰκις ἐπιδράγειτο βρώμα. Huc Poëtarum princeps spectasse videtur, cum lib. 5. Cloantho, qui Scyllâ vehabatur, victoriam tribuit :

— illa Noto citius volucrique sagittâ
 Ad terram fugit, et portu se condidit alto.

In his quidem hæc fabularum figmentis omnia fuerunt involuta. At ille, De navi Argo, et Argonautarum expeditione.
 qui primus omnium longâ nave maria tentavit, quantâ nominis celebritate in hominum sermonibus versatur ? quanto ambitu celebratur ? ut verè poetis fabularum Oceanum reclusisse videatur :

Utinam ne in nemore Pelio securibus
 Cæsa cecidisset abieгна ad terram trabes :
 Neve inde navis inchoandæ exordium
 Cœpisset, quæ nunc nominatur nomine
 Argo, quâ vecti Argivi delecti viri
 Petebant illam pellem inauratam arietis
 Colchis, imperio regis Pelææ per dolum.

Longâ

Longâ enim nave Iasonem primum navigâsse, Philostephanus autor est. Ait Plinius libri septimi capite sexto et quinquagesimo :

Benè dissepti fœdera mundi
Traxit in unum Thessala pinus,
Jussitque pati verbera pontum.

Fuit hæc *πρωτόπλοος* prima, in eam cum Iasone quatuor et quinquaginta Argonautæ sunt ingressi. Hæc nempe erat illa delecta heroum manus, *Θεῶν ῥαυτὴν ἔλκοντες*. In hanc unam flos totius Græciæ co-ivit : Hercules, Castor, Pollux, Diis geniti ; alique et reges et regum filii : Vates Mopsus, et Idmon : Naupegus Argus :

——— fors tibi, nequâ
Parte trahat tacitum puppis mare, fixâque fluctu
Vel pice vel molli concludere vulnera cerâ.

Celeustes ipse Orpheus,

——— qui carmine tonsas
Ire docet, summo passim ne gurgite pugnent.

Navis magister et gubernator Tiphys,

Qui maris insidias, claræ qui sidera noctis
Nôrit, et è clausis quem destinet Æolus antris.
Non metuat cui regna ratis, cui tradere cælum.

Denique navem ipsam fatidicam et humanâ voce locutam, fama est : habuit enim *ῥαῖπις ἐκ τῆς Διὸς ἑρμῆς*, carinam è Dodonide quercu ; aiunt enim Junonem, alii Minervam

——— Cæso, monitu Jovis, augure luco,
Arbore præfagâ tabulas animâsse loquaces.

Unde et Ciceroni divinum Argonautarum vehiculum dicitur. Itineris quoque stadiasmi, et cursus ita accuratè à poëtis proditi sunt, tanquam singulos ex ipsorum diariis expressissent ; per quæ maria, quibus ventis cucurrerint, quæ littora prætervecti sint, quos portus intrârint, quæ freta transferint, quæ promontoria et quorumnam fluviorum ostia præternavigaverint, ad quas gentes et nationes accefferint, quemadmodum in ultimum Euxini Ponti angulum penetraverint,

ἐς Θάειν, ἔνθα ναυτὴν ἔρχεται ἑρμῆς.

Denique quomodo ad Eridani ostia in Peuce insulâ in reditu hyemaverint ; reliqua lubens mitto. Et ista quidem hæcendûs *πρωτῶν* Euxini Ponti nobis universim ob oculos ponunt, et heroum duros labores quos in hæc expeditione pertulerunt ; quæ autem libri finis adeò difficilis expeditionis fuerit, ex Plinio facillè intelliges ; is enim libri tertii et trigésimi capite tertio eas oras auri feraces prodidit. Jam, inquit, regnaverat in Colchis Saluces et Esibopes, qui Terram virginem nactus, plurimum argenti aurique eruisse dicitur in Suanorum gente, et alioquin velleribus

velleribus aureis indycto regno. Sed et illius aureæ cameræ, et argentæ trabes narrantur, et columnæ atque parastatæ. Hinc adeo istis Heroibus sacra illa fames, et cupiditas, et ardor, et pericula, et lues, et diffidia, et cædes, et moræ atque ambitus itinerum. Hoc nempe illud erat ad cuius famam universa Græcia aures arrigeret; hæc illa præda, quam jam animo atque oculis destina-verant, quæque illos patriis sedibus excitos ad terram inhospitam mitteret. Sed fabularum satis est; omnia enim persequi infinitum sit. Quamobrem manifestò hinc liquet, singulas gentes mediterranei maris accolæ navium inventionem sibi vendicare conatas esse, aut singularia sua beneficia et pericula fabularum involucri postérité commendare studuisse. Et quo firmitus ea hominum animis infigerent, non puduit eos sideribus et cælo sua nomina inscribere. Illic enim nobis Aries, Taurus, Pegasus, Cete, Argo, navium nomina, totidem harum expeditionum testes resurgunt, quorum æternis ignibus et fulgore Phœnices, Cretenses, Corinthii, Thessali industriam suam pulteris testatis fecerunt. Et sanè, quàm certum est, ab his singulis navium alias atque alias formas ad expeditiorem cursum, aut onerum subvectiones, aut mercium exagôgas, esse excogitatas, quemadmodum

— etiàm quædam nunc artes expoliuntur,
Nunc etiàm auferuntur : nunc addita navigiis sunt
Multæ :

tam indubitatum quoque nobis est, primam navis fabricam à Noacho, Dei ipsius De arch No-jussu, ejus podissimum et spatia metante fuisse institutam. Cum enim humanum ^{achi.} genus à majorum suorum institutis discessisset, et, deserto divini numinis vero et legitimo cultu, ultimam improbitatis et scelerum omnium lineam transiisset, neque concionantem monentemque Noachum audiret : sed divini senis monitis insuper habitis, etiàm in dies in pejus ruerent, et ab omni virtute pedem retrò referrent : tunc demùm, tanquam malo inveterato, et rebus desperatis, Deus ipse, ut hanc improbitatem æternis tenebris damnaret, Noachum ad ^{avvnyias} instruxit, et quemadmodum unumquodque fieri, et mundi depositum custodiri, vellet, ostendit : quæ animantia in semen reservari : quòque singulorum generum in arcam admitti : ubi, et quemadmodum stabulari oporteret, edocuit. Seculum integrum duravit ea navigii molitio, quam à formâ ^{νεωτος} Arcam, dicimus. Corporaturam et arcæ modum Deus quoque ipse præscripsit. Trecentorum, inquit, cubitorum esto arcæ longitudo, quinquaginta latitudo, et triginta altitudo ejus; amplissimo sanè et maximo spatio; ut podissimum ejus è longitudine et latitudine major sesquijugero Romano deprehendatur. At forsàn hominibus quibusdàm nimium curiosi, quòque diviνα opera ad humani cerebri normam exigere nihil pudet, ista minus fidei habere videantur, si hanc molem ad nostri ævi navigia conferamus; et partium symmetria insuper haud satis ad istam consuetudinem apta. Atqui ludibria regum in mari lascivientium, qui triremes lusorias et cubiculatas exstruxerunt, satis superque silem facient. Hi enim, isto exemplo haud dubiò incitati, eam fabricam sunt æmulati: nam et hæc arca quoque à profanis scriptoribus literarum monumentis fuit celebrata; si quid Josepho credimus, qui Berosi Chaldæi, Hieronymi Ægyptii, Mnafeæ, Nicolai Damasceni, aliorumque complurium auctoritatem appellat. Atque hinc

factum

factum arbitror, quod Ptolomæus Philadelphus (qui studiosissimè munificentissimam bibliothecam, libris undique maximâ impensâ conquisitis, instruxit) ad æmulationem divini operis maxima navigia construxerit. Memoriz enim proditum est, eum omnes, qui ante se fuerant reges, navium copiâ et magnificentiâ superavisse; inter quas cæteris præstabant *τριαντήρης δύο, εὐκέρως μία*, nam aliarum numerus erat infinitus. Sed et Ptolomæus Philopator, hujus nepos, navigij magnitudine eum longè superavit. Unius autem mensuram δ

De magnâ
navi Ptolomæi
Philopatoris,
regis Aegypti.

Callixeno Athenæus ita expressit: τὴν τετραρακκλήτην ναὺν κατιστάσαν δὲ φιλονάτην, τὸ μὲν οὖν ἔχουσαν διακέρων ὀρθόγων τε καὶ, ὅκτ' ἂν καὶ τριάντα ἀπὸ πρὸς εἰς πᾶσιν, ὅκτ' δὲ εἰς ἀκροπόλιν τετραράκκλιαν ὅκτ' τε καὶ. Ptolomæus Philopator *navem quadrangulâ ordinem construxit, longam ducentes et octoginta cubitos, latam intrâ foros octo et triginta, altam ad acrotholion usque octo et quadraginta, à puppi autem summâ ad mare usque cubitos tres et quinquaginta*; cùm in mari cursum expediret, recipiebat remiges quater mille et amplius, miniâs quadringentos, milites classarios serè ter in mille, præter aliam hominum turbam sub transtris, et præterea commeatum maximum; materies ad hanc unam comportata satis fuisset faciendis triremibus quinquaginta. Idem Philopator aliam quoque locavit faciendam, quæ in Nilo navigaret, *ῥαλόμενον* cubiculatam, longitudine dimidii stadij, qui sunt pedes trecenti, latitudine cubitorum triginta, altitudine serè quadraginta; reliquum porro ornatum, partium singularum dispositionem et commensum ibidem video. Verùm has omnes faciliè superaverit Hieronis Syracusarum tyranni navis illa, *ἡ Ἀρχιμήδης ἡ γυναικὴς αἰ ἐνὲν ἡν*, cui faciundæ

De magnâ
navi Hieronis,
Syracusarum
regis.

Archimedes erat præfectus; ejus molis amplitudinem et fabricam Moschion integro libro huic rei dicato explicavit; trecentis enim operis fabricam, præter administratos, tractantibus, ad eam faciendam sexaginta triremium materia fuit congesta; et navis corporatura ita intrâ distincta, et in suas sedes tributa, ut remigibus, nautis, militibus, vectroribusque reliquis recipiendis cuicque sua loca, et sua receptacula essent assignata. In hac eadem navi erant præterea varia cœnacula, cubacula, ambulationes, horti, piscium vivarium, equilia, Scholasteria, Veneris templum, balnea, clibani, culinæ, molæ, et alia compluria; et hæc intrâ quidem. Insuper navis ipsa ferreo vallo circumdata; et turribus octo, duabus in puppi, totidem in prorâ, reliquis autem in lateribus dispositis munita; super navi foris murus erat cum propugnaculis; et in muro, turribus, ac *εὐκαρχειῖς* per totam navem multæ machinæ bellicæ erant dispositæ; et inter reliquas una, quæ saxum trecentarum librarum, atque sagittam duodecim cubitorum ad sexcentorum pedum intervallum vibraret; et ne omnia nimis scrupulosè persequi videar, lectorem ad Athenæum remittam, unde plura ad satietatem et stuporem usque de hac eadem navi sigillatim tibi petere licebit; et quidem ita, ut nobis non navis pictura, sed castelli, aut urbis, vel insulæ alicujus descriptio ante oculos poni videatur:

— pelago credas innare revulsas
Cycladas, aut montes concurrere montibus altos:
Tantâ mole viri turritis puppibus instant.

Hanc tamen Syracusarum *εὐκαρχειῖς* Romani etiâ superavisse videntur fabricatis navibus ad obeliscos ab Alexandriâ ad Ostiam usque tranvehendos. Super omnia (inquit

Plinius lib. 36, cap. 9.) accessit difficultas mari Romam devehendi, spectatis admodum navibus. Divus Augustus priorem advexerat, miraculisque gratiâ Puteolis navalibus perpetuis dicaverat; sed incendio consumta est. Divus Claudius aliquot per annos asservatam, quâ Cajsus Cæsar importaverat, omnibus quæ in mari visæ sunt mirabiliorem, turribus Puteolano ex pulvere exædificatis, productam Ostiam, portus gratiâ merfui. Rem navalem de navium generibus, partibus, ornatu, armamentis satis spisso volumine profecutus est Lazarus Bayfius, vir doctissimus; sed, ut ipse sibi met haud usquequaque satisfecisse videtur, ita non pauca ad accuratiorem judicii limam studiosus, credo, lector exigi postulabit. Maximè in biremium et triremium descriptione, et illis quæ triginta, quadraginta, aut quinquaginta ordinum altitudine confurgunt. Sed cum et hanc controversiam felicissimâ et eruditâ manu explicaverit vir summus, et verè unicus nostri seculi Phœnix, Josephus Scaliger, totum locum, atque adeò illustrem illam eclogam huc transcribere, in tuam gratiam, non pigebit: dignus enim est, qui non uno tantum in loco legatur.

Igitur prima triremis, sive *τρίηρης*, (inquit) Corinthi fabricata est. Antè enim longis navibus tantùm utebantur. Prima longa navis fuit ea, quæ Argonautas vocit, ut Plinius, et alii ferè omnes veteres scribunt. Nos hodiè id genus *Galeas* vocamus, et primùm illud nomen obtinuit adhuc vigente imperio Constantinopolitano. In Taclicis enim legimus, *Γαλαίας μονήρια*. Nostre enim Galeæ, et veterum naves erant *μονήριες*, hoc est, uno versu remorum agitabantur. Quemadmodum enim *πύργος* *μονήρης*, *διήρης*, *τρίηρης*, à numero tabulatorum dicitur, qui aliter *πύργος* *μοιόγυος*, *διόγυος*, *τρίγυος*, ita et *ναὺς* *μονήρης*, *διήρης*, *τρίηρης* dicta, quæ totidem, ut ita loquar, tabulatis constabat, ac proinde totidem utrinque ordinibus remorum. Quum igitur differentiam navium longarum constituat numerus ordinum, primæ in usu fuerunt, quæ unum tantùm remorum versum habebant. Duplex verò earum constituitur differentia, à numero remorum, et à numero ordinum. Quæ à numero remorum dicuntur, earum nomen terminatur in *ερος*, *ιούσπερος*, *τριακόνταρος*, *τετραρακόνταρος*, *πεντακόνταρος*. Quæ vero à numero versuum, in *ηρης*, *μονήρης*, *διήρης*, *τρίηρης*, *τετρήρης*, *πεντήρης*. Vetus auctor Tacitum ad calcem Æliani: *ἡ τριακόνταρος καὶ τεσσαρακόνταρος, καὶ πεντακόνταρος λέγεται κατὰ τὸ πλῆθος τῶν κινήσων, ἢ μονήρης, καὶ διήρης, καὶ τριήρης, κατὰ τοὺς σίχους τοὺς κατὰ τὸ ὕψος ἐν ἀλλήλοις*. Sed scito, citrà *ιούσπερος* non posse componi nomen: hoc est: non dicebatur quot remorum, sed quot scalmorum esset, siquidem minus haberet, quam vicenos remos. *πεντακαίδεκάσκαλμος*, *δωδεκάσκαλμος*, ut *δωδεκάσκαλμος* πάντων Straboni. Est enim *σκαλμός*, paxillus, ad quem alligatur remus. Quare aliâ mente, ut multa, Æschylus Persus *ὅτις τρέσκαλμοι* pro triremibus dixit: quum *τρέσκαλμος* *ναὺς* tantum dici possit quæ trinis remis agitur. Erat et alia differentia, ut naves citrà magnitudinem alicujus, *εἰ* dicerentur *ἡμιόλια*, ut quæ inter magnitudinem *τριακόνταρος*, ea dicebatur *ἡμιόλιος* *τριακόνταρος*, major quidem quàm *τριακόνταρος*, minor verò quàm *τριακόνταρος*. Sic quæ citrà magnitudinem *τριηρης*, ea dicebatur *τρίηρης* *ἡμιόλια*. Nam τὸ *ἡμιόλιον* continet assen cum semisse. Ideo *τρίηρη* *ἡμιόλια* erat magnitudine *τῆς* *διήρης*, et præterea aliquanto amplior. Neque verò necesse erat exactè dimidium accedere, non magis quàm in semitonio musico. Tonus enim bifariam non dividitur. Quare ut dicitur *ἡμιόλιος* *πῶς*, sic eadem

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S

con-

Excerptum
ex Josephi
Scaligeri Li-
bro de Navi-
bus Veterum.

construptione ἡμιολία τρίτης, sed non eadem ratione. Nam potius dici debuerat διπλῆς ἡμιολία, utpote quæ magnitudine διπλῆς esset, aut aliquanto major. Sed usus obtinuit, ut potius vocaretur ab eâ, à cuius magnitudine proximè abesset, quam ab illâ quam non solum æquabat, sed etiâ aliquâ parte superabat. Hæ igitur naves ἰσίοτεροι, τριακόσιοι, πεντακόσιοι, solæ Græcis in usu primum fuerunt, ad illud usque tempus, quod hic ab Eusebio determinatum, earumque ἡμιολία.

Huius generis inventum Sidoniorum est. Græci ante expeditionem: Jafonis Colchicani id genus non tentarunt. Unde ejuscemodi naves Σιδωνία Græcis dicebantur. Euripides in Helenâ :

Χάρη σὺ, καὶ γὰρ τοῦδε πιστοκότερον
Σιδωνίαν δὲ, κρήνην ἐπιβάτας.

Fas diximus esse, quas Græci μακρὰς νῆας, nos *Galeas* vocamus: qualis Jafonia, quæ πεντακόσιος fuit, ut ex numero Argonautarum constat. Nam ipsi Heroës fuerunt ἀντίεταί, ut omnes sciunt. Unde Sereno Poëtæ Hercules dictus Semi-emex Hercules: quod remex quidè inter Argonautas esset, sed imperitus, et ei muneris ineptus. Alii plures Argonautas numerant. Ideò μείζονα πεντακόσιον τὸν Ἀργὸν fuisse necessarium est. Quare Theocritus in Hylâ τριακὸνδύζον Ἀργὸν dixit: hoc est, ἑξακότερον, utrinque tricenis remigibus impulsam. Earum navium, quæ pluribus remorum verbis agebantur, videtur Trieres prima fuisse quæ dicta, ut monuimus, à numero tabulatorum, καὶ τὸν εἶχον τοὺς κατὰ τὸ ὕψος ἢ ἄλλως, ut ait ille vetus Scriptor Ταχικῶν: secundum versùs alium alii juxta altitudinem incumbentes. Itaque imus ordo erat propior aquæ, supremus propior foris navis, medius inter imum et supremum. Remiges imi versùs θαλαμῖται dicebantur, medii ζυγῖται, supremi θραυῖται. Eadem etiam navigia dicuntur μέγιστα, διὰ τὴν, τρίηρα. Biremis, sive διπλῆς, vel διπλῆτα formam ex veteribus marmoribus exhibet Bayfius, et fatetur, eam esse veram Biremem veterum, in quâ superior ordo ternis, inferior binis instructus est. Deinde alias duas Biremis formas profert, quarum uterque ordo quaternis remis constat. Eam, ut diximus, veram Biremem credit esse. Sed (vide desultoriam sententiam) propositâ etiam formâ veræ Triremis, ex veteribus Romæ monumentis, eam negat esse Triremem. Certè si altera, ut ipse fatetur, est Biremis, quæ duplici remorum versù instructa est; hæc ab eadem causâ Triremis fuerit, nulli autem medii. Neque enim necessarium est ut ζυγῖται adsint; sed cum sunt plures ordines duobus, semper medii sunt ζυγῖται, infimus et supremus ordo sunt semper θαλαμῖται et θραυῖται. Si major pars Romanorum, prænomen, nomen, et agnomen habebant, non idè sequitur, ut omnes habeant agnomen. Quid ergo adfert novi ad illam objectionem? Remi ridiculam ac tanto viro indignam. Dividit navem in tres partes: à prorâ, ad malum: à malo, ad initium ἀκρογῶν: inde ad summum ἀπὸ τῆς ἀκρογῶν.

Lazari Bayhi
errorum super
hâc materiâ.

ἀσπογόνων ζυγίτας : inde, ad summum aplussire, θαλαμίτας. Si exemplum Aretalogorum, qui pro commentariis Tomorum decurias edunt, sequi vellem, neque temporis, neque chartae satis esset tam puerili commento confutando. Sed omnibus ineptiis, auctor θαλαμῶν dixit, κατὰ ῥόχους τὰς καλὰ τὸ ὕψος ἐστ' ἀλλήλους ; non dixit, κατὰ μῆκος. Quid melius ejus commentum eluserit ? ῥόχοι κωπῶν aliud est quam κῆται, et κατὰ τὸ ὕψος, quam κατὰ τὸ μῆκος. Vegetius de quinquereimis : *Quinque remigum gradus*. Si gradus, ergo in altum, non in longum. Virgilius :

———— triplici pubes quam Dardana verflu
Impellit, triplici confurgunt ordine remi.

Sanè confurgentes remorum ordines dici non possunt de longitudine ; longitudo jacet, altitudo confurgit. Idem dixit Lucanus de Quadrimem, lib. iii :

Quósq̃ue quater surgens extructi remigis ordo
Commovet, et terno confurgunt ordine remi.

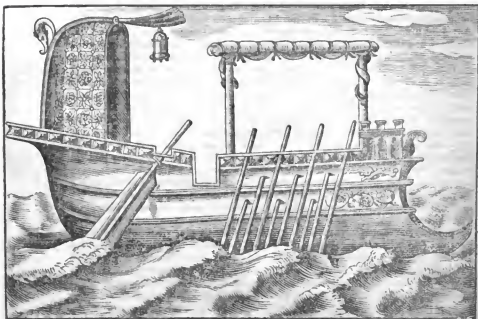
Extructos remigum ordines non possumus accipere in porrectum. Hoc enim non est extruere. Illustis locus ille est in excerptis Memnonis apud Photium : καὶ ὀκτὴς μία ἡ λευκοφόρος καλαμῆς, ἡ μεγάλῃ ἵνικα καὶ κάλλει ἕκαστα εἰς θαῦμα. ἐν ταύτῃ γὰρ ῥ' μὴν ἄνθρωποι ἐκείνοι οὐκ ἔπρεβον, ὥς αἱ ἐκ θατέρῃ μέρους γινέσθαι, ἔξ ἑκατέρων δὲ χιλιῶν καὶ ἑξακοσίων. Octaves dicta ab octo utrinque remigum versibus, centenis remigibus in unoquoque versu : ut essent in universum bis octingenti remiges, hoc est, mille et sex centum. Quid diceret hic Bayfius ? Octingentos homines in porrectum esse ?

Aut quid si non hoc ? Sed illarum navium prodigiosa moles. Ad Triremes, or- De Triremi-
dinarium et usitatissimum navigium, revertor. Thranitarum, id est, supremi ordi- bus.
nis ; remos longiores esse, quàm Zygitarum, et horum, quàm Thalamitarum, ratio convincit. Thalamiæ enim erant propiores aquæ, inde eorum remi κολοβαὶ erant et non longi. Lucanus in descriptione τῆς ἑξέως : *Summis longè petit æquora remis*. Intelligit θραυδικὰς κώπας, quas omnium longissimas fuisse necesse est. Itaque οἱ θραυταὶ majus stipendium percipiebant, quam Thalamiæ. Scholia Βαστράχων' οἱ δὲ θαλαμῆες ὀλίγον διακρίσαντες μισθόν, διὰ τὸ κολοβαῖς χρῆσθαι κώπαις παρὰ τὰς ἄλλας (τερεῖς) τάξεις τῶν ἱερῶν, ὅτι μᾶλλον ἰσχυρὸς εἴσι τὰ ὕδατα. ἦσαν δὲ τερεῖς τάξεις τῶν ἱερῶν, καὶ ἡ μὲν κάτω θαλαμῆται, ἡ δὲ μέση ζυγῆται, ἡ δὲ ἄνω θραυταὶ. Haec tenè bene. Reliqua, quæ sequuntur, sunt illis optimis Scholiis infarsa ex sæce sequioris seculi. Θραυτὸς οὖν ὁ πρὸς τὸ πρῶτον, ζυγῆς ὁ μέσος, θαλαμῆς ὁ πρὸς τὸ ὕψος. Ad hoc præclarum oraculum Bayfius pellibus incubuit stratis, somnolique petivit, unde illud somnium suum protulit de nave in porrectum trifariam dividendâ. Illa quoque vox (τερεῖς) inclusa virgulis delenda. Hæc sunt quæ de triremibus, et aliis veterum navibus in tabulata confurgentibus, breviter ac pro loco et temporis angustiis dicere potuimus. Duo sunt quæ satis mirari non possum ; primum, quomodo singuli homines tam longos remos moliri possent, quum in nostris Galeis aliter fieri videamus. Nam in illâ immensi oëtere, cuius Memnon mentionem facit, singuli homines tantum remis adhibebantur. Et tamèn quo plures erant versus, eo longiores supremè ordinis remos fuisse necesse est. Alterum, cuius non mediocris nobis incescit admiratio, est moles Hexerum, Decerum, et, ut Plinius scribit, etiam illarum quæ ad 15, 20,

30, 40 et 50 ordines remorum constructæ sunt. Quomodo, inquam, eas in bello moliri possent? Unde tam longæ arbores, quæ Thrancitarum remis sufficerent? Virgilius quoque, quas in pugna Actiacâ egit Antonius, Cycladibus comparat. Idem etiâ scribit Dio. Si venum erat, quod Bayfius somniavit, remos in porrectum expansos, non in altitudinem extructos fuisse, quo major remorum numerus foret, eo expeditior fuisset Triremis Bireme, Tetreres Triere, Penteres Tetrere; atqui contrarium apparet ex Floro, libro 4. de illis Cycladibus Antonii et Augusti in classe Actiacâ: Nobis quadringentæ non amplius naves, ducentæ, non minus, hostium; sed numerum magnitudo pensabat: Quippe à fenis in novenos remorum ordinibus, ad hoc turribus, atque tabulatis allevatæ, castellorum et urbium specie non sine gemitu maris et labore ventorum ferebantur; quæ quidem ipsis moles exitio fuit. Cæsaris naves à triremibus in fenis, non amplius, ordines creverant. Itaque habiles in omnia, quæ

Finis excerpti
ex Josephi
Scaligeri li-
bro.

Biremium duarum et unius triremis quarum hic fit mentio, et quæ omnino veræ sunt effigies, huc ex Bayfio transtuli, ut inde facilius navium pluribus ordinibus surgentium molem et altitudinem animo complecti, et cogitatione asse-



qui

qui possis. Unum illud quod Scaliger miratur, quomodo singuli homines tam longos remos moliri possint, ex Athenæo discēs, in descriptione navigii à Philopatore Ptolemæo constructi ordinum quadraginta. Κάρας δὲ Θρασιῶδες ὄντων τῶν τεσσάρων πηχῶν τὰς μίγας, αἱ δὲ τὸ μὲν ἔχον ἐν τοῖς ὑγερύτοις καὶ γρηνοῖσι λίαν ἰσὺν βαρύνει κατὰ τὴν ζήγαν, ἐκείνης ὑπὲρχειν ἐν τῇ χεῖρτι. *Remos autem in supremo thranitarum versu habebat longissimos octo et triginta cubitorum, tractatu et remigio in usu faciles, quia ob plumbum manubriis additum introrsum graves essent; quare pondus hoc, tanquam in librili, cujus hypomochlium in scalmio sit, expeditum præstitit remorum usum. Neque tamen inficias iverim, haud absurdum mihi videri, binos remiges longissimis his remis admotos. Singulos quidem certè remum suum duxisse in longâ navi, ex Argonautarum hîstoriâ discere est.*

Dant remo sua quisque viri, dant nomina transtris.
Hinc lævum Telamon pelagus tenet, altior inde
Occupat Alcides aliud mare, cetera pubes
Dividitur. —

Verùm in tantâ mole et tot ordinibus exstructâ illud non satîs affequir, quomodo non altius multo assurrexerit ordo quadagesimus: ut thraniæ summi remo octo et triginta cubitorum, cujus pars non exigua inrâ navem versaretur, mare potuerint suis palmulis ferire. Quod ut clarius sit, posito abaco calculum exsententiâ subducam; hominis sedenîs altitudo ab offe sacro ad summum capitis verticem, si modicè sumas, tribus pedibus Romanis non est minor: atqui ut affurgas, et totis viribus remum adducas, id fieri nequit, si in solo cruribus porrectis sed eas, sed aliquâ altitudine sedilis opus est: hinc igitur pes unus accedat positæ altitudini, fiunt pedes quatuor; tantoque, ut minimum, intervallo transtrorum altitudo disjungi debuit. Cerè si quatuor pedes quadragies sumas, constantur pedes centum sexaginta, quo intervallo thalamitæ remiges infimi à thranitis, qui in summo remigabant, inter se essent disjuncti; et propterea altitudo, quæ à foris, et summis lateribus navis ad mare pertingeret, longè adhuc major. Atqui constat, hanc navem à summâ puppe ad mare usque, tantùm fuisse altitudine cubitorum trium et quinquaginta, in lateribus autem multò minore. Ipse Athenæus ab acrostolio octo et quadraginta prodidit; qui sunt pedes tantum duo et septuaginta; ut neutiquam dimidiam habuerit eam altitudinem, quam nos suprâ constaveramus. Thalamiæ quidem certè aquæ propiores brevioribus utebantur remis, et idè pars remi interior à scalmio ad manubrium minor; et ipsi in transtris, navis lateribus admodum vicini, quantòque uniusquisque ordo altius assurgeret, tantò longioribus remis opus erat, et ad eos molîendos longiore radio, et à scalmis distantia; ut hoc ordinum recessu aliquid loci accesserit inferioribus, et inferioris remigis vertex absque impedimento altitudine pectus proximè superioris ordinis æquare potuerit; maxinè verò cum alternis ordinibus ita remi sint ordinati, ne omnes ad lineam per sua columbaria, tanquam ad perpendicularum exerti sibi mutuò essent impedimento: sed alternis, ut in possessionem hoc pacto, quasi in vacua maris parte mitterentur: sîque ita in plurium ordinum navibus accuratè observatum, verò videtur non abfimile; quouique remus superior et longior, inferiori et bre-

Conjecturæ
de remigum
in diversis or-
dinibus posi-
torum distan-
tiis à se invi-
cè.

viori,

viori, post tertium aut quartum ordinem ampliùs impedimento esse non posset. Et navis quidem ipsa hoc pacto tanto remigio, et tot navalibus pedibus concitata, per mare curreret. Illud verò etià ad hujus nodi solutionem palmarium mihi videtur: quantumnam ejusdem ordinis remigum, ipsorum inter ipsos intervallum fuerit: ea enim res percepta omnem caliginem amolietur. Ego quidem illud observavisse mihi video, transtra non in omnibus navigiis eodem intervallo disita: sed alibi longiùs, alibi propiùs, à se mutuò abfuisse. Idque duarum argumento ostendam. Philopatoris quadragintarenis à remigibus quater mille incitabatur; qui numerus per quadraginta divisus dabit centum remiges; et idè in singulis lateribus quinquaginta in porrectum. Atqui hoc navigium erat 280 cubitos longum, qui sunt pedes 420: hinc ob carinæ, et tropidis curvaturam in prorà et puppe deducantur 60, reliqui 360 per numerum remigum 40 divisi, dabunt singulis spatium moliendo remi manubrio pedum novem: cum corporis et brachiorum agitatio in adducendis remis quatuor, aut quinque pedum interscalmio duntaxat opus habeat; ut intermedium hoc spatium ordini proximè inferiori lucro accedat. Atque ita in sequentibus porro, alternis vicibus intervalla paria aptando, ut suprà in expressà hac triemi. Atqui quod omnem fidem excedat, neque satis ipse deminari possum; certum est, thalamitis à thranitis, imos, inquam, remiges à supremis, in hac tesseraconteri minùs quadraginta pedum altitudine fuisse disjunctos. Idque ex posità remorum longitudine ita difces. Concipiamus enim triangulum rectangulum æquicrurum, cujus hypotenusa, seu latus recto angulo subtensum, sit ipsa remi exerti longitudo; altitudo verò sit distantia à scalmio ad mare. Constat enim, minore ad navis latus angulo remi molitionem fore nihil molestam, et minùs tractabilem. In hoc igitur triangulo quadratum à crure uno, est dimidium quadrati ab hypotenusa; remi autem longitudo quidem univèrsa datur cubitorum duo de quadraginta, qui sunt pedes septem et quinquaginta: atqui pars remi à scalmio ad manubrium, ut commodè agitari possit, aliquà longitudine esse debuit, et ad reliquam partem exertam rationem habere tolerabilem. Esto itaque ea pars pedum tantum septem, ut quàm parcissimè sumamus; reliquum igitur à scalmio à l. mare fuerit pedum quinquaginta; cujus quadratum sit 2500, dimidium 1250, latus hujus dimidii est minus sex et triginta pedibus. Ita singuli ordines, cum ipsà transtri crassitie, vix unius pedis intervallo fuerint disjuncti: ut nefas sit credere, hæc transtra à remigibus occupari potuisse, nisi forsàn eos, tanquam haleces et salimenta, stipatos intelligas. Jam reliquum porro spatium suprà thranitis ad navis fores, in triclinia, cubicula, et alios usus tributa, ac distincta, non erit dubium ei, qui Athenæi verba paulo diligentius examinare animum induxerit. Neque adeò ex his difficultatibus et angustiis, in quis remorum mensura nos præcipitavit, nullà ratione extricare possumus, nisi illud concedamus, ordines imos rarioribus remis incitatos, summos pluribus; et præterea aliquam partem recessus superiorum concedamus, qui longioribus manubriis suos remos molirentur. Et hæc vià illud eflici video; has quippe moles duntaxat ad ostentationem, nulli usui fuisse institutas; quod utique verissimum est. At in Memnonis apud Photium excerptis, ceteris Ptolemei Cærauni faciliorem habet explicationem, ubi numerus remigum etià longè major

est,

est, centum nempe in versum: fuitque hic remigum numerus, illius duplus sesquialter; et ideo transrorum in altitudine distantia paulo fuerit major: namque hic ob multitudinem remigum, inferiores superionibus omnino subijci potuerunt. Si in hac octeri distantia transrorum in singulis ordinibus, et eodem versu tanta fuisset quanta in illâ tessaraconteri, longitudo huius octeris faciliè 900 pedes exæquasset, et plus quam dupla fuisset illius; quod nemini sano in mentem venire posse crediderim. Quamobrem ut huius longitudo minor quidem fuerit, quam illius quadraginta ordinum, ita distantia inter thalamitas imos et thranitas supremos, non fuerit multò minor quadraginta pedibus, ut inter singulos ordines quinque pedum distantia intercedat. Sin autem minus assumas in singulis, et huic summæ ideo aliquid quoque deceder. At illa tessaraconteris tantùm ostentationis gratiâ facta videtur, cum vix loco suo moveretur, sed tantum ædificium suis sedibus affixum hæreret. Magis mihi placet Archimedis industria, qui tantùm vigintirem in maiore navigio construxerit, quæ aded vastâ mole etiam concinnius expeditiusque per mare curreret: nam Syracusis Alexandriam usque, onusta frumento, Ptolomæo regi dono missa est. Verùm, ut diximus, ista ludibria tantùm fuerunt regum in mari lascivientium. Atque hæc quidem de inventoribus, magnitudine, et formâ navium nunc utcunque dixisse sufficiat. Superest ut doceamus quemadmodum cursum in medio mari regere instituerint,

De modo navigandi, seu cursum navis dirigendi, apud Veteres.

ἄλογα Νηγίης πλοκὸς
τήνοντες, ἀγέῃ πύρον·

ubi nulla pedum apparent vestigia, quæ secuti optatum iter tenerent, et ad portus destinatos pervenirent. Certè satis constat, istos, qui primi maria tentare ausi fuerunt, ei se timidè commississe; et tantùm littora legere, neque è terrarum conspectu longiùs usquam discedere solitos: ut nocte ingruente in portum se reciperent, vel in ipso aded littore quiescerent.

Sol ruit intereà, et montes umbrantur opaci.
Sternimur optatæ gremio telluris ad undam
Sortiti remos, passimque in littore sicco
Corpora curamus: fessos sopor occupat artus.

Quem morem etiamnum hodiè maris mediterranei accolæ usurpant, ut ingruente tempestate sese versùs littora recipiant: cum longè satius sit tunc navem alto mari committere, ubi nulla fluctuum agitatio reciproca, quibus naves quassantur et dissolvuntur. Postquam verò etiam animis ad omne periculum obfirmatis, in alto mari, in navis foris et transis pernoctari cœptum: tunc etiam tempora navigationi opportuna notata, et legibus sunt definita; ut quâ anni tempestate naves subduci, et quâ in mare deduci expediat, omnibus definitè constaret. Ita in actis Apostolicis cap. 27, vides navigationis tempus periculosum et intutum censeri: ὁ δὲ πλοῦς ἦν ἐπιπορευόμενος τῷ πλοῦς, διὰ τὸ καὶ τὸν ἡμεῖς ἦν παρὰ τὸν καιρὸν· cum, inquit, jam esset navigatio periculi plena, quod tempus jejuniti (Judaici intellige, decimo die mensis septimi, quo Ihericus expiationis offerrebat) jam præterisset. Eandem anni tempestatem in animo habebat poeta, apud quem Dido Æneam fugam molientem increpitat:

Quin

Quin etiam hyberno moliris fidere classẽ,
 Et mediis properas aquilonibus ire per altum,
 Crudelis. Quid ? si non arva aliena domotique
 Ignotas peteres ? et Troja antiqua maneret ?
 Troja per undosum peteretur classibus æquor ?

Nam ex die tertio Iduum Novembris, usque in diem sextum Iduum Martiarum, [id est, à die duodecimo Novembris ad diem nonum Martii,] maria claudabantur. Et Plinio quoque, Ver navigantibus aperit maria, cujus in principio Favonii molliunt cælum. Atque tunc, ab eo primùm tempore, mundus in suas plagas accuratius quoque tributus. Ut enim decempedatores in terrâ, agris ad ferri rigorem degrumandis, meridianum et cardinem suum designant : ita miseri isti ἀντιπάρους, undarum fluctibus jactati, signa quærare instituerunt, unde certi essent quas in oras, quo ventorum flatu ferrentur, ut ita regionum maritimarum situm feliciter explicarent. Et ad hanc rem quoque cælum ipsum in partes vocatum, ut ad siderum conspectum cursus regerent.

— — felix stellis qui segnibus usum,
 Et dedit æquoreos cælo duce tendere cursus.

Phœnices in
 hac arte præ-
 cellant.

Phœnicum ea fuit potissimum gloria, qui, ut reliquos mediterranei maris accolæ mercaturæ et mercium exportandarum studio superaverunt, ita fideralis scientiæ gloriâ illis quoque præcelluerunt. Et Solem quidem interdiu secuti meridiei et septentrionis plagas haud difficultè dispunxerunt : noctu autem à Cynosurâ toti pendebant, quod ea à polo proximè abesset. At Græci latiore specie contenti Helicen spectabant ; id enim passim à veteribus inculcatur,

Esse duas Arctos, quarum Cynosura petatur
 Sidoniis, Helicen Graja carina notet.

Hujus fiduciâ freti Tyrii Sidonisque non solum omnia maris mediterranei littora frequentare, et ultimas ejus oras adire ; sed etiam coloniam extrâ ejus ambitum in mari Oceano Gades deducere ausi sunt. Horum operâ hominum sapientissimus, rex Solomon, quoque usus, aurum longè ex Ophiro per mare Æthiopicum petere instituit, et suam classẽ cum classe Hiram Tyriorum regis conjunxit. atque id quidem tanto tempore ante, quàm Taprobane insula in illis partibus vel fando Occidentalibus esset inaudita, quibus post tot secula in admiratione sit, Annii Plocami libertus tempestate in eam delatus. Hæc igitur observatio multa secula duravit, ut, religiosè vitatâ tempestatum sevitia, tantum commodis istis mensibus navigarent. At pirarum primum coëgere mortis periculo in mortem ruere, et hiberna expetiri maria. Nunc idem avaritia cogit. Neque id solum in mari mediterraneo, ejus spatia brevissimo tempore percurri possunt, (In tantum, ut Galerius à freto Siciliae Alexandriam septimâ die pervenerit, Bibulus sextâ, ambo præfecti ; æstate verò proximâ Valerius Marianus ex Prætoris senatoribus, à Puteolis nono die lenissimo flatu, ait Plinius ; Itemque Gadibus extrâ Herculis columnas Osliam septimo die, à citiore Hispaniâ quarto, à Provinciâ Narbonensi tertio, ab Africâ altero petveniri, idem prodit :) Sed in ipso

ipſo vaſtiſſimo Oceani maris æquore; ubi ſæpe aliquot menſium ſpatio nullæ uſquam

Apparent terræ; cælum undique, et undique pontus:

ubi illæ veterum de cælo obſervationes non ſint unius aſſis; cum venis ſævi-
entibus mare ex imo fundo vertitur, coactæque in nubem aëre, nimbis diem
ipſum involventibus: et maximè quoque, cum

— jam occidente ſole inhorreſcit mare,
Tenebræ conduplicantur, noctiſque et nimbū occæcat nigror.

Cum, inquam, cæci in undis obeirant, neque in illo vaſto gurgite viæ amplius
meminiſſe poſſunt; ſed modò huc, modò illuc è deſtinato curſu undarum im-
petu excutuntur. Hic igitur non cæli, non ſiderum conſpectus miſeros illos
vel hilum juvat; longè alio *ἑκαγὼν* utique opus eſt, quiſque ſemper et quo-
ties libet, non è cælo illis petendus ſit, ſed ante oculos et in manibus verſetur;
unde, ſi nihil aliud, ſaltem iſtud cognoscant, quam in plagam navis curſum
modiatur: et quamvis vel undarum vi, vel veniorum furentium fragore et im-
petu, è recto curſu diverſi ferantur, ut nihilominus ipſis è veſtigio conſlet, in
quam plagam puppis ſe verterit, et quanto impetu concitata feratur. Neque
verò iſta operoſius erant investiganda, aut extrà hunc globum longius petenda;

quò ipſa Terra parens nobis duntaxat erat adeunda, illi vota faciunda, quæ
ubique nobiliſſimo magnetæ ſæta, ſponte eum è viſceribus ſuis in lucem pro-
fert, ut ejus ope, tanquam prætorio edicto, humanum genus in ipſius Maris
poſſeſſionem quoque mittatur. Magnetem enim lapidem ferrum trahere, et
eam vim attrahendi in ipſum ferrum tranſfundere, jam olim pro miraculo à
veteribus fuit jaclatum. Plato in dialogo cui Io nomen fecit, *ὡς περ ἐν τῇ λήδῃ*,
ἐν Εὐριπίδῃς μὲν Μαγνήτιν ἀνόμαστον, οἱ δὲ πολλοὶ, Ἡρακλείαν. καὶ γὰρ αὕτη ἡ λήδῃς ἡ μόνον
αὐτὸς τοὺς δακρυλῶν ἀγυὸς τοῦ εὐρημῆς, ἀλλὰ ἡ δύναμις ἐντίθει τοῖς δακρυλῶν ὡς περ ἐκασθεῖαι
ταύτων τῶτο πῶς ὅτι ἐ λήδῃς, ἀλλὰς μὲν ὅγιν δακρυλῶν, ὡς ἐντοὶ ὀρυκτοῦ; μακροῖς παύ
εὐρημῶν καὶ δακρυλῶν ἐξ ἀλλήλων ἔρρηται. πῶς δὲ τότε τοῖς ἐξ ἐκείνης τῆς λήδῃς ἡ δύναμις ἀπορρηται.
Sicuti, inquit, contingit in illo lapide, quem Euripides cognominavit Magne-
tem, plerique Herculeum. Is enim lapis non modò annulis ferreos attrahit,
ſed et ipſis annulis vim communicat idem faciendi ac ipſe lapis, ſcilicet, attrahendi
alios annulos, ita ut longiſſima quidem ferreorum annulorum ſeries con-
tinuo quodam textu aptetur: ad quos omnes ex illo lapide vis ea omnino perti-
net. Hoc quidem noviffe haud magni noſtra intererat: at ejus cauſam intellexiſſe,
id demùm erat naturæ fontes recluſiſſe; unde rei manuum et ma-
jeſtas et uſus in ſolidum dependet. Et, cum illa magnetis efficienſia naturalis
attrahendi ferri, tot jam ſæculis ab hominibus induſtris notata et obſervata ſit;

nemini tamen omnium, *ἔτι κατ' ὄψιν*, in mentem venit, aliam longè nobiliorem
huius efficienſiæ cauſam ſuſſeſſe. Haud multa ſæcula fluxerunt ab eo tempore,
cum in Campaniâ, vel caſu, vel ſtudio, notatum ſit, Magnetem lapidem ſuis
habere polos, quos ad mundi polos convertat; ut et axem ſuum contrà mundi
axem parallelo ſitu dirigat: neque ita tantùm ferrum aut chalybem ſuâ conta-
ctione inficere, ut id ſolum affectet, ut illa ad ſe trahat: ſed multo maximè,
ut vim illam per terrena omnia diſfuſam illi indat, quâ affecta ipſa acus ſemper

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T

ſuam tere.

De lapide qui
magnetem nomi-
natur, et qui
ferrum ad ſe
trahere obſer-
vatur.

Quadringen-
tia ſere abhinc
annis in Cam-
paniâ com-
pertum eſt,
acum ferream
à magnetē vi-
ſificatam ſcilicet
ſemper verſus
Septentrionem conver-
tere.

suam lineam meridianam spectet. Sagacissimum naturæ mystem cum fuisse oportet, cujus operâ et industriâ illud primum è secretionibus naturæ adytis erutum et investigatum sit. Attamen tanti inventi gloria nullo titulo singulari circumferitur; cum minimarum et vilissimarum rerum inventores suos habeant titulos, sua nomina, hæc autem in re adeò illustri, vel solo gentis titulo contenti simus oportet. Et quemadmodum olim Phœnices audiebant multarum rerum inventores, cœli, siderum, navigationis, multarumque aliarum artium: ubi tamen non homini, sed genti ea gloria adscribitur: idem hic usu venit, ut non quis, sed quâ è gente, auctor et inventor adeò divinæ rei exstiterit, nobis duntaxat constet. Vix anni quadringenti sunt, ex quo adeò memorabile et maritimis itineribus opportunum inventum erutum, et in lucem prolatum sit. Acus enim pyxidulâ coercita, suo sese imperio et ductu in eandem plagam constantèr vertit: ut vix ullis aut solis, aut siderum, observationibus sit opus; quin, adeò ita mundi plagæ multo accuratius investigantur, dum solam pyxidulam quæ ante pedes est, identidem inspeçant, ejusque ductum accuratè observant.

— Nec te spectare Bo-oten,
Aut Helicen jubeo, stricturnque Orionis ensẽ.
Hæc duce carpe viam. —

Hujus enim fiduciâ Itali primum, inde etiam Hispani, externa maria tentare, atque insulas Canarias in Oceano, haud longè ab Africano littore distantes, frequentius adire instituerunt: cum ipsi perpetuò in manibus, et ante oculos versaretur norma, secundum quam cursûs plagas regere possent; neque id solum ad trerantes, sed octantes quoque, et alias minores peripheriæ partes. Quod ab eo tempore factitari cœptum, quousque processerit, haud faciliè dixerim. Istud quidem memoriæ proditum est, Navarchum illum (cujus tamen nomen non magis scitur, quàm illius, qui omnium primus in acum nauticam magnetis vim versoriam transfundere docuit) qui primus mortalium ultrâ Maderam, in oppositum Americæ littus, sive studio, sive casu fati ita ducentibus, penetraverit, ejusce rei adumbratam quandam cognitionem habuisse. Hic enim à primâ usque adolescentiâ diversis, et, prout tunc tempora ferebant, longinquis, navigationibus exercitus, in Americam secundo vento delatus; inde converso cursu in Maderam navi appulsâ, plusculis è suorum numero defideratis: ipse quidem redux, sed adeò longo itinere fatigatus, et labore fractus, apud Christophorum Columbum, narione Ligurẽ, qui tum illic chartis hydrographicis delineandis vitam tolerabat, expiravit; lineamentis itineris ex pyxide, et parallelo loci ipsi indicato. Hæc incitamento postea Columbo fuisse, ut tanto studio, manibus pedibusque obnixè eam expeditionem urgeret; ut nihil non moverit; et apud Ligures suos Genuenses; et apud Henricum septimum Angliæ; Alphonsum quantum Lusitaniz; Ferdinandum et Isabel-lam Castiliæ reges; donec tandem, propositi tenacissimus, apud Castiliæ reges voti compos factus, Terras istas, tanto tempore nostro orbi ignotas, primus detexit, et opes immensas, miseris et pacatissimis illis mortalibus ereptas, in Hispaniam intulit. Sed pari planè cum sociis suis fato usus, quo Argonautæ, è Colchide optima spolia referentes, defuncti sunt.

Quifquis

Navarchus
quidam (cujus
nomen igno-
ratur) annis
aliquot ante
Columbum,
in Americæ
littus secundo
vento est de-
latus.

Quisquis audacis tetigit carinæ
 Nobiles rarios, nemorisque sacri
 Pelion densâ spoliavit umbrâ:
 Quisquis intravit scopulos vagantes,
 Et tot emensus pelagi labores,
 Barbarâ funem religavit orâ,
 Raptor externi rediturus auri;
 Exitu diro temerata Ponti
 Jura piavit.

Sed ad generosum Andalusum illum Navarchum revertamur. Omininò extrâ controversiam statuendum est, eum pyxidis nauticæ adminiculo, anguli inclinationis æstimatione, et confecti-itineris conjecturis usum; locorum situm, terrarumque, ad quas appulerat, distantiam utcunque expressisse. Quod acûs magnetice directionem adhibuerit, id quidem ex nupero tum invento assumptum; itineris autem confecti quantitas ex cursûs velocitate aut tarditate jam olim quoque in mari æstimari solita. Disertè id monet Herodotus in Melpomene, ubi Euxini ponti cum longitudinem tum latitudinem nobis hæc ratione definit: Τὸ πρὸς, ποτὶ, μέγεθος ἐστὶν εὐκατὶς ἰκατὸν καὶ χίλιοι καὶ μύριοι. τὸ δὲ ἕρπος, τὸ ἐνυρτάτος αὐτοῦ ὅλῳ, εὐκατὶς διπλοῦσι καὶ τρισχίλιοι. *cujus longitudo est undecim millium ac centum stadiorum: latitudo (quæ mare latissimum est) trium millium et ducentorum.* Inde mensuræ rationem explicat: Μιμέρεται δὲ ταῦτα ὅδε· πῶς ἐπέκεινται μάλα καὶ καθανύει ὁ μακρὸς ἡμῶν ὄργυιας ἑκατομυρίας, νυκτὸς δὲ, ἑκατομυρίας. πῶς ὡς ἐς μὴν Πάσιον ἀπὸ τοῦ εὐκατὶς (ταῦτο γὰρ ἐστὶν τοῦ Πόντου μακροτάτου) ἡμερῶν ὡς πέντε πλάτος αὐτοῦ, καὶ νυκτὶν οὐκ ὅλῳ. αὗται ὅδεκα μυριάδες καὶ ἑκατὸν ὄργυια ὧνται· ἐν δὲ τῷ ὄργυιῳ ταύτῳ, εὐκατὶς ἰκατὸν καὶ χίλιοι καὶ μύριοι ἴσιν· ἐς δὲ Θερμύδοσιν πάλαι ἐκ τῆς Ἰσθμῆς (καὶ τῆτο γὰρ ἐστὶν τὸ Πόντου ἐνυρτάτος) τριῶν τε ἡμερῶν καὶ δύο νυκτῶν πλάτος. αὗται δὲ, τριῶν καὶ τριῶν μυριάδες ὄργυια ὧνται, εὐκατὶς δὲ, τριπέσους καὶ τρισχίλιοι. *Hec autem (inquit) ita dimensa sunt. Navis, diebus longioribus, ut summum quotidid conficit septuaginta millia orgyiarum (orgyia autem sex pedibus definitur) noctu verò sexaginta millia. Itaque è faucibus Ponti ad Phasiā (hoc enim est Ponti longissimum) novem dierum est navigatio, et octo noctium, quæ sunt mille centum ac decem millia orgyiarum, hoc est, stadiorum undecim millia ac centum. E Scythiâ autem usque ad Thermscyram, quæ est suprâ flumen Thermodontem (hic namque ponti latissimum est) trium dierum duarumque noctium est navigatio, quæ sunt orgyiarum trecenta ac tria millia, hoc est, stadiorum tria millia ac trecenta. Hunc igitur in modum Pontus, ac Bosporus, et Hellepontus à me dimensa sunt.* En, suam fidem interponit scriptor accuratissimus et fide dignissimus, non appellat alienam. Vides, eum interdum aërem laxare, noctu contrahere: quod faciliè intelliges, si cogitabis, parallelum quadragesimum quintum per medium Ponti exporrigi, longissimæ autem diei spatium esse horarum quindecim cum semisse: noctisque brevissimæ, octo cum semisse; et quamvis per ventorum vario impetu naves modo vehementius, modo tardius impellantur, Herodotum tamen hic mediocritatem quandam videmus in horum intervallo æstimatione secutum. Haud abîmili ratione distantiam inter Rhodum et Alexandriam quinque stadiorum millibus, inter Carthaginem et Lilybæum (quæ Straboni quidem stadiorum mille quingenta sunt, Polybio autem mille

Arumentis eadem multo fufius defcripfit, et harum linearum proprietates accuratius eft perfecutus. Efti in iis non rarò κέρχρη, minùs feliciter verfatus fit; et in magnas difficultates fele præcipitaverit. Palmarium illud, quod Loxodromiam noftram (Rumbum ille, voce fuæ genit, appellat) lineam verè δυνάμει, et totam fui generis, in Sphæricorum triangulorum τμήματι, et minutal quoddam conciderit. Atque hinc ei prima illa mali labe, ut non rarò hic cefpiter; neque hanc navem falvam in portu collocarit. Quæ res tantum abeft ut ejus exiltimationi quicquam derogaverit; ut etiã alios fummos viros ad eadem accuratius indaganda excitaverit; qui in rebus Mathematicis, et Geographicis verfatiffimi, fingulari ftudio Loxodromiarum affectiones et ufui fcutati funt. Ejus enim explicationem rebus mariimis non utilem folùm; fed quàm maxime neceffariam viri perfpicaces facilè intelligebant; quantumque ab hujus abfolutâ et caftigatâ cognitione abeffent, tantum quoque à perfectione fuâ hanc ipfam met fcientiam abeffe, non erant nefcii. Et eas quidem lineas alii in globo ipfo; alii in chartis Hydrographicis, quæ analogiâ fuâ globo propiùs responderent, fumma cunâ designare et exprimere funt conati. Quâ in re vir doctiffimus, et Geographorum diligentiffimus, Gerardus Mercator, non inutilem operam pofuiffe meritò fit judicandus; et poft eum in Belgio etiam complures alii, et Stevinus quoque nofter in fuo magno opere. In Galliis quoque haud pauci. In Britannia reliquis diligentior Edowardus Wricht; inde Robertus Huës, et, ut audio, alii complures. Sed, ut ingenuè dicam quod res eft, Coronis, et operis perfectâ delineatio defiderari videbatur; ut quantum Loxodromiæ fingulæ, et earum partes in longitudine evariarent ad minutum conftaret: neque id rudi, aut mechanicâ aliquâ fâctione; fed accuratiore illâ, quæ minimas qualque particulas fubducit. Id igitur fecutus fum, ut ab ipfo fundamento omnia folidè demonftrata, et ordine conftituta nunc demùm exhiberem. Utque momenta demonftrationum noftrarum non minus haberent firmitudinis, quam ufus facilitatis in praxi quoxidianâ; antilogias hic omiffimus, quæque ipfam artem fpectant duntaxat tractavimus: illas tamen in Scholis Hiftiometricis poftmodùm relaturi. Tu ifta, benevole lector, tuo ufui et commodo, ad Divini nominis gloriam, quo omnia noftra ftudia contendunt, dicata habeto.

Et, poft eum,
Gerardus
Mercator, et
alii:

Τῷ μὲν δ' ἡγεμῶνι τῷ, καὶ τῷ ἀλλ' χαίρειν ἔσται,
Τῷ δὲ αὖτ' ὁ ὀψωνίος, καὶ ἡρα χρεμάτια πάντα.

CLAUDIANUS PRÆFATIONE LIBRI I.

INventâ secuit primus qui nave profundum,
 Et rudibus remis sollicitavit aquas;
 Qui dubiis ausus committere flatibus alnum,
 Quas natura negat, præbuit arte vias.
 Tranquillis primùm trepidus se credidit undis,
 Littora securo tramite summa legens.
 Mox longos tentare sinus, et linquere terras
 Et leni cœpit pandere vela Noto.
 Ast ubi paulatim præceps audacia crevit,
 Cordâque languentem dedidicere metum,
 Jam vagus irrupit pelago, cœlûmque sequutus,
 Ægæas hyemes, Ioniûmque domat.

PLINIUS, Lib. xxxvi. Cap. 16.

AMarmoribus digredienti ad reliquorum lapidum insignes naturas, quis dubitet in primis Magnetem occurrere? Quid enim mirabilius? Aut quâ in parte major naturæ improbitas? Dederat vocem faxis, ut vidimus respondentem homini, imò verò et loquentem. Quid lapidis rigore pigrius? Ecce sensus manûsque tribuit illi. Quid ferri duritiâ pugnacius? Sed cedit et patitur mores: trahitur namque à magnete lapide, domitrixque illa rerum omnium materia ad inane nescio quid currit: atque, ut propius venit, assistit, teneturque et complexu hæret. Sideritina ab hoc alio nomine appellant, quidam Heracleon. Magnes appellatus est ab inventore (ut autor est Nicander,) in Idâ repertus. Nam et passim invenitur, ut in Hispaniâ quoque. Invenisse autem fertur, clavis crepidarum et baculi cuspide hærentibus, cum armenta pasceret.

WILLE-

WILLEBRORDI SNELLII

T I P H Y S;

SIVE

DE HISTIODROMIÀ,

LIBER PRIMUS.

PROPOSITIO I.

Histiodromice est doctrina, quæ lineæ designatæ à navis cursu, magnetica acûs ductum secute, afflixiõnem et proprietates interpretatur.

NAVIS enim magister ita cursum dirigit ad optatam metam ut semper linearum certi generis ductum sequatur : et ad hanc rem acûs magneticæ situm atque angulos quos cum instituto cursu comprehendit sedulò speculatur. * Acum enim chalybeam, super axiculo cui incumbit mobilem, magnetis lapidis polo notio eâ parte tingi, quâ Boream spectet; et contrâ ejusdem lapidis polo boreo, quâ Notum sit designatura; vulgò, opinor, jam notum est; atque ita eam semper boream inter et meridiem exporrigi. Sed tamen hunc situm aliquando à naturæ legibus descissere, idque oculi et latente aliquâ de causâ, jam diù experientia comprobavit. Quamvis enim magnes, ubi liberè movetur, insitâ sibi vi et facultate suis polis mundi polos affectet, et parallelo ad mundi axem situ, suo axe exporrigatur; ut ita quoque acus legitimè tincta secundum locorum meridianos sese dirigat: nihilominus ab hac cujusque loci meridianâ lineâ, modò à septentrione in ortum, aliquando etiâ in occasum, aliquot gradibus declinare, deprehensum est. Et cum rectâ Septentrionem spectat, eam χαλκίειον, tanquam situm acûs legitimum et proprium vocant: cum autem evariat, χαλκιδέουσα ejus, ad meridianam lineam inclinationem indigitant. Ita in insulâ Corvo (insularum Flandricarum unâ, medio ferè inter

* Guliel. Gilbert, lib. 3. cap. 4.

Europam et Americam (spatio) rectâ septentrionem spectat. Unde à quibusdam idèò meridianorum omnium initium ab hoc meridiano derivatur. Hinc versùs Americam contendentibus apex boreus in occasum declinat: contrâ Europam aut Africam petentibus in ortum evariat, nempe Pleimudæ in Angliâ 13. gr. 24. scr. et Londini 11. gr. 30. scr. Hic apud nos Amsterodami 9. gr. 30. scr. ut vulgò existimatur, atque ita alibi alitèr. Verùm hujus evariationis Synopticam tabellam videto in Limeneureticâ ingeniosissimè Stevini; cujus etiâ causâs pervestigare et eruere subtilissimè conatus est Guilielmus Gilbertus, Anglus, peculiari volumine huic rei dicato.

PROPOSITIO II.

Legitimus acûs magneticæ situs est luci linea meridiana.

HIC duntaxat est legitimus, reliqui ~~voluntarii~~, sunt adulterini. Nam occulta et insita illa magneti mens, ita ejus molem agitat, ut polos suos et axem contrâ Terræ axem parallelo situ exporrigat, nisi fascinatione quâdam naturali ab hoc situ, etiâ aliquantum modò huc, modò illuc, avellatur; et ita acus quoque alibi alitèr atque alitèr declinet; cujus inclinatio tamen sedulò est corrigenda; si ejus ductum cum fructu sequi velimus.

PROPOSITIO III.

Acûs magneticæ situs adulterinus inventâ ebalysoclysi emendatur.

CUM incertum sit, acûs-ne rectâ septentrionem et meridiem spectet, an verò ab eo situ in diversum abeat, id ex ortu et occasu solis siderumque, aut eorum utrimque altitudinibus æqualibus, aut aliis modis denique deprehendi potest; et in mari solet: inde enim constat, an acus magnetica ita porrigatur ut peripheriam ab utrisque his lineis comprehensam in plano horizontis bisecet; aut quantum à medio in partem alterutram discedat. Unde legitimus meridiani situs, et acûs magneticæ in alterutram partem aberratio definiatur. Sed ista in scholas nostras rejecimus; quas vide.

PROPOSITIO IV.

Histiiodromia est circularis, aut Loxodromica.

VELIFICATIONIS cursum, quâ acûs magneticæ ductum sequitur, ita natura composuit, ut vel circulum aliquem, vel helicem quandam lineam delinearet, quam ab obliquitate *Loxodromiam* dicimus.

P R O.

PROPOSITIO V.

Velificationis cursus circularis efficitur, cum recta in Septentrionem et meridiem, aut ortum et occasum dirigitur.

PROPOSITIO VI.

Velificationis cursus in Septentrionem aut meridiem maximi circuli peripheriam describit.

HOC enim pacto Navis semper in eodem meridiano versatur, sub quo inde ab initio constiterat. Atqui meridianos esse maximos Sphæræ circulos, nemini ignotum est qui vel levitèr Sphærica degustaverit. Itaque

Quindecim maritimerum milliarium cursus hic poli Elevationem uno gradu evariat.

Quantum maritima sive nautica milliaria à terrenis et horariis distent, infra propositione octavâ dicitur.

PROPOSITIO VII.

Velificationis cursus in ortum aut occasum, sub æquinoctiali quidem maximum circumulum, extrâ autem cum semper huic parallelum describit.

SIT enim linea carinam mediam à prorâ ad puppim secans, quam *ῥαυτίδα* Græci vocant, meridiano quocunque in loco perpendicularis, tum quidem navis rectâ ortum aut occasum spectabit. Et id meridiani segmentum cum carinâ sibi affixâ super mundi polis circumduci intelligatur; hæc igitur carinæ linea cum omnibus meridianis quibus hoc segmentum congruet angulos normales comprehendet: Et ita circuli peripheriam describet. Idem autem efficitur si carina suo proprioque motu progressâ per omnes meridianos normalitèr, hoc est, ortum aut occasum spectans, transeat. Hæc enim linea illi congruet. Et propterea, cum sub æquinoctiali erit motus principium, per æquinoctialem decurret, donec, obito orbe, ad idem punctum unde moveri cœperat iterum restituatur. Vel, si extrâ æquinoctialem à quocunque tandem loco incipiat moveri, et conversionem motûmque continuet donec ad idem punctum revertatur, hoc utique ductu omnino illius loci parallelum designabit. Itaque constat Propositionio.

PROPOSITIO VIII.

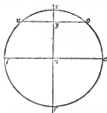
Sub æquinoctiali quindecim milliaria maritima unum gradum in longitudine exarant; in parallelis etiã amplius.

MENSURA itineris in rebus maritimis utramque facit paginam: absque hac enim omnis illorum Hydrographice, et portuum, littorum, atque insularum descriptio, manca esset atque mendosa. Sed neque hæc cum fructu in censum vocari potest, nisi ejusdem comparatio cum terreni globi perimetro, aut perimetri parte aliquâ, legitimè cognoscatur. Hinc adeo vulgò factum est ut gradum in partes secarent, et eas *milliaria*, pro suæ gentis consuetudine, vocarent. Angli enim tot milliaria in gradu maximi circuli statuunt quot in eo scrupula contineantur, nempe sexaginta. Nos hæc in re Germanicam et aliarum gentium consuetudinem secuti sumus, qui quindecim milliaria uni gradui attribuant, cum Anglicum milliare quartam Germanici milliarii partem tantum consticiat. Lusitani quidam, Nonio teste, milliaria $16\frac{2}{3}$, alii $17\frac{1}{2}$, uni gradui imputant. Nos alibi docuimus 19 milliaria horaria uno gradu contineri, quorum singuli habeant pedes nostros Rhiinlandicos 18000; quæ *terrena* indigitare soleo: vel 15 milliaria, ut singulis cedant pedes 22800, quæ tunc *maritima* voco; quia horum usus à nauticâ consuetudine non recedit. Unum tantum addidi, quod palmarium erat, mensuræ modum*. Sed hæc de re vel scholas, vel Eratosthenem nostrum Batavum videto. Quot autem milliaria uni gradui in dato quocunque parallelo cedant, sequenti theoremate ita explico.

PROPOSITIO IX.

Quemadmodum radius ad sinum complementi Latitudinis dati paralleli, ita quantitas unius gradus in maximo parallelo, ad quantitatem unius gradus in dato.

QUOT milliaria uni gradui in singulis parallelis cedant, id hæc expositâ analogiâ invenitur. Sit enim *sr* mundi axis, *ie* meridiani et æquinoctialis communis sectio, *uo* paralleli et meridiani: *ie* igitur et *uo* erunt suorum circulorum diametri, *ae* et *yo* radii. Ut igitur *ae* ad *yo*, ita tota peripheria descripta super *ie* diametrum, ad peripheriam super *uo* descriptam. Sed ut tota ad totam, ita pars ad partem similem. Ut igitur *ae* sinus totus, ad *yo* sinum complementi peripheriæ *oe*, quæ mensura est latitudinis; ita unius gradus quantitas in maximo circulo, 15 milliariis marinis æstimata, ad numerum milliariarum quæ continentur in uno gradu dati paralleli. Ut, si quæretur magnitudo unius gradus sub parallelo 52 gr. 10 scr. complementum hujus est 37 gr. 50 scr. et ejus sinus 61337; fiat igitur, ut 100000 ad 61337, ita 15 ad $9\frac{1000}{100000}$ pro quantitate milliariarum in uno gradu sub dato parallelo comprehensorum. Itaque



* Seu methodum inveniendi longitudinem unius gradus in aliquo Meridiano.

P R O.

PROPOSITIO X.

Quadrans finis complementi dati paralleli per radium divisus dabit quantitatem unius scrupuli.

CUM enim per propositionem antecedentem sit, ut radius ad finem complementi dati paralleli, ita 15 milliaria (tot enim milliaria in maricimis semper pro maximi circuli gradu assumimus) ad numerum milliarium unius gradus in dato parallelo. Atqui, cum ita $\frac{1}{4}$ milliariis fiat $\frac{1}{60}$ gradus, et ideo sit mensura unius scrupuli in maximo gradu; erit quoque ut radius ad finem complementi dati paralleli, ita $\frac{1}{4}$ milliariis ad mensuram scrupuli in dato parallelo; vel per interpretationem transpositis terminis, ut radius ad quadrantem finis complementi dati paralleli, ita 1 milliaria ad mensuram unius scrupuli *. Sit idem exemplum, quod prius: sub parallelo 32 gr. 10 scr. cujus complementi finis datur 61337: ejus pars quarta 15334 per radium divisa dabit $\frac{15334}{15}$ milliariis, spatium unius scrupuli, sive minuti, in dicto parallelo. Et sane id cum antecedente epilogismo congruit. Pars enim sexagesima de $\frac{15334}{60}$ est $\frac{2555}{6}$, seu $\frac{425}{1}$, seu $\frac{15334}{60}$; quemadmodum supra expressimus. Atque ita absque ulla circuituione unius scrupuli, sive minuti, spatium in quocunque parallelo exhiberi potest.

PROPOSITIO XI.

Ut radius ad quadruplum secantis dati paralleli, ita milliaria quotcumque data ad numerum scrupulorum, sive minutorum, ipsi in eo debitorum.

CONVERSUM est utriusque antecedentis, si finem spectes. Illic enim è datis gradibus aut scrupulis, sive minutis, milliaria ipsis congrua quaeruntur. Hic contrà è datis milliariis scrupulorum numerus ipsis sub dato parallelo debitus inquiritur. Est enim demonstratum propositione nonâ et decimâ, radium ad finem complementi dati paralleli esse, ut milliaria quotvis datae partis in maximo circulo ad milliaria simili parti in dato parallelo respondentia. Et inversè igitur, ut finis complementi dati paralleli ad radium, hoc est, ut radius ad secantem dati paralleli, ita milliaria in eodem parallelo ad milliaria simili peripheriae in æquinoctiali debita. Porro autem, cum quindecim milliaria gradum impleant; ideo quadruplum horum milliarium exhibebit numerum

* Erit enim ut radius ad quadrantem finis complementi dati paralleli, ita pars quarta unius milliaria ad quartam partem mensuræ scrupuli in dato parallelo, atque ideo unum milliaria ad totam mensuram scrupuli in dato parallelo. Ergo si productum ex multiplicatione quadrantis finis complementi dati paralleli in unum milliaria, hoc est, iste ipse quadrans, dividatur per radium, quotiens erit æqualis mensuræ unius scrupuli, sive minuti, in dato parallelo, seu numerus milliarium in uno scrupulo, sive minuto, ejusdem paralleli contentorum.

serupulorum, five minutorum, ipsis in æquinoctiali debitorum, atque idem minutorum numerus in dato parallelo simili peripheriæ quoque respondet. Exemplo res erit illustrior. Ut si quæram in sexagesimo parallelo quot serupula 15 nauticis milliariibus respondeant. Fiat ut 10,000,000 ad 8,000,000, quadruplum secantis graduum sexaginta; ita 15 ad 120. Dicam igitur, in sexagesimo parallelo 15 milliaria occupare serupula 120, qui sunt duo integri gradus; quod omnino verum est. Nam in hoc parallelo singuli gradus dimidii sunt eorum qui in maximo continentur: Quia hujus sinus complementum, nempe 30 graduum sinus, dimidius est sinus maximi. Sin verò plura aut pauciora fuerint milliaria quam 15, veritas tamen eadem fuerit: sunt enim milliaria milliariibus et serupula serupulis proportionalia. Tercera insula, Açorum, seu Flandricarum insularum, una, sub parallelo 39, abest ab Ulissipone, Hispaniæ emporio (qui sub eodem quasi parallelo situs est) 250 milliariibus; quæritur quantum sit longitudinis intervallum. Quadruplum secantis dati paralleli est 514,703. quare fiat ut 100,000 ad 514,703. ita 250 milliaria ad 1286½ ser. qui sunt gr. 21. ser. 26½. Tot igitur gradibus Tercera occidentalior est Ulissipone.

PROPOSITIO XII.

Ut radius ad quadruplum summæ secantium inter datos parallelos inclusivè comprehensorum; ita datorum milliariū summæ pars quæta iisdem parallelis cognominis, ad summam serupulorum longitudinis, quanta singulis distinctis parallelis ex datis milliariibus accedit.

THEOREMA est ex antecedente planum et indidè derivatum. Sunt milliaria 10 tribuenda parallelis 20 singulorum serupulorum intervallo distantibus à gradu latitudinis 52^{da} exclusivè. Sunt itaque hic viginti circuli; propterea, ut singulis parallelis æqualem partem attribuam, fiet milliariis ½, et quia dimidium milliare in his omnibus non occupat æqualem serupulorum numerum, sed majorem in minoribus, minorem in majoribus, idè sigillatim erit de singulis concludendum; ut radius ad quadruplum secantis 52 gr. 1 ser. ita ½ milliariis ad minuta in hoc parallelo sibi debita. Et rursum, ut radius ad quadruplum secantis 51 gr. 2 ser. ita ½ milliariis ad serupula sua in isto. Atque ita porro ad ultimum limitem. Erit igitur per Compositionem proportionis arithmeticam, ut Radius ad quadruplum summæ omnium secantium, ita ½ milliare ad summam omnium serupulorum, quantum spatii milliariis semissis in singulis occupat. Quemadmodum hic vides.

Secantes.

Secantes.	gr.	scr.
162487	52	1
162548	52	2
162608	52	3
162669	52	4
162730	52	5
162791	52	6
162852	52	7
162913	52	8
162974	52	9
163035	52	10
163095	52	11
163157	52	12
163218	52	13
163279	52	14
163340	52	15
163402	52	16
163463	52	17
163525	52	18
163586	52	19
163648	52	20

3423749
quadruplum 4

100000 13694956

†

6847498 } 68 $\frac{47}{100000}$
100000

Ut quartus hinc detur 68 scrup. 28 $\frac{1}{2}$ sec. Tantum igitur spatii data milliaria per singulos 500 parallelos æqualitèr tributa occuparent, in singulis tamen scrupulorum numerum inæqualem. Namque in primo parallelo, nempe 52 gr. 1 scrup. milliare dimidium possidet 3 scr. 14 $\frac{9}{1000}$ sec. in ultimo autem 3 scr. 16 $\frac{3}{1000}$ sec. quæ differentia, etsi in tantâ vicinâ exigua sit, in longiore intervallo admodum magna existit. Dimidium enim milliare marinum sub æquinoctiali 2 scrupula duntaxat explet.

Atque ita unicâ multiplicatione, additis secantibus, hoc opere defungeris. Sed tam multiplex additio, quæ semper pro ratione quæstiti denuò est iteranda, plurimum habet tædii, ut vix operæ sit istos labores subire; et utilitas, quæ hinc sperari possit, ipsâ laboris ac molestiarum mole obruta et tanquam oppressa jaceat. Hinc ut subvenirem, quia usus ejus perquam necessarius, et in hoc opere ferè est perpetuus, Canonem construxi, ex additis secantibus constatum, progressu per singula quæque scrupula continuo. Cujus usus postmodum nobis erit longè commodissimus in loxodromiarum calculo, et evariatæ longitudinis spatio æstimando. Tabulam ipsam hic subijcio, cujus constructio solâ additione constat, ut nempe radius ipse 10,000,000 ad secantem unius scrupuli 10,000,001 addatur, unde summa constatur 20,000,001; hæc ad secantem 2 scrupulorum 10,000,002 addita, dabit 30,000,003; et hic numerus cum se-

cante

cante 3 scrupulorum 10,000,004 compositus, conflabit 40,000,007, atque eo porro ordine per singula scrupula ad extremum quadrantis limitem progrediendo. Ego hunc canonem ad septuagesimum gradum produxi, qui ad usum maritimum, (quem hic potissimum spectamus) satis est; et, si ultrâ tendas, non jam per singula scrupula tantum, sed per scrupulorum semisses, trientes, sextantes, imò uncias, uspiam esset progrediendum; denique, cum proximè polui accesseris, etiam per scrupula secunda aut tertia. Sed, quanquam ille labor et infructuosus et inutilis sit futurus; specimen tamen aliquod hujus *descriptio* in scholis infâ exhibemus. Si cui libeat imitari, habet ubi otium suum ponat sine damno: nos instituto nostro insistimus; et quo calculus ille securius subduci posset, et error certius deprehendi qui fortè nobis obrepisset; his eundem subduxi, semel ex tabulis secantium Thomæ Finckii, iterum è tabulis Bartholomæi Pitisei; ut, si quid vitii in ipsas fortè tabulas operarum incuriâ irrepsisset, ex mutuâ collatione faciliè haberet emendationem: cum Pitisei notæ sint etiam ex opere Palatino expressæ; illæ autem Finckii alæ ab his, et ante subductæ.

Tabellam ipsam in libri calcem rejecimus; cui titulum fecimus *Canonica Parallelorum*.

Verùm de constructione et ordine id monendus mihi es, me ab ipso radio additionem idèò auspicatum esse, quod ille ad ipsum æquinoctialem pertineat. Quia enim millaria singula quatuor scrupula illic occupant, etiam quadruplus milliarium numerus numerum scrupulorum ipsis debitorum definiet; et ita nobis eadem analogia redit, ut radius ad radii quadruplum, ita numerus milliarium ad scrupula congrua.

In graduum et scrupulorum notatione eum ordinem secuti sumus, ut æquinoctiali circulo 1 præfixerimus tanquam omnium parallelorum primo principio; reliquos autem parallelos ordine ad hunc disponimus, ut unus gradus sit sexagesimus parallelus, quorum primus sit æquinoctialis: ut ita parallelus qui per undexagesimum scrupulum educitur sit nobis numerus sexagesimus, et qui per scrupulum sexagesimum, sit primus et sexagesimus, sive 1 gr. 1 scr. quod idèò disertis verbis exprimendum duxi, quia aliis canonibus hoc sit dissimile.

Atque hæc canon ipse nobis fuit expressus: sequitur ejus usus, qui etsi ex ipsâ constructionis lege haud obscurus esse possit, placuit tamen eundem hoc theoremate breviter complecti.

PROPOSITIO XIII.

Si inclosæ ab æquinoctiali parallelorum datorum initium ducatur, quadruplum numeri ultimo parallelo unitate aucto respondentis, secundum proportionis terminum explebit: sin aliunde, sed utraq; latitudine simili, differentie inter datos parallelos unitate auctos quadruplum: sin dissimili, parallelorum unitate auctorum summe.

HIC tabularum expositarum usus est primus, ut inde terminus proportionis illius, quam 12 propositione expressimus, assumatur. Sed exemplis res erit

erit illustrior. Quærat enim, quot scrupula longitudinis occupent 750 milliaria, ab æquinoctiali ad parallelum 44 gr. 59 scr. inclusivè, ut singulis parallelis æqualis milliarius portio attribuat. Hic erit tibi assumendus numerus canonicus è regione 45 gr. 30,31.462, nam quem nos vulgò dicimus parallelum 44 gr. 59 scr. is, si æquinoctialem annumeres, quod in istarum tabularum constructione est factum, fiet 45 gr. et comprehendet parallelos 2700. quadruplum porro numeri assumpti est 121,245,848. Jam, milliarius datorum numerus per 2700 parallelos divisus dabit singulis parallelis $\frac{1}{27}$, hoc est $\frac{1}{27}$ unius milliarii. Atque hinc proportio, ut 10,000 ad 121,245,848; ita $\frac{1}{27}$ ad 3367 $\frac{1}{27}$; tot igitur scrupula (qui sunt gradus 56, scr. 7, sec. 55,) data 750 milliaribus, per expositos parallelos æqualitèr tributa, occupabunt. Si data milliaribus solo æquinoctiali fuissent mensuranda, explevisent tantum gradus 50; aut, si in ipso quadragesimo quinto parallelo fuissent æstimanda, hic jam occuparent 70 gr. 42 scr. 38 sec.: ut hæc per singulos intermedios parallelos distributio major sit illà quæ in maximo, minor hæc quæ in minimo, fuit proposita.

Sunt jam iterum à 35 gra. 58 scr. exclusivè usque ad 51 gra. 59 scr. inclusivè (quæ sunt scrupula sive paralleli 960) milliaribus 320 ita tribuenda ut singulis parallelis milliarius portio assignetur, nempe $\frac{1}{32}$ seu $\frac{1}{2}$; numerus itaque canonicus, qui pertinet ad 35 gra. 58 scr. unitate majorem, nempe 35 gr. 59 scr. 23,178,702, de numero canonico 52. 36,665,060 deductus, relinquet 13,486,358, cuius quadruplum 53,945,432. Erit itaque, quemadmodum radius 10,000 ad 53,945,432, ita $\frac{1}{2}$ ad 1798 $\frac{1}{2}$, qui sunt gradus 29, scrupula 58, secunda 11.

Exemplum tertium esto à parallelo austrino 36 inclusivè, ad parallelum boreum 52 inclusivè, et sunt exposita millaria 4000, per singulos parallelos æqualitèr cædenda. Hic quia numerus unitate augendus erit, ut sint 36 gra. 1 scr. et 52 gra. 1 scrup. illi competit numerus Canonicus, 2319,3971. huic autem 3666,5060. summa 5985,9031; et cum in hoc utroque parallelus primus, seu æquinoctialis, sit additus, efficitur in utroque addito, bis adhibitur; qui tamen ideo non sit subducendus ut utriusque latitudinis respectu assumptus intelligatur, quemadmodum in usu infrà planè liquebit. Hujus quadruplum 23943.6123. parallelorum autem numerus 5281, nempe 52 gra. 1 scr. et 36 gra. 1 scr. et quia bis iteratur æquinoctialis, fiunt 88 gr. 1 scr. seu scrupula 5281. ut singulis parallelis cedat $\frac{1}{88}$. milliarius pars quota: unde proportio, quemadmodum radius 10000 ad 23943.6124. ita $\frac{1}{88}$ ad minuta 18156 $\frac{1}{88}$. qui sunt 302 gr. 15 $\frac{1}{88}$ scrup.

Debuisset quidem ex ipso summæ quadruplo radii quadruplum semel subducere, si tantum singulis parallelis semel velis imputare $\frac{1}{88}$ milliariis: sed æquinoctialis sibi bis hanc partem postulat; quod Loxodromiarum usus et demonstratio propositionis 25tæ postea comprobabit.

PROPOSITIO XIV.

Datis evariæ longitudinis gradibus et scrupulis, dabitur quoque mensura milliarium æqualiter parallelis inter datos interjectis debita.

EST conversæ antecedentis, et hæc alter nostri canonis usus est; quâ utrâque omnis Loxodromiarum utilitas clarissimè illustratur. Constat enim ex ipsâ canonis structurâ et propositione 12, eam esse rationem itineris per singulos parallelos æqualiter tributi ad minorum summam, in omnibus parallelis, quantum illud spatium occupat; quemadmodum 10000 ad quadruplum numeri canonici cujusque loci (ut, puta, ab æquinoctiali ad parallelum 44 scr. 59 inclusivè), is locus itaque unitate auctus dabit gr. 45. et hæc erit ratio proposita, ut 10000 ad quadruplum summæ tangentium 30,311,462, nempe 121,245,848: ergo si 1 milliæ in singulis assumatur, erunt minuta 12124 $\frac{1}{8}$, pro longitudinis evariatione. Et contrâ igitur, minorum per omnes parallelos æquali itineris spatio attributorum ratio erit, quæ 12,124 $\frac{1}{8}$ ad 1 milliæ. Itaque si minorum numerus major minörve queratur, utpote si assumantur evariæ longitudinis minuta 1000, et queratur itineris quantitas in singulis parallelis; erit, quemadmodum 12,124 $\frac{1}{8}$ ad 1 milliæ, ita 1000 ad $7\frac{24}{25}$ unius milliæ; atque tantum itineris in singulis parallelis fuerit conficiendum, ut 1000 minuta longitudinis differentiam impleant, cujus veritatem per antecedentem propositionem vicissim è converso experiri tibi liceat.

Si non ab ipso æquinoctiali inclusivè initium assumatur, sed intervallum parallelorum terminetur latitudinis affectione simili, assumatur differentia, et proportio instituat ut prius.

Sin verò latitudo sit dissimilis, altera Borea, et Austrina altera, assumatur parallelorum summa, eaque quadruplicetur, et inde proportio instituat ut ante.

PROPOSITIO XV.

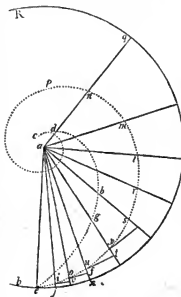
Loxodromia est linea maximè in terreni globi superficie, quam ubique contingens recta linea cum omnibus meridianis per cuncta ejus puncta ductis æquales angulos comprehendit.

SI quando navis rectâ in septentrionem aut meridiem dirigitur, tum per meridianum, et ideo per maximum terræ circulum decurret. Cum autem rectâ vel ortum vel occatum petit, tunc cum omnibus meridianis, per quos fertur, angulum rectum comprehendere, et ideo aliquem parallelum suo cursu describere, suprà docui. At verò cum alium quencunque cursum, ab his duobus diversum, tenebit, atque ad acûs magneticæ situm (qui tamen ante, secundum æstimatam καθύψωθεν emendatus intelligatur) ita se componet, ut τρέψει navis, quæ per mediani carinam à prorâ ad puppim porrigitur, aliquem angulum,

lūm, et semper eundem cum omnibus meridianis quos fecat comprehendit: tūm quidem, quamdiū istum tenebit cursum, helicem lineam describet, quales istæ sunt, quas hic in plano, ad eam formulam ut proximè expressimus; nempe *egbd*, et *eiou*; ubi *efqkb* æquinoctialem lineam, et *a* terræ polum notat; mechanicam ejus, tām in globo quām in plano, delincandi rationem in scholis dicturi. Hanc autem *παρα τῆς τοῦ ὁρίων λοξότητος*, à cursū obliquatione, *λεξιδραμία* voce Græcā indigitamus; qui planè, ut diximus, circulus esse nunquam possit. Nam circuli ad meridianum aliquem obliqui, sive majores ii sint, sive minores, semper alium atque alium angulum, eūmq; varium et reliquis angulis inæqualem, comprehendunt; quemadmodum videre est in ecliptico circulo coluros intersectante. Is enim in communi sectōne cum tropicorum coluro angulos rectos comprehendit; ac circā nodos tantūm complementum maximæ suæ declinationis, nempe 66 gradus et 29 scrupula, intermediis autem locis angulos omnes iater se facit inæquales. In minoribus verò circulis etiā manifestior est evariatio. Loxodromia igitur *ἰσοκύβητος* quædam sui generis linea est, cujus longitudinem et situm in ipso cursu gubernatores, et navium magistri, reliquique navales pedes ad hanc rem idonei, accuratè et diligenter observant, et pro velocitate aut tarditate conjecturis accuratissimè æstimatam seorsim notant. Hinc enim certissima motatæ tūm longitudinis tūm latitudinis indicia, (modò nihil in his conjecturis sit peccatum) ab ipsis derivari solent; ut inde ad quas oras appulerint, quove usque sit perventum, hoc indicio cognoscant et definiant. Hujus itaque lineæ accurata et constans mensura, et angulus, quem ipsa cum acūs magneticæ sitū legitimo, hoc est, cum vero loci meridiano, comprehendet, utramque in hac arte paginam nobis faciunt; cui aded rei, hanc unam ob causam, sedulam aded operam navant. Nam acūs magneticæ situm luxatum, summā, quā possunt, curā quotidianis observationibus restitunt. Et cursū mensuram jactu funis, adhibita clepsydra momentanea, explorant, hoc ferè modo. Funem notæ longitudinis decem aut viginti decempedatum, etiā puppis perpendiculo, tanquam trianguli rectanguli crure æstimato, è puppi præcipitant, cui in fine lignum cimbulamve alligant, quæ transversim ponè in maris sulco, à concitato navis motu facto, innatet; ut ita, dum funem à glomere minimo momento versabili deglomerant, eidem quasi loco affixa hæreat; interim ab ipso initio clepsydra momentanea temporis spatium in ipsā glomeris evolutione elapsū mensurant: et inde, quantum singulis horis tali successu proficiant conjecturam ca-

Vol. IV.

X



Methodus æstimandi velocitatem quæcum navis movetur per jactum ligni parvi in mare ad navem per funem ligati, et deglomerationem funis per parvulum aliquod tempus.

piunt :

piunt : quod ipsum accuratiores navium magistri sæpiùs iterant, pro ventorum vario imperu ; ut hujusmodi observationibus quàm proximè divinando ad verum accedant. Atque ita ex angulo confectæ Loxodromiæ et ejusdem quantitate, postea latitudinis et longitudinis evariatæ rationem ineunt. Tanti sanè fuerat Loxodromiarum rationes habere benè explicatas. Ingeniosum utique commentum, quod tamen hætenùs duntaxat processit, ut ejus affectiones mechanice tantùm usurpari, neque accuratiores calculo explicari potuerint. Nos jam omnes ejus affectiones etiàm numeris, qui ~~axodromia~~ ^{axodromia} sunt geometricis lineamentis, persequi docemus, quæ res singularem habet utilitatem cum summâ facilitate conjunctam. Nam mechanica eorum praxis juxtà est et operosa et lubrica, quæ in chartis hydrographicis illis exercetur, quæ terreni globi situm nuspiàm ad amissum refert ; quæque adeò immane quantum à vero abludit ; ut non tam mirandum quàm necessarium sit hic mechanicen istam suos affectus in errorem manifestarium dare præcipites.

PROPOSITIO XVI.

Loxodromia nulla se in terræ polos induit.

CONSECTARIUM est ex antecedente propositione derivatum. Cum enim loxodromiæ forma ex angulo, quem cum meridiano loci comprehendit, tota censeatur, et eâ sublatâ loxodromia ampliùs nulla sit, materiæ vicem subibit ipse meridianus, quo subducto, tanquam cruribus anguli sublati, angulus planè nullus comprehendi posse intelligitur ; ut ita loxodromiæ et res et ritulus concidant. Atqui sub polis et mundi cardinibus in hoc Terræ globo nullum meridianum esse, et ratio et demonstratio convincit. Neque enim in illo terræ puncto sidus ullum, aut cœli quodcunque aliud immobile punctum vel oritur vel occidit : unde meridiani notio solùm derivatur, quemadmodum nominis definitio arguit ; qui ita Latinis meridianus, Græcis *μεσημβριος*, quasi meridianus aut *μεσημβριος* indigitatur, quod spatium temporis inter ortum et ejusdem puncti occasum medium fecit. Quod autem à solis tantùm ortu et occasu meridianus dicatur, id tantùm ab usu potiore, et ~~φανωμύ~~ ^{φανωμύ} in omnium oculis incurrente, est derivatum.

— *Medio cum sol altissimus orbe
Tantum respiciet, quantum superesse videbit.*

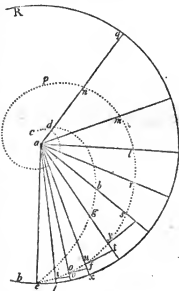
Et si hoc illuc locum quoque habeat, jam istuc meridiani erunt omninò mutabiles et ambulatorii, atque quotannis varii ; cum sol nobis hic modò in cœli medio, inde in ortu, tùm in cœli imo, aliàs in occasu ; denique semper in aliis atque aliis locis in æquinoctialia puncta ingreditur. Atque ita, pro hac aut illâ bisectione, meridiani circuli situs erit inconstans ; nonnunquam enim illis meridianus esset idem qui nobis : aliquando ille circulus horarius, qui nobis horam sextam matutinam signat : aliàs alius, et semper varius. Quamobrem cum hic nullus meridianus detur ; etiàm affectiones, quæ ipsi attribui solent, simul cum ipso evanescent. Et propterea loxodromia (quæ à sectione meridiani

diani nomen et formam fortitur,) planè nulla concipi hoc loco sub polis poterit; ut, in diagrammate propositionis antecedentis, Loxodromia *cio u*, continuata porro in *m*, *n* et *p*, atque inde in *c*, nunquam tamen in *a* solum incurret, sed, infinitis gyris circumacta, ad illum punctum nunquam perveniet.

PROPOSITIO XVII.

Loxodromia est instar basis trianguli plani rectanguli ad sphaera superficiem applicati, cujus crus unum sit distantia parallelorum inter quos intercipitur.

INCLINATIO anguli quem navium carina cum meridiano comprehendit, postquam à perpendiculari demutat, varia omnino esse potest, et ideo loxodromiae helices infinitae, alium atque alium cum meridianis angulum comprehendentes. Angulum autem recto minorem hic inclinationis angulum vocamus. Cum autem singularum una eademque sit ad omnes meridianos inclinatio, et per singula puncta, etiam vicinissima, paralleli circuli meridiano perpendiculares cogitatione traduci possint, vides hic triangulum efformari rectangulum. In partes porro minutissimas et meridianus et loxodromia concidi possunt, ut vix ullo calculo eorum à rectis lineis differentia exprimi aut deprehendi queat; atque hae particulae in alias minores millesimas. Et quamvis rectae et curvae differentia hoc sectionum minutali haud possit tolli, neque unquam quantulacunque curvae lineae pars recta sit, tamen ad sensum et usum omnino evanescit; hoc igitur minimum, quod hic concipimus, loxodromiae segmentum, fiet anguli recti basis, segmentum autem meridiani crus unum. Ad istam planè formulam, quemadmodum accedit triangulo rectangulo, cujus crus unum cylindri peripheriae aequale sit, si alterum lateri cylindri recti applicetur, hoc autem circa cylindrum iassectatur, tum basis trianguli helicem cylindraceam in ejus superficie designabit. Quod Pappus prop. 24, l. 8, disertè docet; ut illa helix nostrae helici loxodromicae, et illud crus trianguli lateri congruens, vel, quod idem sit, axi parallelum, meridiani segmento hic respondeat: quamvis illius designatio et explicatio non paulò sit facilius quam hujus nostrae; quod circuli paralleli illi sit aequales: hic autem longè secùs sit. Intelligamus itaque in loxodromia *cio u* segmenta meridi-



meridianorum *ji*, *vo*, *uf*, esse æqualia, et angulos *rij*, *iov*, *cus* æquales, *ej*, *iv*, *es* parallelorum circulorum segmenta non quidem similia, sed æqualia tantum. Hic basis trianguli totius esset *eiou*; crur. porro reliquum *ux*, è segmentis *ij*, *ov*, *uf*, compositum intelligatur.

PROPOSITIO XVIII.

Ejusdem loxodromiæ segmenta inter parallelos circulos æquali intervallo disjunctis intercepta sunt æqualia.

ET hoc quoque expositam loxodromiarum formam consequitur. Sit enim loxodromia aliqua quæ ad æquinoctialem usque pertingat à gradu latitudinis primo, quemcumque tandem cum meridianis angulum comprehendat. Quod si hic per singula minora paralleli circuli transire intelligantur, sient sexaginta hujus loxodromiæ partes inter hos interceptæ, et ideò per ipsa sectionum puncta ductis parallelis totidem sient quasi triangula rectangula; quorum crur. unum, meridiani nempe segmentum, cruri alterius, et angulus unius, à Loxodromiâ, tanquam base, et meridiano comprehensus, angulo alterius pariter constituto, sit æqualis: et triangula ideò ipsa quoque similia et æqualia. Aut, si ne hoc quidem satis sit, per singula secunda scrupula paralleli ducti intelligantur, et sient rectangula triangula 3600; vel, si etiâ minora segmenta concipias, tanto faciliorem habebit explicationem, et cum planis triangulis affinitatem; ut unus gradus in meridiano non tantum in tertia, aut quarta, sed etiam in decima vel millesima scrupula concisus intelligatur. Atque ita in his minutissimis triangulis singula crura cruribus, et bases, (quæ sunt loxodromiæ istius segmenta,) basibus, æquabuntur. Et istud helici cylindraceæ quoque responder, cujus segmenta inter circulos parallelos, et æquali intervallo disjunctos, æqualia omnino sunt: sed ista segmenta sunt præterea etiam congrua, ob illius lineæ ἀμεσότηταν, ut Apollonius apud Proclum demonstravit; quod in nostrâ helice non assequimur. Quare ejusdem helicis segmentum inter æquinoctialem et parallelum primum tantum erit, quantum inter 45 et 46, aut 50 et 51, vel 80 et 81, atque ita porro de omnibus qui dari possunt parallelis. Id in superiore diagrammate lineis *ji*, *vo*, *fu*, expressimus.

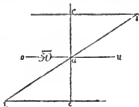
Terræ autem polus punctum est, non parallelus; et nullo loxodromiæ cursu ad eum deveniri posse supra demonstravi, quia helix ista ante deficiat quam ad eum pertingat. Nihilominus loxodromiarum segmenta à puncto polo proximo inter duos parallelos quocunque intervallo (ut, puta, unius minuti) disjunctos tanta sunt, quanta inter parallelos propè æquinoctialem eodem intervallo distantes; sed ita, ut minimum hoc loxodromiæ segmentum illic sæpius revolvatur, antequam intrâ positos parallelos suam quantitatem explicet.

PRO-

PROPOSITIO XIX.

Datâ quantitate et angulo inclinationis loxodromiæ, parallelorum distantiam invenire.

EXPONATUR loxodromia cujus longitudo sit milliarius maritimorum, vel nauticorum, 50, angulus inclinationis 56 graduum et 15 scrupulorum: quæritur latitudinis evariatio. Id totum per triangulorum planorum doctrinam secundum demonstrata explicari potest. Concipiamus enim triangulum rectangulum *aei*; ut *a* sit loxodromiæ initium; et meridianus illius loci *ae*; *ai*, loxodromiæ mensura, milliarius 50; *eai*, angulus inclinationis, 56 gradus et 15 scrupula. Cum igitur, per propof. 17, *ae* meridiani segmentum sit instar cruris, et loxodromia *ai* instar basis anguli recti *a ei*, et angulus *eai*, complementum inclinationis, detur 33 graduum et 45 scrupulorum; erit quemadmodum radius 10,000,000, ad sinum anguli *eai*, 33 grad. 45 scrup. 5,555,702, ita *ai*, 50 milliaria, ad $27\frac{11}{100000}$ milliaria; tot igitur milliarius erit segmentum meridiani *ae*. Ea porro ad scrupula per proportionem reducentur hoc modo: 15 milliaria sunt scrupula 60, hoc est, 1 milliare valet scrupula 4; ergo $27\frac{11}{100000}$ dabunt scrupula $111\frac{11}{100000}$; quæ faciunt unum gradum et 51 scrupula et $\frac{11}{100000}$ seu $\frac{11}{100000}$ partes scrupuli: et propterea *e* punctum, seu parallelus *ei*, in austrum vel in boream à parallelo *au* disjungetur 1 gradu et 51 scrupulis, et $\frac{11}{100000}$ partibus scrupuli. Quamobrem si statuatur *a* punctum in parallelo 51, et *ae* versùs boream porrigi, erit *ei* parallelus per gr. 52, scr. 51,eductus quam proximè. Contrà, si in austrum, jam ad 49 gra. 9 scr. pertingeret; atque ita porro in reliquis. Aut concinnius breviusque hoc modo:



Ut radius ad quadruplum finis complementi inclinationis datæ loxodromiæ; ita milliaria longitudinis ejusdem, ad scrupula evariatæ latitudinis.

Res ex positâ analogiâ manifesta est. Ne longè discedamus, eodem utar exemplo: quadruplum finis 33 gr. 45 scr. est 22,222,808, unde secundum positos terminos 10,000,000 22,222,808, 50, concludes quantum $111\frac{11}{100000}$, ut suprà.

PROPOSITIO XX.

Dato parallelorum intervallo cum loxodromiæ inclinationis angulo, ejusdem mensura queque datur.

UT in diagrammate antecedente, si detur *ea* meridiani segmentum 1 gr. 51 scr. et angulus loxodromiæ *eai* 56 graduum et 15 scrupulorum, sit, per propositionem 18, triangulum rectangulum *eai*, cujus angulus et unum recti crus huic adjacens

jacens datur: nam quaterna minuta in maximo circulo unum milliare nauticum exæquant; et propterea 1 gr. 51 scr., seu scrupula 111, fient milliaria $\frac{1}{4}$ '. Atque inde fit hæc proportio; ut radius 10,000,000, ad secantem 56 gr. 15 scr. 17,999,524, ita milliaria $\frac{1}{4}$ ' ad 497,836,666; tanta igitur erit longitudo loxodromiæ *ai* inter istam parallelorum distantiam, sub hoc inclinationis angulo dato. Sed concinnius et elegantius fuerit si fiat,

Ut radius ad quadrantem secantis anguli inclinationis datæ loxodromiæ; ita scrupula evariatæ latitudinis, ad longitudinem loxodromiæ optatam.

Id quoque ex præmissâ demonstratione fuerit perspicuum. Nam, cum scrupulorum evariatæ latitudinis quadrans sit tertius proportionis terminus, secans autem secundus; nihil refert an hunc, an illum, per 4 dividam. Res experienti manifesta est.

PROPOSITIO XXI.

Dato parallelorum intervallo cum loxodromiæ quantitate, inclinationis ejusdem angulus quoque dabitur.

HIC enim datâ basi et crure quæritur angulus ab ipsis comprehensus. Fiat itaque, ut quadruplum milliarium loxodromiæ datæ, ad scrupula evariatæ latitudinis; ita radius ad sinum complementi inclinationis: vel, ut scrupula evariatæ latitudinis, ad quadruplum milliarium loxodromiæ; ita radius ad secantem inclinationis. Horum veritas ex antecedentium analogiâ est manifesta.

PROPOSITIO XXII.

Crus alterum trianguli loxodromici integrum simul imaginarium est; sed per minimas particulas singulis parallelis æquali intervallo disjunctis æqualiter attribuendum; quod idem vocetur • $\mu\kappa\omicron\delta\omicron\upsilon\alpha\mu\iota\kappa\omicron\delta$.

CUM enim, per propositionem 17, Loxodromia sit quasi basis trianguli, cujus unum crus sit segmentum meridiani distantiae parallelorum inter quos intercipitur æquale, reliquum crus nobis hic quoque quærendum est. Id autem simul totum in superficie sphericâ non datur; quia circuli in spherico inæquales sunt, et polo viciniore minores remotioribus et æquinoctiali propioribus. In helice quidem cylindraceâ et ejus triangulo, in singulis parallelis circulis, si perpendicularare latus in æqualia segmenta tribueretur, crus anguli recti jacens erat segmentum circuli proportionale et æquale reliquis, per æqualia perpendiculari segmenta inter se distantibus. Hic verò etiam si meridiani segmenta, quæ sunt tanquam perpendiculari segmenta, sint inter se æqualia; et loxodromiæ, quæ sunt bases

• Evariationem longitudinis potentiâ complexum.

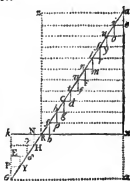
angulo recto subtensæ, etiã inter se æquales sint; atque idẽ reliqua crura jacen-
tia, quæ parallelis imputanda sunt, æqualia esse debeant: quia tamen paralleli suc-
cedentes (si à polis initium ducas) antecedentibus sint majores, segmenta etiã
minima, ut in ipsis necessariò quidem sint æqualia; neutiquam tamen erunt simi-
lia: sed in majoribus circulis pauciora, in minoribus plura, occupabunt minuta.
Hæc autem omninò exemplo proposito erunt illustriora. Sit triangulum loxo-
dromicum è propositione antecedente *aei*, cujus basis *ai* sit milliarius
49⁷/₁₀₀₀₀, crus alterum *ae* (quod erat meridiani segmentum,) dabitur 1 gr.
51 scr. seu milliarius 27¹/₂; itaque reliquum, quasi *ei*, per tabulas tangentium
invenietur milliarius 41¹/₁₀₀₀₀. Hoc quidem uno nomine et sub unâ con-
tinuâ lineâ in sphaeræ superficie nusquam datur. Sed in minimas particulas
per singulos, et quasi infinitos, parallelos distribuendum intelligitur; et in quanto
plura triangula id diffundetur, tanto ea usui erunt aptiora. Nos ab æquinocti-
ali, ad parallelum septuagesimum, parallelos per singula scrupula ducere, et
totidem triangula delineare sufficere existimamus: ut majore *πρωτεύειν* non sit
opus, quemadmodum infrâ demonstrabitur. Atque ita in proposito exemplo
futuri sunt 111 paralleli, quot erunt minuta à principio ad finem. Et idẽ tot
rectangula triangula quoque concipiantur, quot inter hos parallelos efformari
poterunt; ita ut unum crus perpendiculare sit minutum unicum in meridiano
per loxodromiæ et parallelorum communes sectiones eductum, crus reliquum
pars tanta cruris *πεποδωμέναις* in parallelis. Sed, ut dixi, quanto horum paral-
lelorum distantia minor erit, tanto quoque accuratior erit æstimatio minutorum
per singulos parallelos diffusorum; quia ista nobis longitudinis æstimationem
præstare debent. Itaque crus *πεποδωμέναις* constat per doctrinam triangulorum
hoc modo.

PROPOSITIO XXIII.

*Triangula loxodromica unitate sunt pauciora parallelorum numero, quot à primo
ad ultimum intercipiuntur includit.*

SIT in exposito diagrammate *kx* primũ cir-
culus ipse æquinoctialis; et *az* parallelus per
undecimum minutum eductus; loxodromia pro-
posita *aR*. Si ergò per singula minuta paralleli
educantur, (ut hic vides *fp*, *dg*, *ed*, et cæteros,) habebis undecim; quibus si ipsum æquinoctialem
kx annumeres, erunt in universum duodecim
paralleli, triangula autem duntaxat undecim; quia
vertex ultimi trianguli in parallelo *za* tantum
terminatur, ut hic vides. Si aliundè quam ab
æquinoctiali initium ducas, res eadem erit, et
analogia respondet.

Cum autem in parallelorum canonicis initium ducatur
ab ipso æquinoctiali, et nos vulgè ita loquamur,



ut eum excludamus; ut cum dicimus, parallelus per 1 gr. 7 scr. hic eum intelligimus, qui per istud scrupulum transit: atque idè præter æquinoctialem essent paralleli per singula minutaeducti 67; huc si annumeretur æquinoctialis, sunt paralleli 68; et idè triangula 67: ut ita tu'd tibi liceat ex tabulis canonicis assumere quadruplum numeri, cui 1 gr. 7 scr. attribuitur. Idem erit si aliunde quam ab æquinoctiali initium ducas.

PROPOSITIO XXIV.

Si initium loxodromiæ ab æquinoctiali ducatur, totidem erunt triangula loxodromica, quot scrupulis inde extremus parallelus distabit.

ID in superiore exemplo manifestum est; nam cum illic assumam parallelum: ultimum az minutis undecim ab æquinoctiali distantem, primus autem sit æquinoctialis in exposita figurâ, qui tamen in hac vulgari et usitatâ parallelorum numeratione in censum non venit, quia parallelus primus intelligitur, qui minuti intervallo ab ipso æquinoctiali sit disjunctus: efficitur, ut minorum numerus qui hujusmodi parallelorum est index, à numero triangulorum interceptorum non discrepet; sed unus, et planè idem sit.

PROPOSITIO XXV.

In latitudine simili numerus minorum differentie parallelorum, cognominis est numero triangulorum loxodromicorum: in dissimili verò, numerus summæ.

SIT jam in eodem diagrammate kb parallelus 52 gra. et az parallelus 52 gr. 11 scr. Hic igitur si utrumque extremum inclusivè assumas, sunt paralleli quidè duodecim, triangula autem loxodromica duntaxat undecim, quantus est scrupulorum differentie numerus, si 52 gr. 11 scr. deducas. Si latitudo sit dissimilis, altera borea, et reliqua notia; summa scrupulorum latitudinis utriusque in unum conflata triangulorum intermediorum numero erit cognominis. Sit enim kx iterum æquinoctialis, az parallelus boreus per scrupulum undecimumeductus; parte adversâ $a9$ parallelus notius per latitudinis notie scrupulum quartum. Illic igitur, per prop. 23, erunt triangula undecim, hic quatuor, summa triangulorum quindecim, quanta quoque est scrupulorum.

PRO

PROPOSITIO XXVI.

Si trianguli loxodromici crux $\mu\kappa\sigma\delta\omega\alpha\mu\iota\kappa\delta$ per parallelorum minutatim distantium differentiam, in latitudine simili dividatur, quotus erit pars singulis à maximo inclusivè ad minimum exclusivè equalitèr attribuenda.

SIT in eodem diagrammate kx crux $\mu\kappa\sigma\delta\omega\alpha\mu\iota\kappa\delta$ milliarius 17½, et k in æquinoctiali, a in latitudinis scrupulo 11. Itaque, per 23 propositionem, hic erunt triangula loxodromica 11 et totidem crura singulis parallelis attribuenda, quorum primum æquinoctiali debetur, secundum fp parallelo per scrupulum latitudinis primum, tertium bg parallelo per scrupulum secundum, denique undecimum ie parallelo per scrupulum decimum; ut ab hoc numero parallelus ultimus sive undecimus excludatur: quia vertex novissimi trianguli iae eo tantum pertingit. Quamobrem crure mecodynamico 17½ milliarius per 11 divisio, pars quota erit 1½ milliarius singulis parallelis attribuenda; ut kb , fp , bg , reliquaque sigillatim habeant longitudinem sesqui-milliarius. Atque ita hoc totum latus imaginarium per partes æquales singulis, quotcunque concipimus, parallelis attribuitur. Res eadem erit si non ab ipso, sed aliunde à quocunque alio parallelo, initium facias. Cujus veritas è propof. 23, et 25 quoque, manifesta est.

PROPOSITIO XXVII.

Loxodromiæ æquinoctialem secantis crux $\mu\kappa\sigma\delta\omega\alpha\mu\iota\kappa\delta$ ab æquinoctiali in suas utrimque: partes est distribuendum; et æquinoctialis geminam habet hoc casu portiunculam: extremi autem paralleli utrimque excluduntur.

SIT bx æquinoctialis; crux $\mu\kappa\sigma\delta\omega\alpha\mu\iota\kappa\delta$ a 9; cui æquetur Kx . Itaque pars Kt parallelis notis, et kx parallelis boreis, est imputanda, per antecedentem. Atque ita fit ut $k\lambda$, kb geminæ portiunculæ æquinoctiali attribuendæ sint, extremus autem uterque hic æquo jure excludatur.

PROPOSITIO XXVIII.

Datis angulo loxodromiæ et latitudinis evariatione, crux mecodynamicum invenire.

UT radius ad quadrantem tangentis datæ loxodromiæ, ita evariatæ latitudinis minuta, ad milliaria cruris mecodynamici. Sit enim distantia parallelorum oa , ie minorum 32, et angulus inclinationis iae 56 gra. 15 scr. Erit itaque in triangulo rectangulo iae , ut radius 100,000 ad tangentem 56 gra. 15 scr. 149,660, ita evariatæ latitudinis segmenti ae milliaria, quæ sunt pars.

Vol. IV.

Y

quarta.

PROPOSITIO XXXI.

Datâ Loxodromiæ quantitate cum angulo inclinationis, datur euariatio longitudinis.

LONGITUDINIS euariatio, et suprà, et apud geographos semper, est distantia duorum meridianorum, in gradibus et scrupulis, seu minutis, æstimaia. Hujus autem euariationis determinatio est Theorema maximi in re maritimâ et loxodromicis usûs, et hætenûs à multis frustra sollicitatum. Hic autem, tanquam in triangulo rectangulo, datâ basi cum angulo obliquo, utrumque crus querendum tibi est. Hoc quoque perinde ut antecedens theorema, triplici exemplorum genere explicare operæ videtur pretium. Sit igitur primum ab æquinoctiali navigatum per milliaria 100, angulo inclinationis loxodromiæ 33 gr. 45 scr. Hic, per propositionem 19, dabitur euariatio longitudinis $332\frac{1}{2}\frac{1}{2}$ scrupulorum, sive 5 grad. $32\frac{1}{2}\frac{1}{2}$ scrup. Sed, quia plus dimidio scrupulo hic abundat, assumam 5 gr. 33 scr.; ut ad hunc usque ab æquinoctiali parallelum deventum esse constet; et per propositionem 28 cruris $\mu\epsilon\tau\alpha\delta\upsilon\lambda\alpha\mu\epsilon\kappa\alpha$ pars quota singulis parallelis attribuenda $\frac{1}{10}\frac{1}{10}\frac{1}{10}$ unde, per propositionem antecedentem, dantur euariatæ longitudinis 3 gr. 42 scr. 47 scr.

Secundò, sit à parallelo 47 navigatum angulo inclinationis loxodromiæ 33 gr. 45 scr. intervallo milliarium 100: queritur euariatæ longitudinis quantitas. Cum, per 18 propositionem, loxodromiæ ejusdem segmenta inter æquales parallelos sint æqualia, igitur, per 19 propositionem, datur parallelorum distantia $332\frac{1}{2}\frac{1}{2}$ sive, quod idem sit, 5 gra. 33 scr., qui, à 47^m parallelo numerati, incident in 52 gra. 33 scr. Hinc jam pars quota cruris $\mu\epsilon\tau\alpha\delta\upsilon\lambda\alpha\mu\epsilon\kappa\alpha$ singulis parallelis attribuenda dabitur per propositionem 28, nempe $\frac{1}{10}\frac{1}{10}\frac{1}{10}$. Inde per 29 propositionem dabuntur scrupula euariatæ longitudinis; nam differentie numeri in canonicis parallelorum, qui pertinet ad 47 gr. et numeri qui pertinet ad 52 gra. 33 scr. quadruplum est 20,660,132; unde fit hæc proportio, per propositionem 30; quemadmodum 10,000 ad 20,660,132, ita $\frac{1}{10}\frac{1}{10}\frac{1}{10}$ ad $345\frac{1}{2}$ scrupula, qui sunt 5 grad. 45 scrup. 7 scr. Ita vides eandem cursûs hujus inclinatione et mensurâ, inter hos parallelos longitudinis euariationem 2 gr. majorem esse quàm circa æquinoctialem.

Exemplum tertium tale esto: à latitudine austrinâ 36 gr. versûs æquinoctialem et ultrâ navigatum sit milliaribus 1500, angulo inclinationis loxodromiæ 33 gr. 45 scr. ut suprà; queritur longitudinis euariatio. Hic, per propof. 19, datur parallelorum distantia 4989, qui sunt gradus 83, scr. 9, ut ideò terminus loxodromiæ pertingat ad parallelum boreum 47 gr. 9 scr. Jam pars quota cruris $\mu\epsilon\tau\alpha\delta\upsilon\lambda\alpha\mu\epsilon\kappa\alpha$ singulis parallelis attribuenda, per propof. 27, datur $\frac{1}{10}\frac{1}{10}\frac{1}{10}$. Hinc assumuntur numeri è canonicis parallelorum, qui pertinent ad 36 gr. et ad 47 gr. 9 scr. Horum summa quadruplum est 221,342; unde fit hæc proportio; ut 100 ad 221,342, ita $\frac{1}{10}\frac{1}{10}\frac{1}{10}$ ad 3699 $\frac{1}{2}$ scr. qui sunt fanè 61 gr. 39 scr. Atque tanta esset intrâ dictos parallelos euariatio longitudinis, ut proximè.

Si loxodromia data sit aliqua è vulgò usitatis, poterit idem per canones $\mu\epsilon\tau\alpha\delta\upsilon\lambda\alpha\mu\epsilon\kappa\alpha$ facilius explicari, quemadmodum aliquot exemplis, in illarum tabularum constructione et usu, insâ libro secundo demonstratur.

P R O.

PROPOSITIO XXXII.

Dato parallelo cum latitudinis et longitudinis evariatione, loxodromiæ inclinationem et quantitatem invenire.

UT loxodromiæ inclinatio inveniat in longitudinis et latitudinis differentiâ, omnino opus est latitudinis initium consignari, et præterea in quam plagam discedatur à dato parallelo, nempe, an versùs polos, an versùs æquinoctialem; nam pro illius principio et plagâ loxodromiæ inclinatio quoque variari potest. Exemplum primum tale esto. Ab insulâ Divi Thomæ sub ipso æquinoctiali ad ostium Fluvii Argentei in Brasiliâ, vulgò *Rio de Plata* dicti, longitudinis differentia statuitur 60 gr. et hujus est ab æquinoctiali distantia 34 gr. præter præter. Quæritur quâ cursus inelinatione, seu loxodromiâ, ab insulâ D. Thomæ ad dicti fluvii ostium, et quanto ejusdem cursus intervallo sit navigandum. Hic igitur ex duobus cruribus quærenda est basis, eum angulo altero obliquo. Nam differentia latitudinis est totius trianguli loxodromici crus alterum; è differentia longitudinis autem crus reliquum erit investigandum. Primum 34 gr. in meridiano dabunt milliaria 510: tum è differentia longitudinis 60 gr. dabitur, per propositionem, in triangulo loxodromico cruris meodynamici pars quota in milliariis, singulis parallelis attribuenda, $\frac{44444}{100000}$. Hæc per numerum parallelorum, 2040, multiplicata (tot enim trianguula loxodromica interfunt per propof. 23,) dabunt milliaria $845\frac{1111}{10000}$. Additis igitur horum crurum quadratis, nempe à 510 et $845\frac{1111}{10000}$, dabitur etiam basis 987 $\frac{1111}{10000}$ milliaria: atque tanta erit distantia loxodromiæ quantitas. Denique è datis cruribus dabitur quantitas anguli inelinationis: *ut enim milliaria evariatæ latitudinis 510, ad crus meodynamicum $845\frac{1111}{10000}$; ita radius ad tangentem anguli inelinationis, per propositionem*: unde ipse angulus loxodromiæ quæsitæ datur 58 gr. 54 fer.

Exemplum alterum esto, ubi utraque latitudo similis. A *Faiæl*, insularum Flandriarum unâ, ad insulam prope fretum Davis, quam, à Pario lapide Jaspidem referente, *Bellam*, tanquam pulchellam, voeant, quâ loxodromiâ contendendum sit, quæritur: utriusque autem latitudo est Borea, illius 38 gr. 50 scr. hujus autem 52 gr. 21 scr. differentia longitudinis sit 16 gr. 12 scr. Itaque numeri parallelorum eanonici utriusque latitudinis erunt 3699. 1192 et 2531: 2354. Horum differentia autem quadruplum erit 4668. 7352. idèque (per propof. 14.) 4468 $\frac{7}{10}$ minuta evariatæ longitudinis attribuunt singulis parallelis unum milliaria; ergò 16 gra. 12 ser. hoc est, ser. 972 dabunt $\frac{1111}{10000}$ milliaria: parallelorum autem differentia est scrupulorum 811; unde latus meodynamicum dabitur 168 $\frac{11}{10000}$ milliaria, et ex his eruribus basis loxodromicæ longitudine 263 $\frac{11}{10000}$, et angulus inelinationis loxodromiæ 39 gr. 47 scr.

Exemplum tertium tale esto, ubi utraque latitudo sit dissimilis. Bantam in Molucis habet longitudinem à longitudine Capitis de Aguillas, prope Caput Bonæ Spei in extremo Africæ promontorio, secundum accuratioris nostrorum hominum observationes, distantiam 86 gr. 50 scr. Latitudo autem Bantam est 6 gr. 45 scr. austrina, et latitudo Capitis de Aguillas est 35 grad. 0 ser. borea: quæritur

queritur quo loxodromiæ angulo à Bantam ad Caput de Aguillas fit navigandum, et quanto ejusdem intervallo. Hic igitur primum differentia latitudinis, quæ est illic 6 gr. 45 scr. austrina, et hic 35 gra. borea, erit 41 gra. 45 scrup. sive scrupulorum 2505. Unde datur è Canonicis parallelorum primum 405. 9366, deinde 2244. 1764; summæ simul utriusque quadruplum 10,600. 4520. Differentia autem longitudinis 86 gr. 50 scr. efficit scrupula 5210. Unde, per propositionem 14, concludes hoc modo: 10,600. $\frac{4520}{10600}$ scrupula longitudinis occupant in singulis his parallelis 1 milliaria; ergo 5210 scrupulis cedunt in singulis parallelis $\frac{5210}{10600}$ milliariis. Sunt autem paralleli simul in basibus triangulorum mecodynamicorum 2505. Atque ideò in universis simul, pro basi mecodynamico communi, milliaria 1231 $\frac{2505}{10600}$. Latitudinis autem differentia habet milliaria 626 $\frac{1}{2}$. Atque ex his trianguli rectanguli cruribus datur basis, quæ est loxodromiæ longitudo, 1381 $\frac{2505}{10600}$ milliarium. Denique etiam angulus inclinationis: ut enim differentia latitudinis 626 $\frac{1}{2}$ milliarium ad crus mecodynamicum 1231 $\frac{2505}{10600}$, ita radius ad tangentem anguli inclinationis, qui hinc datur 63 gra. 3 scrup. quare angulus loxodromiæ erit tantus.

PROPOSITIO XXXIII.

Dato parallelo, et loxodromiæ inclinationis angulo, cum evariatione longitudinis, loxodromiæ quantitatem et latitudinis evariationem invenire.

SIT igitur ab insulâ Divi Thomæ sub æquinoctiali navigatum versùs ostium fluminis argentei, seu *Rio de Plata*, (quemadmodum Hispanico idiomate vocatur,) sub angulo inclinationis loxodromiæ 58 gr. 54 scr. ut hoc cursu longitudinis evariatio fit facta graduum 60; queritur loxodromiæ quantitas cum evariatione latitudinis. Id erit concludendum hoc modo, ut primum experiaris quantum deni, vel viceni, aut triceni latitudinis gradus sub hac inclinatione longitudinem evariant: quæ res perfacilem habet explicationem è propositionis vigesimæ octavæ confectario. Quadrans enim tangentis datæ inclinationis loxodromicæ, per radium divisus, dabit milliaria cruris mecodynamici ex æquo singulis parallelis æqualitèr attribuenda. Ut si detur inclinatio 58 gr. 54 scr. Nam tangentis hujus 165,772 quadrans 41,443 per radium divisus dabit $\frac{41443}{165772}$ unius milliarii in singulis parallelis. Si itaque per hunc numerum multiplices illius è parallelorum canonicis numeri, qui ad 10 gr. pertinet, quadruplum, dabitur evariatio longitudinis, quanta sub hoc cursu ad parallellum decimi gradus ab æquinoctiali à principio ad finem intercedat, per propositionem 30^{am}, nempe scrupula 398 $\frac{41443}{165772}$. Unde proportio hujusmodi instituetur. Quemadmodum 398 $\frac{41443}{165772}$ scrupula inventa, ad optata 3600 (tot enim scrupula 60 gradibus evariatæ longitudinis continentur;) Ita assumptus numerus parallelorum canonicus 10 grad. 240. 1940 ad numerum quantum 2171 $\frac{41443}{165772}$: qui omnino querendus tibi fit in canonicis parallelorum; cui illic respondent 34 gr. 0 scr. 19 scr. Aio itaque hoc cursu navigandum ab æquinoctiali ad parallellum 34 gr. ut longitudinis evariatio fit 60 gr. Si enim numerum inventum 2171 $\frac{41443}{165772}$ per $\frac{111111}{1000000}$ multiplices, sunt scrupula 3600.00; hoc est, gradus evariatæ longitudinis 60 exactè,

exactè, quemadmodum postulabatur. Veritas hujus factionis liquet ex ipsa canonicorum nostrorum numerorum affectione, et terminis multiplicationis. Namque canonici 10 gra. numeri quadruplum $960\frac{77}{10000}$, multiplicatum per milliari-um numerum mecodynamicum $398\frac{17}{1000}$, dat scrupula evariatæ longitudinis $398\frac{17}{1000}$. Atque in secundo quadruplus numeri canonici, quem jam assumimus pertinere ad 34 gr. nempe $8686\frac{6}{10000}$, multiplicatus per eadem milliaria mecodynamica $398\frac{17}{1000}$, dat evariatæ hujus latitudinis scrupula 3600. Hoc modo

$$\begin{array}{r} 960\frac{77}{10000} \\ \times 398\frac{17}{1000} \\ \hline 398\frac{17}{1000} \end{array} \qquad \begin{array}{r} 8686\frac{6}{10000} \\ \times 398\frac{17}{1000} \\ \hline 3600 \end{array}$$

Quamobrem facti erunt proportionales multiplicatis. Erit igitur, ut $398\frac{17}{1000}$, nempe scrupula longitudinis inventa, ad suum numerum $960\frac{77}{10000}$: ita scrupula proposita 3600 ad suum numerum, cujus quadrans est canonicus, et in tabulâ 2171 $\frac{17}{10000}$, ostendit parallelum 34 gr. unde, per propositionem, loxodromiæ ipsius longitudo invenietur per quadrata crurum trianguli totius mecodynamici. Crus longitudinis mecodynamicum è 34 graduum parallelis datur $845\frac{17}{1000}$ et milliaria evariatæ latitudinis 510. Inde loxodromiæ ipsius longitudo milliarium $987\frac{17}{1000}$.

Esto exemplum secundum, ubi cursus à parallelo aliquo versùs polum proximum dirigitur. Sit itaque idem cursus 58 gr. 54 scr. datus, et eadem longitudinis evariatio graduum 60. Initium autem à parallelo 24 gr. versùs polos. Itaque paralleli alicujus, ut puta 10 graduum, aut 20 graduum, numerum canonicum assumito; atque inde, ut ante, ejus numeri quadruplo per cruris mecodynamici milliaria, quantum in singulis parallelis sub hac inclinatione occupat, $398\frac{17}{1000}$, multiplicato, dabuntur minuta illi evariationi debita, ut suprà $398\frac{17}{1000}$. Atque inde fiet hæc proportio; quemadmodum ante $398\frac{17}{1000}$ dant numerum; ergo 3600 dabunt $2171\frac{17}{10000}$, numerum canonicum; qui, additus ad canonicum 24 gr. $1484\frac{6}{10000}$, conflât summam $3655\frac{17}{10000}$, qui pertinent in Canone parallelico ad 51 gr. 54 scr. 19 sec. Atque ideò secundum demonstrata in exemplo primo à 24 gra. ad 51 gra. 54 scr. navigandum esset hoc cursu antequam istam longitudinis evariationem assequaris. Unde, per propositionem 32^{am}, datur ipsius loxodromiæ longitudo milliarium $987\frac{17}{1000}$.

Tertium exemplum esto ubi cursus à parallelo boreo ultrà æquinoctialem excurrat. Sit à parallelo 23 gr. 30 scr. navigatum in austrum sub inclinatione loxodromiæ 58 gr. 54 scr. et longitudinis evariatione 120 gr. Hic, ut antè, paralleli alicujus numerum canonicum, ut puta gradus decimi, assumito, nempe 240.1940; eoque per cruris mecodynamici milliaria in singulis parallelis sub hac inclinatione multiplicato $398\frac{17}{1000}$, dabuntur minuta $398\frac{17}{1000}$ longitudinis evariationi ad decimi gradus parallelum ab æquinoctiali debita. Atque inde fiet hæc proportio; quemadmodum priùs, $398\frac{17}{1000}$ dant numerum canonicum 240.1940; ergo gradus evariatæ longitudinis, quæ sunt minuta 7200, dabunt $4343\frac{17}{10000}$. Quia autem numerus iste major est numero canonico paralleli propositi 23 grad. 30 scr. $1451\frac{17}{10000}$, id argumentum est, illam loxodromiam ultrà æquinoctialem protendi: ideòque hunc de illo deducam: reliquis $2892\frac{17}{10000}$ erit canonicus numerus in canone inquirendus, cui respondent 43 gr. 21 scr. 10 sec. Ut sub hæc

hâc inclinatione à boreo parallelo 23 gr. 30 scr. initio factô ad austrinum usque 43 gr. 21 scr. decurrendum sit, ut longitudinis evariatio 120 graduum efficiatur. Unde per propositionem 30^{ma} datur ipsius loxodromiæ longitudo milliaria 1974 $\frac{1}{1000}$; quemadmodum erat propositum.

Atque hinc expeditissimè adco invenias quantum in latitudine evarietur continuè ab æquinoctiali, si eadem assumatur ubique longitudinis differentia. Ut si quæretur sub inclinatione loxodromiæ 33 gr. 45 scr. et evariatione longitudinis duorum graduum continuâ, quanta sit evariatio latitudinis. Hic primùm, quemadmodum ante, per proportionem concludes, quantum quilibet parallelus assumptus evariet sub hâc inclinatione, crus in singulis parallelis mecodynamicum est milliaria $\frac{1}{1000000}$: inde si assumatur evariatio latitudinis unius tantum gradûs, cui competit numerus canonicus 60.0030, à cujus quadruplo dabuntur per propositionem 30^{ma} scrupula evariatæ longitudinis 40 $\frac{1}{1000000}$. Atque hinc fit hæc proportio; quemadmodum scrupula 40 $\frac{1}{1000000}$ ad optatæ evariationis scrupula 120, ita numerus canonicus assumptus 60.0030 ad quæsitum canonicum 179.6192; qui dat evariationem latitudinis 2 grad. 59 scrup. 32 $\frac{1}{2}$ sec. Inde, si evariationem 4 graduum longitudinis quæras, numerum inventum duplicato; si sex, triplicato; si octo, quadruplicato: atque ita porro, quemadmodum hic vicies continuatos numeros canonicos vides 179.6192, 359.2384, 538.8576, 718.4768, 898.0960, 1777.7152, 1257.3344, 1436.9536, 1616.5728, 1796.1920, 1975.8112, 2155.4304, 2335.0496, 2514.6688, 2694.2880, 2873.9072, 3053.5264, 3233.1456, 3412.7648, 3592.3840. Qui omnes numeri cum canonicis comparati dabunt parallelos optatos; inter quos hoc cursu longitudinis evariatio ubique par, nempe duorum graduum, inveniatur; paralleli autem quæriti sunt isti, quemadmodum hic ordine dispositos vides.

gr.	scr.	sec.
2	59	32 $\frac{1}{2}$
5	59	35 $\frac{1}{2}$
8	56	40
11	53	19
14	48	4
17	40	30 $\frac{1}{2}$
20	30	14
23	16	53 $\frac{1}{2}$
26	0	8 $\frac{1}{2}$
28	39	42
31	14	17
33	46	45 $\frac{1}{2}$
36	14	53
38	36	31 $\frac{1}{2}$
40	54	32
43	8	2
45	16	47
47	20	53 $\frac{1}{2}$
49	20	14
51	14	59

Atque

Atque ita porro in quâlibet loxodromiâ; ut jam perquam expeditum sit, hanc continuationem longitudinum quamlibet longè producere.

Idem ALITER et facilius.

IDEM problema etiâ aliter, et quidem multo facilius, absque ullâ divisione absolvetur, si id modò rectè expenderis, quod sit, Quemadmodum radius ad tangentem complementi anguli datæ inclinationis, ita scrupula propositæ longitudinis ad canonicum.

Unde "quantum denis, centenis, aut millenis denique, scrupulis cedat" facilis est explicatio.

Ut si proponatur anguli loxodromici inclinatio eadem quæ priùs, initio ab ipso æquinoctiali derivato, nempe 58 gr. 54 scr. tangens complementi hujus fuerit 6,032,386. Is numerus, per radium divisus, dabit $\frac{6,032,386}{10,000,000}$ numerum canonicum debitum uni scrupulo; et $\frac{6,032,386}{1,000,000}$, pro scrupulis 10, et $\frac{6,032,386}{100,000}$ pro centum, atque ita in millenis deinceps. Cujus ratio est propositionis factioe et demonstratione manifesta est. Cum enim numerus canonicus datæ latitudinis, per datæ loxodromiæ crus mecodynamicum singulis parallelis debitum multiplicatus, exhibeat quadrantem scrupulorum evariatæ longitudinis: idè quadrans scrupulorum evariatæ longitudinis per idem crus mecodynamicum divisus dabit numerum canonicum suæ latitudini competentem. Atqui crus mecodynamicum in singulis parallelis sub datâ inclinatione est quadrans tangentis per radium divisi. Quamobrem erit, ut quadrans tangentis anguli inclinationis ad radium, ita quadrans scrupulorum evariatæ longitudinis ad numerum canonicum latitudinis evariatæ; hoc est, (si primi et tertii assumatur quadruplum) ut tangens ad radium, ita scrupula evariatæ longitudinis ad numerum canonicum evariatæ latitudinis. Atqui, quemadmodum tangens ad radium; ita radius ad tangentem complementi; quare ex æquo quoque erit, *Ut radius ad tangentem complementi anguli inclinationis datæ, ita scrupula evariatæ longitudinis ad numerum canonicum evariatæ latitudinis.* Hoc est, si totus radius intelligatur scrupula evariatæ longitudinis exprimere, jam canonicus numerus (qui ex divisione quartæ partis eorundem scrupulorum per quantitatem cruris mecodynamici existisset) esset tangens complementi inclinationis datæ. Ergò, si scrupulum assumatur evariatæ longitudinis 1, 10, 100, 1000, dabitur crus mecodynamicum eidem evariationi debitum; unde latitudinis evariatio ex ipso canone dabitur.

Hinc itaque propositi problematis absque ullâ divisione solutio erit perfacilis. ut si proponatur inclinatio 58 gr. 54 scr. et longitudinis evariatio graduum 60, quæ sunt scrupula 3600. Erit igitur, Quemadmodum 10,000,000 ad tangentem complementi 58 grad. 54 scrup. nempe 6,032,386, ita 3600 scr. evariatæ longitudinis ad numerum canonicum parallelorum 2171, $\frac{6,032,386}{2,800,000}$, seu, quod idem sit (quia numeri canonici tantum ad 10,000 sunt expoliti), ad 2171.6589, cui in tabulis canonicis competit latitudinis evariatio 34 gra.

Eslo etiâ exemplum alterum cursûs à parallelo aliquo versûs posum proximum. Sitque eadem loxodromiæ inclinatio 58 gra. 54 scrui, cujus initium sit à

parallelo 24 graduum; atque eadem longitudinis evariatio ac prius, scilicet, 60 graduum. Quare hic proportionis termini erunt iidem qui prius, nempe, 10,000,000, et, tangens complementi, 6,032,386, et scrupula 3600; unde datur numerus canonicus 2171. 6389. Huic addendus est numerus canonicus 24 gr. nempe 1484. 0097. Summa hinc conflata dabit numerum canonicum 3655. 6686, qui è canone nostro dabit parallelum 51 gr. 54 scr. et pauxillo amplius. Huc usque igitur isto cursu, atque ista longitudinis evariatione, devenum fuerit.

Denique si eodem cursu, eademque longitudinis evariatione, ab eodem parallelo versùs æquinoctialem cursus fuisset institutus, jam à termino canonico 2171. 6389 invento canonicus 24 gr. erit deducendus, et dabitur reliquus 687. 5462, cui adjacent 11 gr. 23 scr. Et huc usque à parallelo boreo 24 gr. deventum sit hoc cursu ad parallelum austrinum 11 gr. 23 scr. Atque tanto hæc factio expeditior est priore; cumque illic geminum opus et multiplicationis et divisionis subeundum sit, hic solâ multiplicatione totum negotium absolvitur.

PROPOSITIO XXXIV.

Dato parallelo, longitudinis evariatione et loxodromiæ mensurâ; ejusdem inclinationem et latitudinis evariationem invenire.

ETIAMSI et hoc et antecessens problema ad usum nauticum parùm habeat momenti, tamen ut artis nostræ integritatem et perfectionem quàm evidentissimè demonstrarem, utrumque proponere et proposito exemplo illustrare placuit. Hoc verò novissimum non nihil amplius difficultatis habet; quò adèd Robertus Hues, (qui hæc à nobis proposita problemata dumtaxat in iis septem loxodromiis, quæ in sphericâ superficie describuntur, mechanicè tantùm explicat, et quidem admodùm *ἀπορρητῶς*,) huic novissimo tale adscribit eulogium. “ Non est quidquam in totâ hæc arte, quod difficilius sit inventu, quam Rumbus (ita, Nonium secutus, loxodromiam * vocat) ex dato intervallo et differentia longitudinis, neque aliâ quàm operosa et sæpius iteratâ praxi et multis dimensionibus exquiri potest per globorum usum. Cùmque sit hæc praxis tam proluxa et tanti laboris, minùs est necessaria, aut prorsùs inutilis, quoniam longitudinis differentia difficultè investigatur, ut suprâ monuimus. Cujus inventionem utinam præstarent magni nostri ostentatores, ut liceret tandem aliquid ab his expectare præter nuda verba, vanas pollicitationes, et inanem spem.” Et hæc ille; cum tamen nihil præter rudem factionis typum in omnibus ferè aut exhibeat, aut requirat, et qui ad loxodromiarum proprietates et affectiones exprimendum utcunq; sit opportunus, in usu autem ipso parùm aut nihil habeat momenti. Nos hic loxodromiarum inclinationes ad scrupula prima usque, et ad horum particulas quoque, si ita cuiquam libeat, inveniendi modum demonstrabimus.

Esto itaque primò cursus instituti initium ab ipso æquinoctiali, ut loxodromiæ longitudo sit milliarium 800, et longitudinis evariatio 50 gr. 19 scr. Quæri-

* Potius legendum videtur, Loxodromiæ angulum seu inclinationem.

tur hujus loxodromiæ inclinationis angulus, et latitudinis ejusdem evariatio, huic longitudini et simul loxodromiæ magnitudini debita. Hic ante omnia periculum est faciendum, utrum sub eodem parallelo sit navigatum, an verò per aliquam loxodromiam cursus sit institutus. Nam si propositi cursûs milliaria, per milliaria unius gradûs in parallelo unde initium factum est divisa, eandem evariatæ longitudinis quantitatem exhibeant, utique res ipsa loquitur, sub eodem parallelo cursum esse institutum, et in latitudine hic nihil esse evariatum. Nam per propositionem undecimam, ut radius ad quadruplum secantis dati paralleli, ita quoque numerus milliarium ad scrupula in eo parallelo ipsis congrua. Ergò si sub æquinoctiali iter sit institutum, numerus milliarium quadruplicatus dabit scrupula ipsis debita: ut hic 800 milliaria facerent scrupula 3200, qui sunt gradus 53 scr. 20. Atqui evariatio longitudinis exposita est 50 gr. 19 scr. unde efficitur, versùs minores parallelos cursum fuisse inclinatum, id est, per loxodromiam aliquam hoc cursu esse decursum. Reliquum est igitur, ut tentando et conjectando ad quæsiûm scopum collineemus, assumptâ primùm loxodromiâ * quâlibet, ut, putâ, inclinatione 45 gr., ut constet suprâne an infrâ hanc quæsitâ inclinatio pertingat. Id autem facile est. Cum enim per secundam praxin propositionis antecedentis sit; ut Radius ad tangentem complementi inclinationis, ita scrupula evariatæ longitudinis ad numerum canonicum; unde latitudinis evariatio quoque cognoscatur; hic autem radius et tangens complementi æquantur: etiâ scrupula datæ longitudinis numero canonico erunt æqualia, quæ nunc sunt 3019. His porrò ad 10,000 reductis, quemadmodum numeri canonici sunt expressi, fiunt 3019.0000. Huic competunt latitudinis gr. 44, scr. 52 $\frac{1}{2}$, et inde dantur sub hâc inclinatione milliaria loxodromica 867 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; quæ sunt plura datis: ea enim sunt tantum 800. Quare illud indicii satis est, angulum inclinationis majorem esse loxodromiâ assumptâ; et contrâ, si loxodromia inventa minor sit datâ, tùm minorem esse reverâ quàm sit assumpta. Quamobrem assumatur alia, et quidem major, inclinatio quàm prius, scilicet, angulus 60 graduum. Hinc eâdem viâ, per secundam antecedentis propositionis factionem, dabitur numerus canonicus 1743.0205, cui cedunt 27 gr. 52 $\frac{1}{2}$ $\frac{1}{2}$ scr. evariatæ longitudinis; atque inde loxodromiæ longitudo milliarium 811 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$, quæ etiamnum major est propositis 800. Rursùm igitur majorem, 10 gr. nempe 70 graduum, inclinationem assumam (hoc enim incremento eam inclinationem intrâ denos gradus primùm concludes haud admodum operosè) unde numerus canonicus exisset 1098.8260, et unde dabitur latitudinis evariatio 18 gr. 07 $\frac{1}{2}$ $\frac{1}{2}$ scr. atque loxodromiæ quantitas 789 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$, quæ est minor datâ; quod ipsum quoque indicium est, assumptam inclinationem majorem esse quæsitâ. Quamobrem constât intrâ 60 et 70 gradus eam cõcereri. Hic deinceps per quinos primùm tentandum censeo; unde constet, suprâne an infrâ 65 gr. inclinatio quæsitâ consistat. Ego 66 gr. 15 scr. hic assumpsi: unde numerus canonicus datur 1328.3901, et inde latitudinis evariatio 21 gra. 36 $\frac{1}{2}$ $\frac{1}{2}$ scrup.; loxodromiæ autem longitudo milliarium 804 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$, quæ est major datâ. Quare loxodromiæ quæsitæ angulus inclinationis hâc major erit; et tamen minor quam 70 gra. Assumantur igitur 68 gr. unde canonicus datur 1219.7516, et latitudinis evariatio 19 grad. 55 scrup. loxodromiæ verò longitudo millia-

* Hic per vocem *Loxodromid* auctor intelligit angulum, seu inclinationem, Loxodromiæ, seu curvæ Loxodromicæ.

rium $797^{\frac{1043}{10000}}$; quæ est minor datis 800: unde planum est, hanc novissimè assumptam inclinationem majorem esse quæsitâ. Quamobrem cum ex istâ inductione liquidò constet angulum inclinationis quæsitæ intrâ hos terminos 66 gra. 15 scr. et 68 grad. coerceri, licebit eadem viâ semel iteratâ adhuc angustiores limites haud difficulter præfigere; ut intrâ unius gradûs magnitudinem inclinationis quæsitâ cogatur: aut vel nunc jam per regulam falsarum positionum ex inventis ad minutum usque propemodum definire, hoc modo. Crus mecodynamicum in singulis parallelis sub inclinatione 66 gr. 15 scr. est milliarium $\frac{10000}{100000}$, et inde loxodromiæ quantitas suprà inventa $804^{\frac{10000}{100000}}$ excedit summam propositam 800, milliariibus $4^{\frac{10000}{100000}}$: et, secundò, sub inclinatione 68 grad. datur crus mecodynamicum $\frac{10000}{100000}$; unde loxodromiæ quantitas suprà fuit exhibitâ milliarium $797^{\frac{1043}{10000}}$, quæ deficiunt ab expositis, sive datis, (scilicet, 800,) quantitate $2^{\frac{10000}{100000}}$ milliarium. Hos autem numeros, ex hujus regulæ falsarum positionum formulâ, ordinabimus hoc modo.

$$\begin{array}{r} \frac{10000}{100000} \div \frac{4^{\frac{10000}{100000}}}{100000} = \frac{2^{\frac{10000}{100000}}}{100000} \\ \frac{10000}{100000} - \frac{2^{\frac{10000}{100000}}}{100000} = \frac{10000}{100000} - \frac{2^{\frac{10000}{100000}}}{100000} \end{array}$$

Unde, multiplicatione ad decussim factâ, dabuntur numeri $2^{\frac{10000}{100000}}$, et $1^{\frac{10000}{100000}}$; et, quia signa sunt diversa, fiat additio. Unde dabitur divisor $7^{\frac{10000}{100000}}$ et dividendus $4^{\frac{10000}{100000}}$; quâ divisione peractâ et ad suas millesimas revocatâ, dabitur quotus $\frac{10000}{100000}$, pro milliariibus evariatæ longitudinis in singulis parallelis. Cujus quadruplum 240,572 est tangens inclinationis per conicatarum propositionis 28: unde ipse inclinationis quæsitæ angulus datur 67 gra. 26 scr. Atqui hinc dabitur numerus canonicus 1254. 6360, et evariatio latitudinis 20 gr. 27 scr. atque loxodromiæ longitudo $799^{\frac{1043}{10000}}$ milliarium, cum debeant esse 800. Ergò paucis scrupulis inclinatio erit major, utpote 67 grad. 30 scr. exactè, quem terminum calculus iteratus exhibebit. Eadem via obtinet in reliquis, si forte loxodromiæ initium ab ipso æquinoctiali non sumatur. Atque ita hujus theoriæ, et simul libri primi, finis hic esto *.

* Hoc Problema à doctissimo viro, Edmundo Halley, anno Domini 1695, seu plusquam septuaginta annis post hujusce Libri, à Snellio scripti, impressionem, eruditus Anglice Mathematicis propositum est tanquam novum omnino problema, quod nunquam antea aut solum aut etiam tentatum fuerat: unde colligi potest hunc Snellii tractatum ab illo nunquam fuisse visum: quod tamen mirum quoddammodò videri necesse est in homine qui non solum in Mathematicis Scientiis erat multum et feliciter versatus, sed etiam navigandi artem percalluisse censebatur, immò et ipse, ut navis magister, exercuerat in longâ navigatione ad Angliâ usque ad insulam Sanctæ Helenæ ultra Equatorem itam. Vide initium tractatus antecedentis hujusce tomî quarti *Scriptorum Logarithmicorum*: in quo tractatu Solutio directâ et generalis hujus Problematis, sine admodum subtili et abstrusâ, exhibetur, et plenissimè et luculentissimè explicatur: quæ à solertissimo viro, Georgio Atwood, A. M. (ante annos aliquot apud Cantabrigienses Collegii Sanctæ Trinitatis Socio,) nuper inventa fuit, et mecum communicata. Hæc quidem Solutio generalis est omnino scientifica, et est talis, ut opinor, qualem Halleyus videre optaverat, et à quâdam serie infinitâ Logarithmicâ, à Jacobo Gregorio olim, anno Domini 1671, inventâ, derivatur. Sed, licet verè scientifica sit ista Solutio generalis, ad usus tamen nauticos non tam commode esse mihi videtur quàm præcedens Snellii Solutio ejusdem problematis his paginis declarata, quæ non nisi tentando perficitur: quoniam hæc tentamina, quanquam sæpius iteranda, minus exigunt laborem quam simplex et directâ applicatio solutionis illius generalis, et tamèn, in fine calculi, magnitudinem anguli quæsitæ, seu inclinationis Loxodromiæ ad Meridianum, non minus exactè definiunt. In materiâ autem tam subtili et à difficilâ utraque Solutio valde utilis est censenda, quoniam altera alteram confirmat.

F. M.

WILLE

WILLEBRORDI SNELLII
HISTIODROMICES

LIBER SECUNDUS,

DE EJUS PRAXI:

SIVE,

DE NAUTICÆ DIVINATIONIS EMENDATIONE.

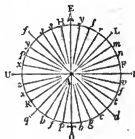
HACTENUS loxodromiarum affectiones omnes inspeximus, et accuratè sumus contemplati, ut earum inclinationes per gradus et scrupula, atque alias minimas quasque particulas, explicare et persequi liceat. Sequitur earum usus; ut quantum maritima praxis nobis addicat, quantumque fidei ei sit habendum, deinceps videamus. Et primùm quidem de numero loxodromiarum vulgò recepto.

PROPOSITIO I.

Loxodromiæ principales in singulis quadrantibus ita ordinantur ut inter meridianum et loci parallelum septem intercident, quæ angulum rectum in octo æquales partes dissepseant; quæque et ipsæ iterum in semisses et quadrantes subdividuntur.

LOCI cujusque parallelum ab ejusdem meridiano normalitèr secari, ex elementis sphaericis notum est; atque hac sectione mundi plagas in quatuor quadrantes disparari, lib. 1, propositione 7, demonstravi: à meridiano nempe septentrionem et austrum, à parallelo autem ortum et occasum designari. Id in plano rectis sese normalitèr secantibus exprimi solet, quæ lineæ sunt tanquam tangentes hos circulos in communi illarum sectione. Ut, si propositi loci linea meridiana sit EO, et ejusdem verticalis primarius UI: tùmque E septentrionem spectet, O meridiem, I ortum, U occasum. Hic OE lineæ ductum secutus meridianum

meridianum obobis, et maximum idèò circum describes à meridie in septentrionem, aut contrà. Si verò sequaris lineam UI, et A punctum sub ipso sit æquinoctiali, tum UI continuata terræ æquatorem designabit: aut si extrà eum alibi sit in sphaeræ obliquæ situ, circum aliquem ipsi parallelum. Atque ita uterque hic cursus circum nobis designat per prop. sextam et septimam libri primi; anguli autem intermedii in singulis quadrantibus non item, per propositionem 15. Hos vulgò rei maritimæ periti ad usum navalem in octò æquales partes distribuunt, ut anguli inclinationum, facto initio à meridiano, inter se 11 graduum et 15 scrupulorum intervallo distent. Ut hic EAy, yAs, sAr, atque ita porro. Atque hinc adèò in singulis quadrantibus octò constituuntur cursus, ut primus à meridiano distet intervallo 11 gr. 15 scr. E quibus septem primi loxodromici sint, octavus autem, qui per UI educitur, loci ipsius parallelus; et angulus inclinationis primæ loxodromiæ sit EAy 11 gr. 15 scr. secundæ EAs 22 gr. 30 scr. tertiæ EAr 33 gr. 45 scr. quartæ 45 gr. 0 scr. quintæ 56 gr. 15 scr. sextæ 67 gr. 30 scr. septimæ 78 gr. 45 scr. octavus autem cursus, qui nonagesimum implet gradum, est *meridianus*. Atque ita totius mundi conspectus per quatuor quadrantes in duas et triginta plagas tribuitur. Naucleri sua habent vocabula quibus singulas plagas pro consuetudine gentis appellent; quarum nomenclaturas singuli è vernaculo idiomate exprimunt. Nobis autem in arte fatis erit ab ordine et successione illas indigitare, vocando eas primam, secundam, tertiam, atque ita porrò; quemadmodum paulò ante expressi. Nomina et appellationes usitatas, et quemadmodum eandem felicitatem græcâ compositione assequamur, videto in scholis. Verùm ista unius quadrantis octifaria distributio accuratioribus de arte navigandi scriptoribus non satisfacit; sed ab aliis octava quævis pars quadrantis etià bissecatur; ab accuratissimis verò etià quadrifecatur, ut ita in singulis quadrantibus existant cursus 32, et in toto circulo 128: sed has tantùm per plagarum quadrantes, semisses et dodrantes explicant, neque ullis novis nominibus designant, ne sectionis subtilitas memoriæ gravis sit. Sequitur canonum ad has inclinationes et inclinationum particulas ordinatio et constructio.



PROPOSITIO II.

Loxodromici canones ἀπόκρυφοι habent in consuetis inclinationibus Ictitudinis evariationem, atque loxodromiæ ipsius et cruris mecdynamici quantitatem ei debitam in miliaribus è regione expressam.

LIBRI primi propositione vicesimâ demonstratum est, quomodo datâ latitudinis evariatione seu differentiâ, cum angulo inclinationis, cursus quantitas intrâ positos parallelos deprehendatur. Ut si exponatur intervallum latitudinis gr. 40, et inclinatio loxodromiæ 60 gr. erit igitur, quemadmodum radius ad quadrantem

quadrantem secantis anguli inclinationis; ita scrupula evariatæ latitudinis ad milliaria loxodromiæ propofitæ. Nempe ut 100,000 ad 50,000, ita 40 gradus, hoc est, scrup. 2400, ad milliaria 1200, quæ loxodromiæ quantitas benè accurata est. Cruris autem mecodynamici longitudo invenietur per propositionem 27. Ut enim radius ad quadrantem tangentis datæ loxodromiæ, ita latitudinis evariatæ minuta ad milliaria cruris mecodynamici. Utpote, quemadmodum radius 100,000 ad quadrantem tangentis 60 graduum, hoc est, ad 43,301; ita scrupula evariatæ latitudinis, ad milliaria mecodynamica $3039\frac{11}{1000}$. Et contrà; datis milliariis cursûs loxodromici, dabitur evariatio latitudinis in gradibus et scrupulis, cum longitudo cruris mecodynamici per propositionem 19, et . . . Verùm ut hoc tabularum tædio leventur ii, qui minùs his tractandis adfueverunt, et ad usum quotidianum expeditius sit: visum est hos numeros, in consuetis loxodromiis, et earum semissibus et quadrantibus, subductos, exhibere; quos, quia semper in promptu et ad manum navarchis esse debent, *Canones πωροχάρτες* indigitavimus. Harum autem ordo et dispositio ita habet, ut anguli inclinationis differentia per quadrantes unius octantis recti anguli increseat. Cum enim primariæ loxodromiæ per 11 graduum et 15 scrupulorum intervallum inter se disjungantur, hic canones exhibemus per hujus intervalli quadrantes, nempe 2 gr. 48 scr. 45 sec. distinctos. Ergò primus canon habet titulum $\frac{1}{4}$; secundus, (cujus inclinatio est 5 gr. 37 scr. 30 sec.) habet titulum $\frac{1}{2}$; tertius, (cujus inclinatio est 8 gr. 26 scr. 15 sec.) habet titulum $\frac{3}{4}$; quartus habet titulum 1, seu primæ loxodromiæ, cujus angulus inclinationis est 11 gr. 15 scrup. Verùm omnium conspectum unico canonio exhibere satius sit, ubi loxodromiæ earumque partes cum suis inclinationum angulis è regione annotati exhibentur, quemadmodum hic vides.

inclinatio.				inclinatio.				inclinatio.				inclinatio.			
lox.	gr.	scr.	sec.	lox.	gr.	scr.	sec.	lox.	gr.	scr.	sec.	lox.	gr.	scr.	sec.
$\frac{1}{4}$	2.	48.	45.	$2\frac{1}{4}$	25.	18.	45.	$4\frac{1}{4}$	47.	48.	45.	$6\frac{1}{4}$	70.	18.	45.
$\frac{1}{2}$	5.	37.	30.	$2\frac{1}{2}$	28.	7.	30.	$4\frac{1}{2}$	50.	37.	30.	$6\frac{1}{2}$	73.	7.	30.
$\frac{3}{4}$	8.	26.	15.	$2\frac{3}{4}$	30.	56.	15.	$4\frac{3}{4}$	53.	26.	15.	$6\frac{3}{4}$	75.	56.	15.
1	11.	15.	0.	3	33.	45.	0.	5	56.	15.	0.	7	78.	45.	0.
$1\frac{1}{4}$	14.	3.	45.	$3\frac{1}{4}$	36.	33.	45.	$5\frac{1}{4}$	59.	3.	45.	$7\frac{1}{4}$	81.	33.	45.
$1\frac{1}{2}$	16.	52.	30.	$3\frac{1}{2}$	39.	22.	30.	$5\frac{1}{2}$	61.	52.	30.	$7\frac{1}{2}$	84.	22.	30.
$1\frac{3}{4}$	19.	41.	15.	$3\frac{3}{4}$	42.	11.	15.	$5\frac{3}{4}$	64.	41.	15.	$7\frac{3}{4}$	87.	11.	15.
2	22.	30.	0.	4	45.	0.	0.	6	67.	30.	0.				

Tot enim canones ordinavimus, ut cursûs navalis minimas quoque particulas, quas in praxi notant, exprimeremus. Omnium autem ordo est consimilis, ut unum cum nôris, omnes nôris. Exemplo esto nobis Canon primus cui titulus $\frac{1}{4}$, pagina 30, ubi primâ felide habes latitudinis evariationes per gradus singulos; inde à decimo per denos, usque ad octogésimum inclusivè: è regione felide secundâ, milliaria loxodromica hoc cursu primùm uni gradui debita, nempe 15.0180, quæ sunt $15\frac{11}{1000}$ milliaria. Ita enim milliariarum particulas ad 10,000 quoque sumus profecuti. Tum felide tertiâ quantitatem milliariarum mecodynam-
amicorum

amicorum sub suo titulo, nempe 7. 3690, five milliaria $7\frac{3690}{1000000}$; atque ita deinceps quantum duobus, tribus, decem, viginti, gradibus competat. Inde felide quartâ, quintâ, sextâ, septimâ, octavâ, nonâ, quantum scrupulis 1, 2, 3, 4, 5, cæterisque deinceps ad 60 usque, debeatur. Similis omnium est syntaxis. Ipsos autem canones in libri calcem rejecimus, et post *Parallelorum canonicarum* ordinavimus. Usus autem est perfacilis ex ipsâ *diaboli*.

PROPOSITIO III.

E loxodromiarum usitatarum inclinatione et magnitudine, exariate latitudinis quantitate, cruris $\mu\kappa\omicron\delta\omicron\nu\alpha\mu\kappa\epsilon$ longitudine in milliaribus, datâ unâ reliquis duas per canones $\pi\rho\omicron\chi\eta\mu\alpha$ exhibere.

HÆC praxin et usus facilitatem spectant: primo enim libro ista particulariter et fusè sumus in minutissimis quibusque particulis prosecuti. Nunc istos limites et angulos tantum sequimur, quos consuetudo recepit; qui cum certo numero comprehendantur, totidem canones concenturiavimus, unde uno dato reliqua in conspectu sint. Hi ab expeditâ facilitate canones $\pi\rho\omicron\chi\eta\mu\alpha$ nobis dicuntur, quia semper ante oculos et *in manibus* hic versari debent. Quamvis enim (ut sæpiculè jam à nobis dictum est) loxodromiarum numerus sit infinitus, et per singula scrupula scrupulorumque partes multiplex ejus angulus esse possit; tamen, cum neque magneticæ acûs fallacia, neque ponti agitato, præcisam angulorum in gradibus et minutis æstimationem admittat: idèd usui satis opportunum, quantam quidem hæc res $\alpha\pi\epsilon\lambda\epsilon\gamma\epsilon\iota\varsigma$ admittet, in praxi judicatur, si singulæ loxodromiæ intervallo 2 gr. $4\frac{1}{2}$ scr. inter se dissent; ita enim totus quadrans in duas et triginta partes dispelcitur. Atque ita singulis suos propriosque canones destinavimus, secundùm leges antecedente propositione expressas. Usus itaque in promptu erit. Ut si quæretur angulo loxodromiæ septimæ, intra parallelos 50 scrupulis distantes, quantum itineris sit confectum, quantumque sit crus mecodynamicum in milliaribus æstimatum; inquirito itaque in canonicis istis sub loxodromiâ septimâ scrupulum quinquagesimum; ubi è regione in laterculo primo dabitur loxodromiæ quantitas quaesita milliarium 64. 0729, seu $64\frac{729}{1000000}$; et crus mecodynamicum $62\frac{8008}{1000000}$.

Si simul uno nomine numerus differentiæ latitudinis in canonicis loxodromicis non detur, per partes erit inducendum; quæretur quantitas loxodromiæ et cruris mecodynamici inter parallelos 27 gr. 35 scr. disjunctos; inductio fiet per partes. Hoc modo.

	loxod.	crus mecodyn.
Gr. { 20	701. 6639	634. 2667
7	245. 5824	222. 0038
scr. 35	20. 4652	18. 5073
	967. 7115	874. 8008

His

His in unam summam additis dantur milliaria loxodromica $967\frac{700}{1000}$, et mecodynamica $874\frac{800}{1000}$.

Et contrà, datà inclinatione loxodromiæ atque ejusdem quantitate in milliariis æstimatâ, dabitur è proclîris his canonibus evariatio latitudinis, cum milliariis mecodynamicis. Eîso navigatum angulo inclinationis loxodromiæ $5\frac{1}{2}$, et cursûs longitudo æstimata sit milliarium $697\frac{700}{1000}$, et quærat parallelorum distantia à principio, ad cursûs finem, seu latitudinis evariatio. Id subductionis contrariâ viâ absolvetur per partes, quia totum simul non datur. Hoc modo.

loxodrom.

<u>967. 7115</u>	20	} grad.
<u>701. 6639</u>		
266. 0476	7	
245. 5824		
<u>20. 4652</u>	35	scrup.

Hinc cruris mecodynamici quantitas dabitur, ut ante. At si loxodromiæ quantitas non exactè inveniatur in ipsis scrupulis, opus erit parte proportionali, ut solet, si forte tanti erit, ut proximè vicinum scrupulum assumere nolis. Erit autem tanti in obliquioribus cursibus, ubi inter parallelos uno scrupulo distinctos loxodromiæ pars non contemnenda intercipitur; et idè quoque cruris mecodynamici pars majuscula: quod hinc ipsa longitudinis evariatæ æstimatio à legitimâ dissidens existat: maximè verò, si in parallelis æstioribus ea sit æstimanda. Illic enim unum aut alterum milliare non parùm longitudinem à legitimo situ abduceret. Sed hæc non sunt nimis scrupulosè inquirenda, ubi opus non erit; aut contrà nimis negligentèr habenda, quando alicujus erunt momenti. Ego unico exemplo præibo. Sub loxodromiâ $7\frac{1}{2}$ cursûs longitudo sit milliarium $82\frac{800}{1000}$, hic è canonibus proclîris datur evariatio latitudinis o gr. 32 scr. hoc modo.

82. 8375	} 35 scr.
81. 6183	
1. 2192	

Et redundant milliaria $1\frac{100}{1000}$. Inde verò fit hæc proportio; 2.5505 dant 60 sec. ergò 1.2192 dabunt 29 sec. Ego ad 100 scrupulorum reducere soleo, nempe ut 2.5505 ad 100; ita 1.2192 ad 48. erit ergò vera latitudinis evariatio o gr. $35\frac{1}{4}$ scr. Unde cruris mecodynamici quantitas dabitur.

81. 2253	} 35 scr.
1. 2183	
82. 4436	

VOL. IV.

A a

Vides

Vides itaque, amplius $1\frac{1}{2}$ milliaris accedere cruri mecodynamico; quæ ut sub ipso æquinoctiali 5 scrupulis longitudinem evariant, ita sub parallelo sexagesimo faciunt scrupula penè 10. et ultrà parallelum sexagesimum faciunt adhuc amplius. Hæc obiter tantum monere visum est. Tu, pro tui iudicii æquitate, quanti ea res sit, tanti quoque suo loco æstimato, et cum modo.

N O T A.

Cruris mecodynamici usus rarus, aut omnino nullus est in cursibus singularibus; ubi è singulis longitudinem debitam invenire cupias; ut in ipso mari nautæ quotidie affolent. Illic enim segmentum cruris mecodynamici, nisi, manente eadem cruris mecodynamici quantitate, latitudini differentiam aliam à datâ inducas, uni parallelorum congruum satis est longitudini inveniendæ: aut cum è pluribus et diversis cursibus longitudinem constare voles, et quidem semel ad extremum tantum: illic enim necessario adhibebitur, quemadmodum ea generalitèr lib. 1, dicta nobis sunt. Sed ante nautici diarii formula nobis erit explicanda, in quod cursum æstimata quantitas et inclinatio ab ipsis quotidie, et quoties opus est, referri solet.

PROPOSITIO IV.

Ephemerin itinerariam nauticam ordinare.

DIARIA nautica à peritis et diligentibus navium magistris sedulò et accuratè quotidie conscribuntur, et per suas selides ac laterculos dispunguntur, ut inde cursûs confecti longitudinem, varietatem, inclinationem, observatam de cælo latitudinem, ventorum status, impetum; aëris quoque temperiem; et, si quæ alia sint, quæ accuratiùs notari expediat, ut conspectum promontiorum insularumve, curiosè exprimant; idque isto ferè modo, quem hic verbi gratiâ ex parte expressum vides; ut singulis selidibus suis sit titulus in fronte, laterculi diebus mensium anni cujusque, quo hæc itinera ab ipsis sunt confecta, respondeant. Ita enim semper itineris confecti spatia, si forsàn uspiam titubatum sit, relegere, et in tabulis suis hydrographicis dispungere ac emendare, ipsis perfacile licet.

TABELLA

TABELLA DIARIA,

SIVE

EPHEMERIS NAUTICA.

Julii

Dies	Vclificationis cursus.	conf. mil.	altitudo poli observata.
16	O. Z. O.	18	
17	O.	4	
	Z. O. t. O.	13	
18	O. Z. O.	14	
	O. Z. O.	3	
	W. Z. W.	11	
	W. t. Z.	1 $\frac{1}{2}$	
19	Z. O. t. Z.	2 $\frac{1}{2}$	
	Z. O. t. Z.	5	
20	Z. Z. O.	14	
	Z. Z. O.	2	48, 38
	Z. W.	7 $\frac{1}{2}$	
21	Z. O. t. O.	4	
22	O. Z. O.	24	
23	Z. O.	29	
	Z. O. t. Z.	31	
24	Z.	5	42, 54
	Z.	24	
25	Z. t. W.	5	
26	Z.	24	
	Z. Z. W. $\frac{1}{2}$ Z.	12 $\frac{1}{2}$	
27	N. O. t. O.	5	
28	O. N. O. $\frac{1}{2}$ N.	16	
29	O. t. N. $\frac{1}{2}$ N.	16	30, 16
	O. $\frac{1}{2}$ Z.	6 $\frac{1}{2}$	
	O. t. Z.	5	
30	O.	8 $\frac{1}{2}$	39, 24
1	O.	23 $\frac{1}{2}$	
2	O.	38	10, 13
	O. t. Z.	17	
3	O.	14	38, 58
4	O.	5	38, 50

August.

Quartâ Augusti appulimus ad insulam *Fojael*, unam earum, quas Flandricas vocant.

TABELLA DIARIA, SIVE EPHEMERIS NAUTICA.

	Curſus plaga.	milli aria.	latitudo obſervata.
Auguſt.	23 O. N. O. $\frac{1}{2}$ O. 4 $\frac{1}{2}$		
	24 O. N. O. $\frac{1}{2}$ O. 10		
	Z. O. 1 $\frac{1}{2}$		
	Z. O. $\frac{1}{2}$ Z. 2		
	25 N. O. $\frac{1}{2}$ N. 4		
	26 N. N. O. $\frac{1}{2}$ O. 16		
	27 N. O. t. N. 16		
	28 N. N. O. 14		
	29 N. N. O. 17		
	30 N. t. O. 9 $\frac{1}{2}$		
	N. N. W. 1		
	31 O. Z. O. 9		
Sept.	Z. O. t. O. 4 $\frac{1}{2}$		
	1 Z. 7 42, 52		
	2 O. t. Z. 5		
	3 O. t. Z. 18		
	4 O. t. Z. 31		
	5 O. t. Z. 37		
	O. t. N. 22		
	6 N. O. $\frac{1}{2}$ N. 9		
	7 N. N. O. $\frac{1}{2}$ O. 26 42, 42		

Hoc curſu denique octavâ Septembris appulimus ad promontorium Hiſpaniæ vulgò dictum *Caput finis Terræ*, quâ longiſſimè in occidentem procurrit in orâ ſeptentrionali; Ptolomæo Nerium promontorium.

Nos tres duntaxat felidas exhibemus: qui volent, plures adjiciant, quemadmodum initio monuimus; has enim tanquàm maximè neceſſarias, et quæ omnino negligi non debent, *ὡς ἐν τῇ αἰ* tantum propoſuimus. Nam in maritimis itineribus illis longioribus, quibus Indiā utramque adeunt, etiā ſuam ſelidem acūs magneticæ χαλκῶδους dari, quam maximè expediet. Eā enim res in medio mari errantibus naucleris, tanquàm ſignum et ſcopulus proponitur, ex quo de curſu ſui rationibus æſtimationem faciùs capere poſſint. Cum enim ſub eodem parallelo eandem chalybocliſin deprehenderint, etiā de longitudinis et meridiani ſitu eodem certiores ſiunt. Verùm hæc materia ab aliis occupata eſt; et Stevino noſtro occaſionem ſcribendæ ſuæ Limeneuretics ſuppeditavit: quem

quem vide in opere Hypomnematum Mathematicorum ILLUSTRISSIMI Principis MAURITII, quod olim à nobis latinè redditum et publicatum est.

PROPOSITIO V.

Curfus maritimus est simplex, aut compositus.

ID de nauticis ad magneticam acum examinatis omnibus curfibus intelligatur: aut enim eodem tramite à principio ad finem decurritur; aut pluribus et diversis inter se permixtis. Ita enim dispunximus, ut emendationi nauticæ, ad quam contendimus, tractandæ suus ordo constet.

PROPOSITIO VI.

Simplex est, qui unicum et eundem sui ductus cursum sequitur.

UT si rectâ eatur à septentrione in meridiem; aut per eundem parallelum decurratur; aut denique secundum unam aliquam et eandem loxodromiam cursum dirigas.

N O T A.

Quemadmodum è singulorum cursuum datâ quantitate, latitudinis aut longitudinis evariatio inveniat, libro primo accuratè fuit à nobis expositum. Et primò quidem:

1. Quomodo datâ cursûs quantitate à septentrione in austrum, latitudinis evariatio detur, (nam hic longitudinis evariatio nulla est) propositione 6. Nempe 1 milliare facere scrupula 4, et 15 milliaria gradum unum.
2. Quomodo datâ cursûs quantitate sub eodem parallelo longitudinis evariatio detur, (nam hic latitudinis evariatio nulla est) propositione 11. Nempe, ut radius ad quadruplum secantis dati paralleli, ita milliaria quotcunque data ad numerum minorum. evariate longitudinis.
3. Quomodo datâ inclinatione et cursûs loxodromici quantitate, latitudinis et longitudinis evariatio inveniat; propositione 19 libri primi de latitudinis evariatione, et propositione 30, de evariatione longitudinis. Nempe, ut radius ad quadrantem tangens loxodromiæ, ita numerus canonicus intervallo evariate latitudinis debitus ad scrupula evariate longitudinis.

Hæc prima in loxodromicis factio est; altera quoque per canones *περυχίως* explicatu est perfacilis. Atque hæc quidem ex observatione inclinationis acûs magneticæ, et itineris confecti æstimatione; sequitur utriusque cum observatione cœlesti comparatio.

P R O.

PROPOSITIO VII.

Est cursus, et plagæ atque inclinationis ejusdem æstimata quantitas, ex loci latitudine de cælo observatâ comprobatur, aut secundum eandem emendatur.

CUM navium magistri in fluctuante sælo diligentissimè cursûs velocitatem, et ejusdem plagæ situm observaverunt, qui aliquando in ipso meridiano instituitur, aut, ut frequentissimè, aliquem angulum cum meridiano comprehendit: tamen eos persæpe in hæc conjecturâ falli experientia comprobavit. Hujus autem hallucinationis examen est paralleli è cursu et loxodromiâ æstimati consensus, aut discrepantia ab latitudine loci de cælo observatâ. Cum enim observatio, quam è sole aut sideribus derivant, alium ostendet parallelum, quam ipsi ex æstimato itineris cursu, vel circulari, vel loxodromico deducunt; necesse est, alicubi errorem hærerè, nempe vel in itineris æstimati longitudine, vel in cursûs situ et angulo loxodromico, vel denique in observatione cælesti. Sed cum istæ quotidie possint iterari, et dies dièni docere; illa autem non possint relegi; hanc tanquam normam, ad quam illæ componi debeant, et ducem minus fallacem, meritò sibi delegervnt. Certè adeò arctius omnium consensus esse debuit, ut positis duobus, tertium constantèr addiceretur: ut, nempe, cursûs plagâ à dato principio, et ejusdem quantitate datâ, loci latitudo inde concludatur: aut ex inclinatione et latitudinis evariatione, ipsius cursûs quantitas: vel è latitudinis evariatione et cursûs quantitate, ejusdem inclinatio. Verùm ut ista ad amissum consentiant, vix est ut in tantâ lubricitate sperari possit. Itaque remedium huic errori quæsitum, quod minimum à vero abludat. Nam vel ex variato non-r:hil angulo; vel cursûs quantitate evariatâ; vel ex utroque id erat constituendum. Sed anguli evariatio, manente cursûs observati quantitate et latitudine inventâ ex observatione, in rectioribus loxodromiis nimium est periculosa; et in reliquis quoque; si observatio à conjecturâ plusculum abludat, uti in scholis docebitur: quantitatis autem cursûs loxodromici variatio, manente tamen angulo in loxodromiis magis obliquis, maxime lubrica est, quemadmodum ibidem ostendemus. Superest itaque unica hæc factio, quæ viam inter utramque mediam insitit, et utrobivis æquè sit tuta. Ea autem gemina esse potest: ut et angulo inclinationis, et cursûs æstimatæ longitudini hic error pariter imputetur. Receptum autem navcleris-et usitatum est, ut retentâ longitudinis evariatione; latitudinis evariatio tantummodò mutetur. Nos autem factionem paulo ab hæc ratione diversam sequemur, quæ minimè intuta aut periculo obnoxia videatur: reliqua in scholis examinaturi, et, quod illi mechanicè ac nimium lubricè conantur, per tabulas nostras quàm accuratissimè explicaturi.

PROPOSITIO VIII.

Si recta in septentrionem aut meridiem contenti cursûs quantitas æstimata ab observatæ latitudinis evariatione discrepet, cursûs limes in observato parallelo erit constituendus.

NAVIS ex *a* sub parallelo 50, digressa, acu monstrante visa sit tenere cursum sub eodem meridiano, quem cursum navis magister milliaribus quidem 27, taxet. Ut tamen suæ æstimationis veritatem exploret, consiliuit cœlestes observationes adhibere in consilium; è quibusprehenderit se isto momento versari sub parallelo 52. Cum igitur milliaria 27 maritima æqualia sunt 1 gr. 48 scr. efficitur ex hac cursûs æstimatâ quantitate, ipsum versari sub latitudine 51. gr. 48. ut, putâ, in *i*; ex observatis autem in puncto *e* sub parallelo 52. octo scrupula ultrâ. Contrâ, si æstimata latitudo major sit observatâ, conjecturæ erit derogandum, et locus sub observato parallelo collocandus. Sit enim cursûs æstimatus *ao* milliarium 51 seu 3. gr. 24 scr. et *o* punctum igitur sub latitudine 53 gr. 24 scr. at observatio tantum addicat 53 gr. 12 scr. Jam tanto minorem assumam latitudinem, et navem statuum in puncto *e*. Atque ita de cœlo conjecturam enendabis. Neque ideò tamen hæc cursûs confecti divinationi frustrâ, aut incassum instituta videri debet, quamvis hic cursûs mensura, cum rectâ in septentrionem aut meridiem contendas, ex ipso cœlo inveniri possit; quia ex hac comparatione paulatim quantum conjecturis reliquis, et ælimato itineri sit tribuendum ex usu cognosco. Quin adeò, si aliquot diebus et noctibus nulla serenitas affulgeat, neque de cœlo observandi potestas detur; huc utique tanquam ad sacram anchoram erit confugiendum: ne imprudens te tuâque eas perditum, cum eo casu nullum certius argumentum uspiam suppetat. Cæterum cum hæc discrepantia vel à cursûs inopinatâ intensione aut remissione, vel ab occultâ maris agitatione et motu obrepserit, id quidem ut ignores, cœlestium tamen observationi subscribes, et tuis conjecturis fidem derogabis. Certè quidem, cum cursûs æstimatio observatâ latitudinis evariatione major erit, idque aliquot continuis diebus eveniat; evidens est, errorem hunc in conjecturâ et æstimatione tuâ hære; et à maris impetu cursûs velo citati remoram factam. At verò cum minor erit æstimata cursûs quantitas, quàm evariata latitudinis observatio addicat, hic cursûs non semper in eodem meridiano ad illum parallelum revocandus tibi erit.

Quamobrem CAUTIO est. Si id sæpius, et quasi continuis observationibus contingat, ut æstimati cursûs quantitas continenter minor deprehendatur observatâ latitudinis evariatione; id enim indubitatum erit indicium, te maris motu transversim abreptum; quamvis acus magnetica sub uno eodémque meridiano cursum institutum arguat. Atque eo casu prudenter et circumspexit ex quantitate cursûs ad observatum parallelum, loxodromiam pro meridiano assumes, et angulum loxodromicum quoque talem, quemadmodum id quantitas cursuum æstimatorum.



æstimatorium postulat. Neque tamèn hæc adhiberi emendatione satis constat, utrum oculo illo maris motu in ortum an in occasum, hoc est, dextrorsum an sinistrorsum loxodromia tua inclinet: donec aliquo insularum aut terrarum occurfu, aut ex observationibus navarchorum qui id pelagus sæpiùs frequentaverunt, didiceris in quam partem aquarum cumuli ferantur: id enim ferè in illis locis certis temporum intervallis perpetuum est et solemne. Talis utique maris impetus naves à septentrionali Americæ litore versùs Angliam furim abripit, qui in sinu Africano intrà Guineam verùs ortum omnia agit et rapit; atque ita aliis in locis etiàm aliter. Hujus exemplum tale esto. Ab insulâ *Beils*, vulgò ita dictâ, sub parallelo 52 in litore Americano, navigarum sit rectâ in meridiem, ut videbatur, idque iter nauticâ æstimatione taxetur milliariibus 50, quæ faciunt evariationem latitudinis 3 gr. 20 scr. Unde datur loci latitudo 48 grad. 40 scr. De cælo autem observata poli altitudo sit 49 gr. 50 scr. Cum hic adeò enormem discrepantiam inter æstimatam et observatam latitudinem deprehendam, non continuò ad observatum parallelum transibo; sed iterùm æstimatam latitudinem cum denuò observatâ postero die contendam. Hic si rursùm conjectura mea abluat ab observatione, et latitudinis æstimatæ differentia iterùm major sit observatâ: Id mihi satis erit indicii maris latente impetu navem in latus furim abreptam, neque sub uno eodémque meridiano cursum hunc continuatum: sed loxodromico aliquo iter confectum. Atque ideò ex datâ latitudinis secundùm observationem differentiâ, et itineris æstimatâ quantitate angulum inclinationis investigabo per propositionem 21^{mam} libri primi; *ut enim cursus æstimati milliaria, ad quadrantem scrupulorum evariate latitudinis, ita est radius ad sinum complementi anguli inclinationis.* Milliaria cursûs hic dantur 50, scrupula evariate latitudinis 130. Unde proportio, ut 50 ad $\frac{1}{2}^\circ$, ita 100.0000 ad sinum 65.0003, nempe 40 gr. 33 scr. cujus complementum 49 gr. 27 scr. datur pro angulo inclinationis quæsitæ. Atque hinc jam, per propositionem 30 lib. primi, dabitur evariatio longitudinis quæsitæ. Nempe, ut 10.000 ad quadruplum numeri canonici, qui huic parallelorum intervallo competit à 49 gr. 50 scr. ad 52 gr. scilicet, 824.8580, ita quadrans tangentis anguli inclinationis $\frac{1}{4}^\circ$ ad scrupula longitudinis evariate 176 $\frac{1}{2}$, qui sunt gradus 3, scr. 16 $\frac{1}{2}$. Tanto itaque intervallo à tuo meridiano abivisti diversus. Neque tamen ullo indicio adhuc constat, in ortumne an verò versùs occasum latens et inobservabilis maris vis navem subduxerit à tuo meridiano, unde primùm discefferas. Ut si scopuli aut faxa latentia et vada in propinquo sint, aut insula aliqua vel denique terra continens, in cuius viciniam noctu aut admodum nebuloso ære delatus sis, hic verò, quod aiunt, intra facrum et faxa confluitus, summo in discrimine verferis. Cui enim in mentem veniat tantillo intervallo tantum periculi ita latentèr objectum, quod nullâ ratione vel à peritissimo nauclero vitari aut declinari possit. Hinc adeò palàm fit, quanti etiàm illud sit in diariis nauticis id quoque accuratè annotari, qui venti anniversariâ periodo in his aut illis oris recurrant, qualis maris et undarum sit agitatio, et quanto tempore ea illic duret; ut ex anni tempestate et ab experientiâ edocti, cognoscant in quam plagam maris unda abripiatur, in ortumne, an verò in occasum. Sæpe enim mare ipsùm per dilucida intervalla suam etiàm habet quietem nonnullis in locis. Exemplo unico rem explicabo.

Marinæ

Marinæ agitationis et motûs ratio inter Brasiliam et Angolam in Africano opposito litore sitam sub parallelo octavo ita obtinet.

A vicesimâ die Aprilis ad vicesimam Julii mensis, maris fluctus feruntur in Zephyroboream, nobis Noori-West.

A 20 Julii ad 20 Octobris hæc admodum in ullam plagam moventur.

A 20 Octobris ad 20 Januarii maris fluctus feruntur versûs Austrozephyrum, nobis Zuyt-West.

A 20 Januarii ad 20 Aprilis iterum nullo motu cientur.

Atque hæc momenta à navium magistris sedulò et diligentèr sunt observanda, cum itineri compendium quærunt: Horum ignorantia aut negligentia partim morbis, partim inedia, plurimos letho dedit, deficiente comiteatu, aut morbis è diuturnâ morâ eorum membra depascensibus; nam et venti ferè hos motus in eandem plagam aut antecedunt aut comitantur; ut frustra huic monstro obnitaris. Hujusmodi maris agitationes et motus in variis locis admodum sunt varii et inter se dissidentes. Sed ut ad institutum redeam, motus maris ab insulâ *Bellâ*, et suprâ à freto *Davis*, in orientem et meridiem ferè fertur. Itaque in nostro exemplo statuum meridianum ad quem pervenerim orientiorem. Atque hæc causa est, cur, qui illic Britanniam aut Galliam petunt, mirentur has terras tanto ante se aperire, quàm ipsi, suis vulgò usitatis conjecturis ducti, existimârent.

PROPOSITIO IX.

Si acus magnetica cursûs plagam in ortum, vel in occasum, dirigî ostendat, et observatio eandem addicat, nullus correctio erit locus.

NAM hic observatio æstimationi et conjecturæ consentiet: atque ita emendandi argumentum, (quod è solâ istarum discrepantiâ derivatur,) nullum hic nobis suppetet. Quin adeò hujus cursûs ratio est lubrica ut, cum rectâ in orientem aut occidentem proram diriges, et observatio eundem parallelum addicat, tamen occultâ maris agitatione contrarium evenire nacleri Indiam orientalem frequentantes constantèr nuntient. Ventorum enim ea illic est ratio, ut tanquam etisæ (ipsi *massonas* indigitant) certâ anni tempestate continuos aliquot menses ultrò citròque flent: maris verò agitatio suis motibus est obnoxia; quemadmodum antecedente propositione à nobis dictum est: atque ita sit, ut, cum disstentis velis in ortum videaris contendisse; contrâ tamen retrò in occasum sis abreptus. Ita olim quidam S. Helenæ insulam in medio mari aditurus, postquam ad ejus parallelum, graduum nempe 16 austrinæ latitudinis, devenisset, et inde rectâ ipsam insulam sub isto parallelo peteret, cum secundo venio hanc propedièm aditurus videretur: tandem ultrâ ducentiesimum milliare occidentem versûs distare, et in viciniâ Americæ et Brasilie versari est deprehensus; maris latente motu tanto intervallo retrorsum abreptus, ventis

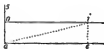
frustrâ obliquantibus. Quamobrem hoc casu periculosa pectâpe est hujusmodi navigatio, ubi nullum tui erroris argumentum appareat, aut quemadmodum te hinc expeditas liquebit. Atque adeo non utile tantum, sed etiâ necessarium videtur, cursum ultrò citrôque obliquare, ut correctioni sit locus: cum autem te non eundem parallelum tenuisse, quem existimabas, observatio arguet; cursus æstimatus secundum observata, ita ut deinceps dicetur, erit tibi corrigendus.

PROPOSITIO X.

Si acûs magneticæ ductum secutus æstimatione tuâ sub eodem parallelo decurrisset, ex observatis autem te eum non tenuisse, deprehendas; manente cursûs æstimatâ quantitate tanquam crure mecodynamico, latitudinem secundum observata mutabis, et inde longitudinis evariationem definies.

ID enim solum maximè consentaneum suprâ dictis propositione sextâ. Cum enim in cursûs plagâ necessariò hic sit erratum, pro parallelo æstimato, loxodromia erit substituenda, quod ipsum evariata latitudo coarguit: sed istum errorem soli plagæ imputare et latitudinem duntaxat mutare iniquum sit, cum itineris æstimati quantitas perinde sit lubrica: quamobrem utrique errori compensando aliquid dandum videtur, ut æstimatâ itineris longitudine retentâ pro crure mecodynamico navis locus referatur ad observatum parallelum, et miliaria cruris mecodynamici ad longitudinem ipsi inter istos parallelos debitam, per 12 propositionem libri primi, revocentur; atque hoc pacto et plaga et itineris quantitas simul variabitur. Ut, si navis magister ab *a* ad *e* sub eodem parallelo cursum peractum existimet, eumque 60 miliaribus taxet sub parallelo 52; inde observatione institutâ, cum antea in *e* navis locum constituerit, postmodum 25 scr. borealiorem deprehendat. Tum 60 miliaribus uteris tanquam totius cursûs crure mecodynamico, et, per propositionem 12 libri primi, erit, ut 10,000 ad quadruplum canonici parallelorum differentie latitudinis 1,631,584, ita $\frac{4}{7}$ miliarium pars quota ad 391 $\frac{1}{2}$ scrupula, qui sunt gradus 6. scr. 31 $\frac{1}{2}$ evariatæ longitudinis. Cum igitur navis mihi, propter observationis à conjecturâ discrepantiam, sit constituenda in parallelo *ai*; longitudinem quoque inventam mutabo, quæ mihi alia fuisset si sub eodem parallelo esset decursum, nempe 6 gr. 30 scr. et pro parallelo *ae* statuum nie decurrisset loxodromiam *ai*. Ita enim manifestum est, me non tantum cursûs quantitatem variare, (cum pro crure anguli recti *ae*, assumam basia *ai*;) sed insuper ab angulo plagæ me quoque discessionem facere; cum pro recto *ae* assumam angulum inclinationis *cai* recto minorem. Ista enim hujus emendationis principia tanquam minùs lubrica supra nobis erant proposita.

Atqui CAUTIO est, si, cum sub eodem parallelo decurres, sæpiùs et quasi continuis observationibus deprehendas, te in eandem plagam (utpote vel in meridiem,



meridiem, vel in septentrionem,) à parallelo quem decurrere institueras abduci, modò itineris confecti æstimatio cum curâ sit facta; tum quidem secundùm observationes de cœlo factas latitudinem censei, itineris verò confecti milliaria pro loxodromiæ quantitate assumi, et inde longitudinis evariationem investigari debere. Ut si in eodem diagrammate acûs magneticæ situs arguat me ab *i* in *o* rectâ ab occasu in ortum cursum meum direxisse, ab insulâ Bellâ sub parallelo 52 gr. et cursûs æstimati spatium sit milliarium sexaginta; observatio autem de cœlo facta arguat me versari sub parallelo 51 gra. 35 scr. ilque continnò deprehendam me ab æstimato parallelo in meridiem declinare, id tantis indicii erit motu maris latente navem à recto cursu in latus abduci. Itaque hic 60 milliaria itineris erunt pro longitudine loxodromiæ; atque hinc et è datâ latitudinis evariatione 29 scr. dabitur angulus inclinationis 84 gr. 1½ scr. per propositionem 21 libri primi, et inde per propositionem . . . crûs mecodynamicum . . . et longitudinis evariatio 38½ scr. qui sunt 6 gr. 2½ scr. Ego parçè psumpsi, cum hallucinatio quæ à maris motu in latitudinem existit, illic tanta sit, ut post peractâ 24, 25, aut 30 milliaria facilè integro gradu latitudinem evariaveris; in 60 autem milliariibus ampliùs gradibus duobus. Si ita sumamus post 60 milliaria confecta 1 gradu et 30 scrupulis in latitudine præter opinionem esse variatum, fiet angulus inclinationis 67 gr. 59 scr. et longitudinis evariatio 5 gradus et 55 scrupuli; cum sub eodem parallelo 52 gr. ea itineris longitudo æstimata det longitudinis evariationem 35 scr. majorem. Tanti interest has medicas manus suo loco et ordine isti negotio adhiberi. Tu ad horum normam reliqua facilè comparabis.

Atque hætenùs de cursuum emendatione, qui in circulo aliquo, monstrante acu magneticâ, instituti videbantur: superest ut ad loxodromiarum æstimationes cursus, et eorum cum observatâ latitudine comparationem, deinceps transeamus.

PROPOSITIO XI.

Si loxodromiam aliquam et ejus quantitatem, æstimationem tuam secutus, ad invenientiam Latitudinis evariationem addideris, et ejusdem evariationem exinde derivatam ab observatâ diversam deprehendas, crûs mecodynamicum secundùm æstimatum cursum retento, latitudinem in observatum parallelum transferes, et hinc longitudinis differentiam investigabis.

NORMA cursûs quantitatem augendi minuendive est latitudo de cœlo observata: hæc enim (ut dictum est,) observationibus sequentibus emendari, aut confirmari et refelli potest; reliqua autem nec iterari, neque relegi possunt. In quibus quidem universis si peccatum non sit, omnium arctissimum consensum esse oportet: sin verò aliqua inter ista differentia existat, unica datur latitudo, quæ nobis ad emendationem reliquorum faciem alluceat. Ut si ab a cursum loxodromiæ quintæ iniverim, quemadmodum acûs magneticæ situs observatus dictare mihi visus est, æstimata autem itineris longitudo sit milliarium 77, à parallelo 30 versùs boream, observata latitudo 52 gr. 20 scr. unde nobis è canone $\omega\epsilon\pi\alpha\upsilon\mu\epsilon\tau\alpha$ datur evariatio latitudinis 2 gr. 40 scr. et crûs mecodyn-

B b 2

amicum

amicum $59^{\frac{1}{10} \frac{1}{10} \frac{1}{10} \frac{1}{10}}$ milliarium. Hoc igitur crure mecodynamico tanquam accurato et bene iusto seorsim reservato, latitudinis evariationem secundum observata mutabo; atque ita mecum statuere habeo necesse; ut assumam latitudinis eam evariationem quam observatio addicit, quæ est in exposito casu tantum 2 gr. 20 scr. ut æstimata latitudo major sit quam observata scr. 20. Secundum hanc ex supra invento crure mecodynamico longitudinis evariationem concludam per propositionem 12 lib. 1. hoc modo. Ut 10,000 ad quadruplum numeri parallelorum canonici, qui ad differentiam latitudinum 50 gr. et 52 gr. 20 scr. nempe, datæ et observatæ, pertinet, scilicet 893. 1516; ita cruris mecodynamici pars quota interjectis parallelis cognominis $1^{\frac{1}{10} \frac{1}{10} \frac{1}{10} \frac{1}{10}}$ ad scrupula evariatæ longitudinis $381^{\frac{1}{10}}$, qui sunt 6 gradus et 21 scrupula; et, cum secundum anguli loxodromici et æstimati itineris quantitatem me in puncto *e* versari existinem, navem tamen in puncto *i* colloco, et pro loxodromiâ *ae*, substituam loxodromiam *ai*: atque ita et angulo loxodromico, et itineris conjectaneo intervallo, aliquid vix obrepisse pariter statuam, neque uni tantum soli hunc errorem imputabo.

N O T A.

Hanc factionem summa illa *ἀκρίβεια* secundum leges dictatas postulat: Attamen in singulis cursibus et totius partibus, vulgò usitatam rationem, (quia expediri, neque ab hac ipsâ diligentissimâ æstimatione ullâ differentiâ abeat diversa, quæ quidem digna sit majore *παραγρη*.) in praxi quoque sequemur. Et, ut tanto evidentius hoc ipsum liqueat, utramque hic inter se contendemus. Sit enim idem exemplum quod prius, et detur à parallelo 52 quintæ loxodromiæ quantitas 77 milliarium; inde per canones *εποχίου* evariatio latitudinis conjectaneæ jam data est 2 gr. 40. et crus mecodynamicum $59^{\frac{1}{10} \frac{1}{10} \frac{1}{10} \frac{1}{10}}$ milliarium. Ut secundum hanc æstimatam, non autem veram, latitudinis differentiam, longitudinis evariationem invenias, duas ostendimus vias; quarum prior utitur cruris mecodynamici parte quorâ è latitudinis evariatione, per propositionem 12^{ma} libri primi. Latitudinis differentia hic est 2 gr. 40 scr. Erit itaque ut 10,000 ad quadruplum numeri parallelorum canonici inter 50 gra. et 52 gra. 40 scr. interjecti, nempe 1024. 5416, ita cruris mecodynamici $59^{\frac{1}{10} \frac{1}{10} \frac{1}{10} \frac{1}{10}}$ pars quota interjectis parallelis (qui hic sunt 2 gr. 40 scr. hoc est, scrupula 160) cognominis, utpote $1^{\frac{1}{10} \frac{1}{10} \frac{1}{10} \frac{1}{10}}$, ad scrupula evariatæ longitudinis hujus loxodromiæ quantitati debita $383^{\frac{1}{10}}$, qui sunt 6 gr. $23^{\frac{1}{10}}$ scr. Tanto igitur hic plus sit iusto, cum antea invenerimus ex iisdem datis et latitudinis differentiâ observatâ 2 gr. 20 scr. tantum 6 gr. $21^{\frac{1}{10}}$ scr. Differentia, quæ inter utrumque calculum intercedit, est duntaxat scrup. $1^{\frac{1}{10} \frac{1}{10}}$, et quidem in cursu 77 milliarium sub loxodromiâ quintâ, ubi observata latitudo ab æstimatâ distabat gradus triente: quæ differentia, in tantâ fluctuum agitatione et inconstantia, nobis tanti esse non debet; et quidem eo magis, quod in singulis cursibus æstimandis hanc jam nauticæ sequi habent necesse, ut securi sint suarum rationum, quemadmodum in singulis cursibus, quamdiù de cælo nondum sit observatum, longitudinis locus sit consignandus.

Atque

Atque id quidem tanto magis retineri expedit quod ista via vulgò usitata solâ multiplicatione absolvatur. Nam crus mecodynamicum singulis parallelis debitum est primus cruris mecodynamici numerus uni scrupulo appositus per radium 10,000 divisus; quemadmodum constat per propositionem 28 libri primi. Itaque ut hoc ipsum facilius abfolvami.

Fiat, quemadmodum 10,000 ad 10,245,416, nempe quadruplum numeri parallelorum canonici; ita in hac quintâ loxodromiâ (nam ea datur) primus numerus mecodynamicus per radium divisus $\frac{10,000}{10,245,416}$ ad scrupula evariatâ longitudinis 6 gr. 23 $\frac{1}{2}$ scr. ut priùs. Hoc est, ut quadratum radii 100,000,000 ad 10,245,416, ita crus mecodynamicum unius paralleli in datâ loxodromiâ inclinatione 3741, ad quæsitam longitudinis evariationem. Hanc igitur cervam pro istâ Iphigeniâ, ut tragædici solent, hoc loco subponemus: non tantùm quia hæc facilius litatur; verùm etiâ quòd inter se hæud admodùm dissideant; ut illam quidem cum ratione constitutam, hanc verò ob facilitatem et affinitatem ei substitutam, concedamus. Quamobrem ista hætenus de simplicium loxodromiarum emendatione dicta sufficiant. Sequitur emendatio loxodromiarum compositarum: quæ in latitudinis evariatione, et cruris mecodynamici quantitate, in singulis quidem Loxodromiis singularum Loxodromiarum seorsim leges sequitur; at junctim suis quibusdam præceptis regitur, in quibus ne peccetur, cautio est.

PROPOSITIO XII.

Cursus compositus est, quando priusquam emendationi secundùm observata sit locus, plures continuantur.

FIT non rarò, quin adeò ferè semper, ut cursus ab observatione altitudinis antecedente usque ad proximè sequentem, non sit ejusdem generis; sed ex variis cursuum plagis illud itineris intervallum componatur: ut modò in meridiem vergat; inde vicissim in septentrionem, secundùm aliam atque aliam loxodromiam, sit obliquandus, vel sub aliquo parallelo dirigendus. Instituti enim itineris cursus, ob terrarum in maria excurrentium margines, sæpe articulatim delumbatur, et per diversas plagas inflectitur, frangiturque. Ut si hinc ex Hollandiâ sub parallelo quinquagesimo secundo Gades atque in extremam Hispaniam navigare instituas: primùm quidem in occidentem loxodromiâ quasi quintâ à meridie erit tibi contendendum, usque ad summum et extremum Galliæ promontorium Gobzeum à Ptolomæo dictam, cui insula prætenditur nostris nautis *Eysant* dicta, sub parallelo 48 $\frac{1}{2}$, qui tamen cursus interea aliquoties deartuandus est. Hinc jam propiùs ad meridiem inflectendum, usque dum perventum sit è regione *Cabo Sant Vincent*, quod Ptolomæo est Promontorium sacrum, sub parallelo 37 $\frac{1}{2}$. Hinc denique cursus in ortum loxodromiâ quasi septimâ erit inflectendus, ut Gades sub parallelo 36 $\frac{1}{2}$ propè Herculis fretum (per quod mare mediterraneum se in Oceanum effundit) adeas; et id quidem si ventus opatus euntes prosequatur. Quin adeò in medio mari, ubi jam altum tenuere rates, neque ullæ apparent aut metuuntur terræ, vel scopuli, vel vada, aut saxa.

latentia,

latentia, haud rarò tunc quoque evenit, ut majore tempestate ingruente ^{πῦρ ἀνέμους ἀντεφθιδάμην τῷ ἀνέμῳ}, cum non possint contrà tendere, ita poscentibus ventis, viam velis flectant, tanquam ageretur illud Poëtae

*Nec nos obviti contrà neque tendere tantum
Sufficimus: superat quoniam fortuna, sequamur;
Quoque vocat, vertamus iter.*—————

Hic enim vehementioribus procellis ingruentibus ^{ἐκδύοντες φέρονται}, venti arbitrio se permittunt: alioquin etià adversissimis ventorum flatibus non concessuri, ut retrorsum agantur; sed cursùs obliquatione modò hâc, modo illac, proram obverientes, aliquà ex parte institutum iter promoturi. Atque hinc breves illæ inflexiones 2, 3, 4, 5, plurium aut pauciorum milliarium, quibus primarii cursùs directio deartuatur et infringitur. Hic igitur cursus è variorum generum miscellâ, utpote duorum triumve, aut etià plurium, conflatus, à nobis nunc compositus dicitur.

PROPOSITIO XIII.

Si in cursu composito continenter omnes à communi parallelo in septentrionem vel meridiem vergant, latitudinis evariationem augebunt; si qui ex iis in contrarium reflectantur, ii pro ratâ parte eam imminuent.

CUM omnis cursus, excepto octavo, qui sub eodem parallelo manet, latitudinis et parallelorum evariationem inducat; sive ea à meridie in septentrionem porrecta augeatur, seu contrà à septentrione in meridiem minuatur, et hoc quidem in singulis sigillatim. In conjunctis autem et compositis cursibus res se paulo secùs habet. Illuc enim latitudinis evariatio eadem esse, vel etià augeri, nonnunquam et minui etià potest. Ut si à parallelo 52 quocunque cursu perventum sit ad 50, atque inde cursum rectâ in ortum vel occasum flectas; jam cum in eodem parallelo continuò decurratur, latitudinis evariatio hinc nulla inducetur major; at si porrò inde ultrà in meridiem inflectas ut pervenias ad 48 parallelum, jam tanto erit auctior latitudinis evariatio: nam principium et hic finis distant 4 gradibus. At si à parallelo 50 flexisses in septentrionem donec perventum esset ad parallelum 50 graduum et 45 scrupulorum, hinc evariatio latitudinis fuisset tantum imminuenda. Exemplum tale esto; à Mosæ fluminis ostio, sub parallelo 51 navigatum sit ad promontorium Angliæ.

<i>De Singels vento Zuyt-West ten Westen</i>	milliaribus	33
<i>A Singels ad Bevesier West-Zuyt-West</i>	milliaribus	6
<i>A Bevesier ad Lezert * West ten Zuyden</i>	milliaribus	54½
<i>A Lezert ad Caput Finisterre Zuyd-Zuyd-West</i>	milliaribus	113½

Primus cursus est loxodromia 5 à meridie versùs occasum, ut infrâ patet ex typo ventorum; et hic designatur cursùs *ae*.

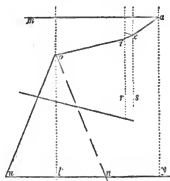
* Anglicè, *The Lizard point*.

Secundus curfus est loxodromia 6 à meridie versùs occasum; et hic designatur cursu *ei*.

Tertius est loxodromia 7 à meridie versùs occasum, et hic exprimitur cursu *oe*.

Quartus est loxodromia 2 à meridie versùs occasum, et hic exprimitur cursu *ou*.

Vides itaque omnes loxodromias, à parallelo sub quo inceperant, versùs meridiem inclinare, et ideò à parallelo 52 unde initium ductum est, longius discedere, latitudinem imminuendo; latitudinis autem evariatio singulis debita invenietur, per propositionem . . . libri primi, aut per canones *περιχύρως*, in singulis ista:



numerus loxod.	cursus nomina.	milliaria.	evar. lat. gra. scr.
5	Z. W. t. Z.	33	1. $13\frac{1}{2}\frac{1}{2}$
6	W. Z. W.	6	0 $9\frac{1}{2}\frac{1}{2}$
7	W. t. Z.	$54\frac{1}{2}$	0 $2\frac{1}{2}\frac{1}{2}$
2	Z. Z. W.	$113\frac{1}{2}$	6 $9\frac{1}{2}\frac{1}{2}$
		<i>summa</i>	9. $4\frac{1}{2}\frac{1}{2}$

Addendæ igitur sunt hæ singulæ, quæ constant 9 grad. $4\frac{1}{2}\frac{1}{2}$ scr. Et quia curfus expositi versùs meridiem vergunt, de propositâ latitudine 52 gra. deducendi sunt: unde datur, secundùm has conjecturas, latitudo *Capo finisterre* 40 gr. $55\frac{1}{2}\frac{1}{2}$ scr. Non opus erat istâ scrupulositate ut etiã primorum partes centesimas inquirerem, potui in primis subsistere, et terminum vero proximum asserere; tanta enim maris inconstantia est, ut istæ conjecturæ itinerum ne quidem ad unum aut alterum scrupulum, verum nobis addicere possint. Quemadmodum propositione . . . exemplis et observationibus in medium prolati ostendetur. Quod si aliquis horum cursum in Boream esset inflexus, is pro ratâ parte latitudinis evariationi tantum derogaret, quintiã possset latitudinem in Boream ultrâ parallelum rejicere; ut si hæc novissima loxodromia *A Lesfert* ad *Cabofinisterre* quæ est *Zuyd-Zuyd-West*, mutetur et fiat à *Lesfert Noordt-West* $113\frac{1}{2}$, dabitur in hoc cursu latitudinis evariatio 6 gr. $59\frac{1}{2}\frac{1}{2}$ scr. ut ante, sed in Boream à suo principio inflexa. Erunt itaque tres priores latitudines

tudines addendæ, nempe 1 gr. $13\frac{1}{2}$ scr. et $9\frac{1}{2}$ scr. et $42\frac{1}{2}$ scrup. quarum summa est 2 gradus et 5 scrupula: tantum igitur à cursuum omnium initio, Mosæ ostio, itum est in meridiem. Atqui inde rursùm in Boream 6 gr. $59\frac{1}{2}$ scr. Quare si illa evariatio minor de hac maiore deducatur, reliqua est evariatio 4 graduum et 54 scrupulorum, sed Borea, numeranda à 52 parallelo, et incidet in 56 grad. 54 scr. Similis ratio erit reliquarum in aliis.

PROPOSITIO XIV.

Si in cursu composito continentèr omnes à communi meridiano in ortum vel occasum vergant, longitudinis evariationem adaugebunt: si qui ex iis in contrarium reflectantur, isti pro rati parte eandem imminuent.

QUEMADMODUM propositione 13 latitudinis evariatæ prosthaphæresin explicuimus; ita hîc longitudinis prosthaphæresin prosequemur: quæ utraque in usu, et cursibus compositis sæpe permiscetur. Longitudinis evariatio, et incrementum aut decrementum, hîc non numeramus à primo meridiano; sed à meridiano illius loci unde iter aut computationem instituis, ad eum locum quo contendis; sive iste illinc in orientem, seu in occidentem diffusus vergat. Ut si ab *a* puncto meridiani *ay*, in schemate antecedentis propositionis, discedas in occasum cursu *ae*; et sequens iterùm vergat à terminò *e* in *i*; et tertius ab *i* in *o*; quartus ab *o* in *n*; hîc cursus continentèr longitudinis evariationem adaugebunt: omnes enim ultrà novissimum meridianum in occasum porriguntur. At si novissimus cursus ab *o* reflectatur in *n*, ut *on*, is factæ evariationi tantum derogabit, quantum pars ipsius quantitati congrua sibi vindicabit: ut hîc *ln* de *ly* deducta relinquit *ny*, longitudinis evariationem inter cursûs principium *a* et hunc finem *n*. Cum autem sub eodem meridiano rectà in Austrum aut Boream decurres, longitudinis evariationi nihilum accedere aut decedere, est manifestum.

Assumantur enim cursus, et cursuum inclinatio, et quantitas, eadem quæ suprâ in propositione antecedente; et quærat quanta sit singulorum evariatio longitudinis. Id fiet per propositionem . . . si proximorum parallelorum differentiæ quadruplum multiplicetur per quadranto tangentis inclinationis, seu numerum nicodynamicum cuiusque uni scrupulo debitum. Ego hîc tabellam subijci, in quâ in selide secundâ loxodromiarum ordinem, in tertiâ ventorum nomina ac cursûs plagam, in quartâ milliaria æstimata, in quintâ latitudinis evariationem, seu parallelum in quo singuli terminantur, in sextâ quadruplos canonicos parallelis interjectis debitos, et in septimâ evariationem longitudinis singulorum, expressi. Primâ autem selide numeros canonicos multiplicantes, seu quadrantes tangentis inclinationum, collocavi. Quemadmodum hîc vides.

multi-

Multipli- cantes.	Loxo- dromiz.	Cursus plaga.	Milliaria æstimata.	Evariatio latitu- dinis.		Numeri Canonici quadruplicati.	Evariatio lon- gitudinis.		
				gra.	min.		gra.	min.	
3,741	5	Z. W. t. Z.	33	48	46 $\frac{6}{100}$	674,272	2.	48 $\frac{1}{2}$	Ad.
6,035	6	W. Z. W.	6	48	36 $\frac{4}{100}$	134,088	0.	33 $\frac{1}{2}$	Ad.
12,568	7	W. t. Z.	54 $\frac{1}{2}$	47	54 $\frac{1}{100}$	1,283,796	5.	21	Ad.
1,035	2	Z. Z. W.	113 $\frac{1}{2}$	40	54 $\frac{6}{100}$	974,288	4.	3 $\frac{1}{2}$	Ad.
12. 46 $\frac{1}{2}$									

Evariatio longitudinis itaque erit 12 gr. 46 $\frac{1}{2}$ scr. atque tanto occidentalius est acroterium Hispaniæ *Cabo finisterræ* hodiè dictum, quàm sit Mosæ fluminis ostium. Ego hic in latitudine confignandâ etiam particulâ sum confectatus, ut isto exemplo doceam, quàm parùm ad rhombum eâ faciant; nisi forte obliquissimi sint cursus, ut loxodromia 7 $\frac{1}{2}$, 7 $\frac{1}{4}$, aut in parallelis minimis propè et ultrâ circulum arcticum verferis. Hic enim in eodem exemplo si proximi veris assumantur, nempe 48 gr. 47 scr. et cæteri deinceps, dabitur in singulis cursibus evariatio longitudinis quantam hic expressimus.

paralleli		evariatio long.	
gr.	scr.	gr.	scr.
48	47	2	48
48	38	0	33
47	55	5	25
40	56	4	3

12 49

Ut differentia non sit trium scrupulorum in tam longo cursu, quæ vel conjecturæ in longitudine itineris, vel errori in observandâ poli altitudine summo jure imputari possunt; ut in istâ *λεπτομετρία* faciliè sine ullâ jacturâ manus de tabulâ tolli possit.

Si cursus ultimus pro *Zuyd-Zuyd West* fuisset *Zuyd-Zuyd-Oost*, hoc est, pro *ou* fuisset *on*; jam ultima longitudinis evariatio 4 gr. 3 scr. de reliquis esset subducenda, nempe de 2 gr. 46 scr. et relinqueretur evariatio longitudinis *ny* 4 gr. 43 scr. tantum.

Denique si ab *e* rectâ in meridiem esset decursum in *l*, nullam aliam esse longitudinis evariationem in *l* puncto, quam ante fuerat in *e*, manifestum est; nempe 8 gr. 46 scr. Sequitur comparatio conjectantæ evariationis compositi cursus cum eâ quæ de cælo observatur.

PROPOSITIO XV.

Si in cursu composito aestimata latitudo ab observata discrepet, retenta longitudinis variatione latitudinem in observatum parallelum transferes.

SIMPLICES cursus suas habuerunt leges, secundum quas ipsæ ad observationes de cælo factas regerentur, et nonnullas earum, si nimia enormitas incidere, emendarentur. In compositis autem cursibus difficilior singulos emendandi est ratio, et magis intricata: eam itaque ob rem, nisi alia evidens causa aliter suadeat, (quam quidem nos postea exponemus,) longitudinis variatione reienta, latitudinis differentiâ inter conjectaneam et observatam possit haberi, ad observatum de cælo parallelum transibimus. Res posito exemplo fiet clavior. Sint enim cursus iidem qui supra propositione 12 sunt expressi, à Mosæ fluminis ostio sub parallelo quinquagesimo ad acroterium Hispaniæ *Cabo finis terre* dictum; atque illic de cælo sit observatum, et latitudo inventa 42 gr. 42 scr.: Atque à loxodromiis et itineris ratione ante datur nobis latitudo 40 gr. 55 scr. ut differentiâ inter illam et hanc sit 1 gradus et 47 scrupula; quam loxodromiarum hallucinationi tribuam, ut illarum inclinationes alias reverà fuisse necessum videatur, aque in chalybocliis observatione, aut aliâ quavis occasione non nihil peccatum esse. Neque omnino mirum est, aut inauditum, in tantâ itineris longitudine, ac tot dierum intervallo, cum inierâ de cælo observatum non erit, tantam à latitudine verâ differentiam intercedere: quod infra quoque altero exemplo patebit. Hæc enim non ficta, sed re ipsâ ex nautarum Diariis excerpta proferimus. Porro autem cum ad longitudinem emendam subsidii nihil detur, ante inventam relinebo, quæ nobis inter Mosam et dictum acroterium supra fuit 12 gr. 46 scr. Atque ita in isto meridiani et paralleli communis sectionis termino confecti itineris limitem statuere habeo necesse.

Atque, ut diximus, CAUTIO est, si enormis sit ea differentia, emendatione tali opus esse, quâ anguli quibus loxodromiæ ad meridianum inclinantur, mutantur; hoc est, aliæ loxodromiæ assumantur, quam acus magnetica, in ipso cursu designaverit: quia id à maris oculo impetu factum sit, quo navis transversim ab inslituto tramite sit furtim abrepta. Verbi gratiâ, maris fluctus adeo violento, sed tamen inobservabili, impetu propè fretum *Davis* à Boreâ versus Austrum feritur, ut qui istinc cursum in ortum dirigunt monstrante acu magnetica, (etiam legitime secundum suam chalybocliam emendatâ,) perpetuò deprehendantur, cum non tenuisse cursum, sed pro parallelo hoc, loxodromiam quam inter ortum et meridiem, (nostris de *Zygt Oost streck* dictam) decurrisse. Id quemadmodum in hoc simplici cursu, cum ab observatione incipiet et terminabitur, emendari possit, supra explicui. Itaque semel cognito isto subreptitiæ loxodromiæ angulo reliquos omnes ad hunc temperabo. Nam cursum observatum hic alium esse quam verum, cum haud sis nescius, et maris motum à Septentrione in meridiem ferri: si tu igitur sub eodem meridiano versùs boream contendas, sequitur, iter tuum minus esse quàm conjecturâ factâ æstimes: si in meridiem, multo esse majus; atque ideo illic observatam latitudinem minore

æstimatâ,

æstimatâ, hic contrâ majorem esse, mirari defines, cum causâ pernoveris. Inter meridianum et parallelum, loxodromiarum anguli et quantitas cursus na erunt temperandi; prout diligens observatio, et sedula itineris confecti æstimatio arguet, secundum præcepta in simplicibus posita, ubi singulis cursibus singulas observationes tribuimus, et secundum eas emendavimus quando esset necesse. Cæterum, cum in compositis, ubi post aliquot cursus continuatos denum observatio et emendatio adhibentur, ea res plus habeat difficultatis; necesse est ut cognoscas quænam in singulis cursibus loxodromia pro aliâ sit substituenda, atque ita paratus ac instructus venias: ut in eo quod modò recensuimus, cum in ortum ire visus sis, tamen non referas in ephemerin tuam nauticam *Oeff*, sed ut diximus, *Zuydcoest* milliaria tot; atque ita in cæteris ad eandem normam. Quod si in pelagus aliquod imparatus venias, et præter opinionem in id sis delatus, ejus neque agitationem neque impetum sandò audiveris; opus erit ut summâ cum curâ atque industriâ ipse huic rei animuni advertas, atque æstimata cum observatis conferas, unde correctione loxodromiarum factâ aliquid sani de tuo profectu, aut defectu, statuas; nam id gnavi et industrii naucleri munus est: ejus fidei non tantum merces istæ, quantumvis pretiosæ; verum etiam navis universa, quâ tot hominum vita et salus continetur, est concedita.

Visum est huic theoremati luculentissimum itineris longioris exemplum et calculi rationem subjicere; unde praxis nostra nautica pernosci possit: et totum illud quod suprà in ordinando nautico diario adhibuimus huc transferre, atque ad abacos nautici epilogismi revocare. Ubi hoc mihi monendus es in ipso vestibulo, Loxodromiarum angulos, et cursuum quantitatem à pericissimis naucleis, qui unâ vehebantur, et harum difficultatum haud essent expertes, ita secundum illorum consuetudinem emendatos, quemadmodum ipsi, usu edocti, vero maxime assidere existimarent; idque illi voce technicâ *bebeuden coursen en mylen* indignant. Id significat *cursus et milliaria facta tella*, quo innunt, cum fide, industria, et legitimâ utriusque taxatione, quibus hæc in medio mari sunt obnoxia, ista pro visis et apparentibus esse substituta, prout quotidiana eos experientia, omnium rerum magistra, erudit.

Sit itaque nobis exemplum è Diario nautico propositionis quartæ, ubi è datâ inclinatione et cursuum quantitate longitudinis et latitudinis evariationem per artem, vel si libeat, potius e canonio *προχῆρος* primùm erues. Sed id quoque notandum tibi est, quod loci situs ad quem itur, hic orientior sit eo unde disceditur, et præterea quoque meridionalior; quod idem monere oportuit, ut inde longitudinem et latitudinem loci optati postea rite et sine errore constituas. Diarii hujus nautici secundum divinationem et observata hæc est formula, ut primæ felici mensis dies sit adscriptus; secundæ nomina ventorum quibus cursus inclinatio et plaga definitur; tertiæ ejusdem quantitas in milliariis; quartæ singulorum cursuum latitudinis evariatio de cælo observata; quintæ latitudinis evariatio secundum cursûs æstimatam quantitatem; sexto milliaria mecodynamicis eidem congrua; septimæ longitudinis evariatio è milliariis mecodynamicis derivata. Horum duorum, quæ ultimo loco recensuimus, magnitudinem, ex canonio procliro, aut per caleulum explicare licebit, prout opportunum videbitur, quemadmodum hic vides. Cujus rationem deinceps sigillatim etiâ expli-
camus. Initium navigationis factum est à parallelo boreo 52 gr. 25 scr.

Dies.	Lascio nauticus	Verificationis cursus.	Mil. con- fect.	Alti. Poli. observata. gra. min.	Latitudo à loco diemius derivata gra. min.	Poli alti- tudo con- jectanea.	Evariatio longitudinis.		Longi- tudo 339. 31.
							gra. min.	A	
Julii	6	O. Z. O.	18		0 28	S	1 50	A	
	17	O.	4		0 0		0 26	A	
	5	Z. O. i. O.	13		0 29	S	1 10	A	
	18	O. Z. O.	14		0 21 $\frac{1}{2}$	S	1 23	A	
	6	O. Z. O.	3		0 4 $\frac{1}{2}$	S	0 17 $\frac{1}{2}$	A	
	6	W. Z. W.	11		0 17	S	1 4 $\frac{1}{2}$	S	
	7	W. t. Z.	1 $\frac{1}{2}$		0 1	S	0 8	S	
	19	Z. O. i. Z.	2 $\frac{1}{2}$		0 8 $\frac{1}{2}$	S	0 8 $\frac{1}{2}$	A	
	3	Z. O. i. Z.	5		0 16 $\frac{1}{2}$	S	0 17 $\frac{1}{2}$	A	344. 24
	20	Z. Z. O.	14	48. 38	0 51 $\frac{1}{2}$	S	49. 27	0 33	A
	2	Z. Z. O.	2		0 7 $\frac{1}{2}$	S	0 4 $\frac{1}{2}$	A	
	4	Z. W.	7 $\frac{1}{2}$		0 21 $\frac{1}{2}$	S	0 32 $\frac{1}{2}$	S	
	21	Z. O. t. O.	4		0 8 $\frac{1}{2}$	S	0 20	A	
	22	O. Z. O.	24		0 36 $\frac{1}{2}$	S	2 11 $\frac{1}{2}$	A	
	23	Z. O.	29		1 22	S	2 0	A	
	3	Z. O. t. Z.	31		1 43	S	1 38	A	
	24	Z.	5	42. 58	0 20	S	43. 58	0 0	350. 5
		Z.	21		1 24	S	0 0		
	25	Z. t. W.	5	41. 4	0 19 $\frac{1}{2}$	S	41. 10	0 5 $\frac{1}{2}$	S
	26	Z.	24	39. 24	1 46	S	39. 18	0 0	350. 0
	1 $\frac{1}{2}$	Z. Z. W. $\frac{1}{2}$ Z.	12 $\frac{1}{2}$		0 47	S	0 21 $\frac{1}{2}$	S	
	27	N. O. t. O.	5		0 11 $\frac{1}{2}$	A	0 22	A	
	28	O. N. O. $\frac{1}{2}$ N.	16		0 27 $\frac{1}{2}$	A	1 13 $\frac{1}{2}$	A	
	29	O. t. N. $\frac{1}{2}$ N.	16		0 15 $\frac{1}{2}$	A	39. 31	1 22 $\frac{1}{2}$	A
	7 $\frac{1}{2}$	O. $\frac{1}{2}$ Z.	6 $\frac{1}{2}$		0 1 $\frac{1}{2}$	S	0 35	A	352. 36
	7	O. t. Z.	5		0 3 $\frac{1}{2}$	S	0 25	A	
	30	O.	8 $\frac{1}{2}$	39. 24			39. 11	0 44	A
Aug.	1	O.	23 $\frac{1}{2}$				2 1 $\frac{1}{2}$	A	
	2	O.	38	39. 13			39. 11	3 16 $\frac{1}{2}$	A
	7	O. t. Z.	17		0 13 $\frac{1}{2}$	S	1 26	A	
	3	O.	14	38. 58			39. 0	1 2	A
	4	O.	5	38. 50			39. 0	0 25 $\frac{1}{2}$	A

Quartâ Augusti appulimus ad insulam *Fajael*, *Açorum*, sive *Flandicarum*,
 unam; atque inde 23 ejusdem mensis solvimus Hispaniam versùs navigaturi.

Dies.

Dica.	Longitudo numeris.	Verificationis cursus.	Mil. con- fect.	Altit. Poli observata.	Latitudo à lon- gitudine derivata.		Milliaria mecodyn amica.	Evariatio longitudinis. gr. mi.	Longi- tudo.
					gra. min.	gra. min.			
Aug.	23	6 $\frac{1}{2}$ O. N. O. $\frac{1}{2}$ O.	4 $\frac{1}{2}$		0	6	A		
	24	6 $\frac{1}{2}$ O. N. O. $\frac{1}{2}$ O.	16		0	18 $\frac{1}{2}$	A		
		4 Z. O.	6 $\frac{1}{2}$		0	18 $\frac{1}{2}$	S		
		3 $\frac{1}{2}$ Z. O. $\frac{1}{2}$ Z.	2		0	6 $\frac{1}{2}$	S		
	25	3 $\frac{1}{2}$ N. O. $\frac{1}{2}$ N.	4		0	12 $\frac{1}{2}$	A		
	26	2 $\frac{1}{2}$ N. N. O. $\frac{1}{2}$ O.	16		0	56 $\frac{1}{2}$	A		
	27	3 N. O. t. N.	16		0	53 $\frac{1}{2}$	A		
	28	2 N. N. O.	14		0	51 $\frac{1}{2}$	A		
	29	2 N. N. O.	17		1	3	A		
	30	1 N. t. O.	9 $\frac{1}{2}$		0	3 $\frac{1}{2}$	A		
		2 N. N. W.	1		0	37 $\frac{1}{2}$			
	31	6 O. Z. O.	9		0	13 $\frac{1}{2}$			
		5 Z. O. t. O.	4 $\frac{1}{2}$		0	10			
		Z.	7	42 52	0	28	S		
Sept.	1	O. t. Z.	5		0	37 $\frac{1}{2}$	S		
	2	7 O. t. Z.	18		0	14	S		
	3	7 O. t. Z.	31		0	24	S		
	4	7 O. t. Z.	37		0	28 $\frac{1}{2}$	S		
	5	7 O. t. N.	22		0	17 $\frac{1}{2}$	A		
	6	3 $\frac{1}{2}$ N. O. $\frac{1}{2}$ N.	9		0	27 $\frac{1}{2}$	A		
	7	2 $\frac{1}{2}$ N. N. O. $\frac{1}{2}$ O.	26	42 42	1	29 $\frac{1}{2}$	A		
	3	N. O. t. N.	5		0	16 $\frac{1}{2}$	A		

Novissimo hoc cursu tandem octavâ Septembris appulimus ad promontorium Hispaniæ, vulgò *Caput finis terre* dictum, quâ Hispania longissimè in occiden-tem in orâ Septentrionali procurrit. Ptolomæo esset Nerium promontorium.

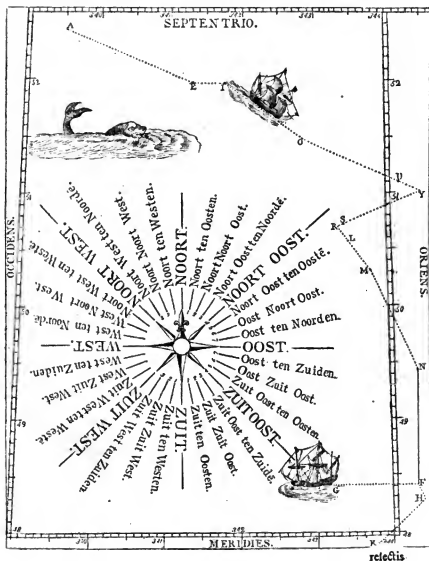
Primi cursûs mensura et plaga in hoc diario exprimitur; longitudo quidem milliarius 18 felide quartâ; plaga autem O. Z. O. felide terciâ, loxodromiæ numerus, memoriæ juvandæ causâ, felide secundâ. Ita enim omnia felicidus accuratiusque absolventur; neque in lubrico consistes quando semel hoc labore eris defunctus, ut felidi secundæ loxodromiæ numerum adjicias. Secus enim etiâ hic memoriâ sollicitare opus esset: quod, ut in tantâ varietate periculo-sum, ita quoque minùs exercitatis haud adeo expeditum fuerit. Licebit itaque, ut suprâ jam expressimus, canones *περὶ κύβου* adhibere, atque illinc latitudinis evariationem secundùm loxodromias, et milliaria mecodynamica depromere; si forsân

forsàn quis in calculo minùs sit exercitatus. Ego verò et tutissimum et expeditissimum iudico ei qui epilogismo utcumque adlueveir, calculum ab ipso fonte areeffere: atque, in cum finem, canonion exiguum ad manus habere, in quo, in primo loco, loxodromiarum ordo, et earum quadrantes; in secundo lox, angulorum, qui ad earum complementa pertinent, sinuum quadriplum; et in tertio loco, tangentium, qui ad ipsarum loxodromiarum angulos pertinent, quadrantes; sint ordine digesti. Secunda enim seils latitudini inveniendæ serviet per confectarium propositionis 28, et per 30 prop. libri primi. Canonion ipsum hîc quoque subiecti ad radium quinque circularum, ut te quoque calculi hujus molestiâ liberarem.

Loxo- drom.	Sin. compl. quadrupl.	$\frac{1}{2}$ tangent.	Loxo- drom.	Sin. compl. quadrupl.	$\frac{1}{2}$ tangent.
$\frac{1}{2}$	390,518	1,228	$4\frac{1}{2}$	268,623	27,583
$\frac{1}{4}$	398,074	2,462	$4\frac{1}{4}$	253,757	30,462
$\frac{3}{4}$	395,670	3,708	$4\frac{3}{4}$	238,280	33,709
1	392,314	4,973	5	222,228	37,415
$1\frac{1}{2}$	388,012	6,262	$5\frac{1}{2}$	205,641	41,710
$1\frac{1}{4}$	382,776	7,584	$5\frac{1}{4}$	188,559	46,805
$1\frac{3}{4}$	376,624	8,945	$5\frac{3}{4}$	171,021	52,858
2	369,552	10,355	6	153,072	60,355
$2\frac{1}{2}$	361,596	11,824	$6\frac{1}{2}$	134,756	69,870
$2\frac{1}{4}$	352,768	13,363	$6\frac{1}{4}$	116,114	82,414
$2\frac{3}{4}$	343,091	14,984	$6\frac{3}{4}$	97,192	99,806
3	332,587	16,704	7	78,036	125,683
$3\frac{1}{2}$	321,283	18,541	$7\frac{1}{2}$	58,692	168,536
$3\frac{1}{4}$	309,204	20,517	$7\frac{1}{4}$	39,207	253,829
$3\frac{3}{4}$	296,380	22,659	$7\frac{3}{4}$	19,627	508,887
4	282,843	25,000			

Nunc itaque numerum sinus qui ad sextum cursum pertinet, (iste enim hîc datur O. Z. O.) excerpam, nempe 153,072; qui, per milliaris multiplicatus, et per radium divisus, dabit scrupula evariatæ latitudinis $27\frac{153072}{1000000}$; aut, quod factis sit, $27\frac{1}{2}$, per propositionem 19 libri primi. Hanc evariationem in opposito laterculo in suâ selide inferibam, sub latitudinis evariatione selide sextâ; et, quia versùs æquinoctialem profectionem promovet et latitudinem minuit, selidi sequenti è regione adscribam S litteram, subductionis notam. Dehinc longitudinis evariationem investigabo per propositionem trigessimam libri 1, hoc modo. Parallelus unde navigari cœptum, erat 52 gr. 25 scr. hic novissimus autem 28 scrupulis est minor; atque ideo 51 gr. 57 scr. differentia numerorum è *canonicis parallelorum*, qui ad hos pertinent, est 45.6587; ejusdémque quadriplum 182.6348. Tùmque quadrans tangentis qui ad hanc loxodromiam percinet, per radium divisus, dabit crus inecodynamicum $\frac{1826348}{1000000}$ quantum latitudinis

nis scrupulo erit consentaneum, per confectarium prop. 28, l. 1. Hinc itaque per propositionem 30, fiat: ut 10,000 ad Canonicum 182.6348, ita $\frac{611111}{777777}$, five,



refectis abundantibus, $100.18263\frac{4}{11}$, hoc est, ut 100,000, ad 182.63, ita 633 ad scrupula evariatæ longitudinis $100\frac{1}{11}$, qui sunt 1 gr. 50 scr. Atque hunc numerum felidi, quæ est à regione, inscribam; et quia cursus hic iter versùs optatum æquinoctialem promover, eidem in felide sequenti adscribam S literam. Cursus iste in appositâ tabellâ exprimat ductu lineæ *ae*. Ego hic milliarium mecodynamica neglexi, quia ita faciliùs negotium absolvitur. Sed, si cui libeat, et ea quoque ratio placeat, per me id ei erit liberum: is enim tum, per propositionem 13 libri primi, quoque longitudinis evariationem in singulis invenire possit. Verùm id mihi videtur operosius: nos certè istam, quam jam institimus viam, porrò sequemur. Quod si tamen velis cognoscere quo cursu simplici principium et finis compositorum cursum copulari et conjungi possit; illic sanè usum habeat hæc res. Sed idem quoque suprâ, propositione 32 lib. 1, nobis est expositum; quare huic felidi suum quidem locum assignamus, etiam si nunc eâ non utamur.

Sequitur cursus secundus, qui ab *e* puncto sub parallelo 51 graduum et 57 scrupulorum, Ortum versùs, per 4 milliarium est continuatus; qui designatur lineâ *ci*. Hic laterculo latitudinis nihil, sive 0, inscribam; quia in eam partem nihil est mutatum: et è regione in laterculo longitudinis 0 gr. 26 scr. 17 sec. Quæ sub isto parallelo quatuor milliaribus per propositionem undecimam libri primi debentur.

Tertia est loxodromia Z. O. t. O. quæ ordine quinta est, ut indicatur è regione felide secundâ, et quidem milliarium 13, qui cursus hic expressus est per lineam *io*; cui in evariatione latitudinis cedunt scrupula 29, rursùm subducenda; atque hæc ita cum his notis suis laterculis inscribantur. Inde quoque ut suprâ longitudinis evariatio datur 1 gr. 10 scr.

Quarta est loxodromia O. Z. O. hoc est, quemadmodum secundâ felide est expressum, loxodromia ordine sexta, qui cursus hic expressus est lineâ *ou*, milliarium 14; unde evariatio latitudinis datur 0 gradus sed 21 $\frac{1}{2}$ scrupula: quam ita suæ felidi inscribam; et, quia latitudinem loci, unde discessum est, minuit, è regione adscribam notam subductionis S. Hinc, ut ante, longitudinis evariatio dabitur 1 gr. 23 scr. addenda: quæ singula in suas propriasque felides referam, ut priùs.

Quintus cursus est VY iterùm O. Z. O. seu loxodromia ordine sexta milliarium 3; unde latitudinis evariatio datur 0 gr. 4 $\frac{1}{2}$ scr. subducenda: quæ ita suis laterculis inferam. Porrò longitudinis evariatio hinc existit 0 gr. 17 $\frac{1}{2}$ scr. addenda antecedentibus; atque ita suis laterculis inscriptas vides.

Sextus cursus est W. Z. W. Loxodromia itidem ordine sexta, milliarium 11, quæ designatur in tabellâ per lineam YS; unde latitudinis evariatio datur 0 gr. 17 scr. quæ adhuc in austrum tendit, atque idè subducenda; unde parallelus per S punctum datur 50 gr. 45 scr. Et hinc evariatio longitudinis invenitur effe 1 gradus et scrupula 4 $\frac{1}{2}$; sed, quia hic cursus in occasum reflectitur, subducenda. Quare longitudinis differentia inter meridianos per A et S transeuntes erit tantum 4 grad. 1 $\frac{1}{2}$ scrup.

Septimus cursus est W. t. Z. loxodromia itidem ordine septima, milliarium 1 $\frac{1}{2}$, quæ exprimitur lineâ SR; unde latitudinis evariatio invenitur 0 gr. 1 sc. subducenda,

ducenda, quia cursum promovet in austrum. Et inde longitudinis evariatio o gr. 8 scr.; quæ est itidem ab antecedentibus subducenda, quia cursum in occidentem contrâ scopum adauget. Atque ita utroque hos numeros suis locis referam.

Octavus cursus est Z. O. t. Z. loxodromia tertia, quæ notatur lineâ RL, est milliarium $2\frac{1}{2}$; unde latitudinis evariatio datur o gr. $3\frac{1}{2}$ scr. quæ itidem, ut omnes antecedentes cursum in meridiem promovet. Hinc longitudinis evariatio invenietur o gr. $8\frac{1}{2}$ scr. quæ rursùm ex instituto cursum versùs opiatum scopum promovet. Atque ideò ita suis locis referam cum notis, illiè S, hic A.

Nonus cursus Z. O. t. Z. est Loxodromia ejusdem generis cum antecedente, scilicet, Loxodromia tertia, et designatur per lineam LM, milliarium 5; atque ideò dabit latitudinis evariationem o gr. $16\frac{1}{2}$ scr. subducendam; et inde evariationem longitudinis o gr. $17\frac{1}{4}$ scr. addendam.

Decimus cursus MN, Z. Z. O. loxodromia secunda milliarium 14, dabit latitudinis evariationem o gr. $51\frac{1}{2}$ scr. ab antecedentibus subducendam, et longitudinis evariationem o gr. 33 scr. quæ itidem cursum versùs locum optatum promovet. Atque ideò cum suis notis suis locis inferam. Hactenus integro quadratuo in mari Oceano jactati, nullam de cælo observandi occasionem habuere; nunc tandem redditâ serenitate, et discussis nebulis, observatio ipso meridie ad solem instituta exhibet poli altitudinem 48 gr. 38 scr. Hanc cum eâ, quam ex loxodromiarum cursibus et quantitate derivavi, contendam. Additis enim latitudinis evariationibus ad singulos cursus pertinentibus, (quæ è cursum quantitate jam sunt derivatæ, et omnes homogeneæ in meridiem à septentrione continuatæ,) dabitur earum summa, 2 gradus et 58 scrupula; quæ, de primâ poli altitudine in puncto A 52 gr. 25 scr. deducta, relinquet altitudinem poli N puncti æstimatam 49 gr. 27 scr. Atqui observata erat tantùm 48 gr. 38 scrup. Differentia utriusque est 49 scrupulorum. Si igitur huic cælesti observationi fidam, et de ejus diligentia certus sim (eam enim postridiè ad solem, aut noctu ad fixas, iteratâ observatione explorare poteris,) hic, repudiatâ latitudine ex cursibus æstimatâ, navem in tabellâ ex N puncto in parallelum observatum FG transferam sub eodem meridiano NG; ut latitudinem quidem ad cælestem observationem componam; longitudinem autem, quam ex loxodromiis derivavi, servem eandem; quæ hic item ex singulis cursibus eruetur, additis addendis, et subductis subducendis; quemadmodum hic vides.

gr.	scr.		gr.	scr.
1	50		1	$4\frac{1}{2}$
0	26		0	3
1	10			
1	23			
0	$17\frac{1}{4}$		1	$12\frac{1}{2}$ S.
0	$8\frac{1}{2}$			
0	$17\frac{1}{4}$			
0	33			
5	$15\frac{1}{2}$ A.			

Vol. IV.

D d

Summa

Summa addendorum est 5 gr. $15\frac{1}{2}$ scr. et subducendorum 1 gr. $1\frac{1}{2}$ scr. Si itaque hunc de illo subducam, relinquunt longitudinis evariatio 4 gr. 3 scr. tantumque ab insulâ et loco A versùs ortum profecimus in F. Si igitur A statuatur sub meridiano 344. N et F erunt sub meridiano 344; adeò ut hæc translatio terminum F tantum longitudinem spectet, manente longitudine æstimatâ.

Verùm hic primum illud occurrit, quod differentia inter observatam et conjecturam altitudinem intercedat 49 scrupulorum; quæ satis magna differentia esse videatur; et proprièrè observatio minùs diligentèr fuisse instituta. Sed in observatione nullum subesse vitium, consensus arguit. Cum enim plures, et quidem periti naucleri eadem veherentur navi, quorum major pars et Orientalem et Occidentalem Indiam non semel adierant; quique hujusmodi observationum essent callentissimi; tamen omnes infrâ nonum et quadragesimum gradum subsistunt, et eâ gratiâ horum omnium observationes accuratè et distinctè adnotabo.

1	2	3	4	5	6
48. 34	48. 8	48. 58	48. 38	48. 20	48. 7

Ego observationem quartam in omnibus sum secutus, cujus autor, ut cæteris in arte peritus, ita summâ diligentia et sedulitate in hoc negotio ad æmulationem usque sit versatus; vides enim è reliquis quatuor longè etiam infrâ ipsum subsistere, unum autem tantum excedere; quod argumenti satis est eam observationem esse vero proximam; quare ea nobis pro legitima esto; et inter conjectaneam atque observatam latitudinem tanta differentia, quantam suprâ expressimus, nempe 49 scrup. Quamobrem illud nunc hinc constanter arguitur, maris agitationem illic à Borcâ in meridiem ferri occulto motu, atque ideò tanto plus ipsos profecisse quam existimarent, aut conjecturis assequi possent. Etiam si enim hujus motûs non essent ignari, et conjecturas suas in itineris conspectu æstimatione, et cursûs plagâ secundùm acûs magneticæ situm factâ, nonnihil secundùm eam maris agitationem definiverint expresserintque; tamen nondum ipsum verum usquequaque assecutos res ipsa loquitur: quod idem quoque, observatione ad diem 24. institutâ, infrâ confirmabitur. Et quamvis hoc motu longitudinis differentiam quoque non nihil luxatam existimem; quia tamen pedum nulla apparet via, ex quâ eadem quoque instaurari corrigique possit, ea tanta nobis reinenda est, quantam calculus è loxodromiis derivatus suppediavit. Et hinc à puncto F reliquæ deinceps loxodromiæ sunt ordinandæ posthabito N termino, tanquam erroneo.

Ab F itaque sit initium cursûs undecimi, qui est Loxodromia Z. Z. O. iterum secunda, 2. milliarium; unde latitudinis evariatio datur 0 gr. $7\frac{1}{2}$ scr. augens itineris profectum in meridiem; et evariatio longitudinis 0 gr. $4\frac{1}{2}$ scr. itidem adjectiva, ut in canonio expressum vides; quæ in tabulâ designatur à lineâ FH.

Hinc duodecimus cursus Z. W. loxodromia quarta, milliarium $7\frac{1}{2}$, ut HK; unde latitudinis evariatio dabitur 0 gr. $21\frac{1}{2}$ ablative; et longitudinis evariatio 0 gr. $32\frac{1}{2}$ scr. ablative, quia in occasum reflectitur, cum noster scopus, *Faiate*, in

in ortum à nobis absit; atque ita suis notis in selides et laterculos debitos hos numeros referam. Et quia reliqua tabella non capit, cum his exemplis satis clarè omnia expresserimus, solos numeros, quibus longitudinis et latitudinis evariatio explicatur, in canonio cum suis notis calculo nostro subductos annotabo, et sinem hinc faciam: ubi illud ante monuero, me scrupulorum particulas in latitudine quoque adhibuisse, et illam *κατασκευασμένην* secutus; non quod tanti ea res sit, sed duntaxat ut hoc exemplo docerem ad summam *ἀκριβειαν* hic nihil esse reliqui, quod jure desiderari possit. Et sanè quamvis abique damno illæ particulæ impleri, et scrupulum proximum assumi possit in quatuor primis, seu minùs obliquis loxodromiis; certè in reliquis, maximè post quintam aut sextam Loxodromiam, minùs tutò negliguntur; cum illæ facilè unius aut alterius scrupuli differentiam, in obliquissimis etiàm complurium, earum neglectus inducere possit.

Hic ita porrò calculum latitudinis et longitudinis ac singulos cursus subluxi, usque ad diem quartum et vicesimum, ubi rursùm de cœlo observatum est, atque loci seu illius paralleli latitudo inventa 42 gr. 54 scr. cum ex conjecturâ et itineris confecti æstimatione deprehenderim 43 gr. 58 scr. ergò nunc fit manifestum quòd integro gradu in conjectaneâ latitudine erratum sit. Atque ita à conjecturis ad observationem transibo, et pro æstimato parallelo 43 gr. 58 scr. statuam navem in parallelo 42 grad. 54 scr.,

Quamobrem, cùm, et suprâ et nunc quoque, bis jam sit observatum navem utrobique integrum gradum suprâ æstimationem profecisse in austrum; cùmque observationes de cœlo factæ sint (quantum maris agitatio sinit) satis accuratæ, insupèr itineris quoque æstimata quantitas et plaga communi omnium consensu approbata; atque ita quidem ut à peritissimis solet, quique hujus agitationis non essent ignari; experimur tamen in isto exemplo mare adhuc potentius à Septentrione in meridiem cieri quàm ipsis visum sit: Optandum utique esset frequentes peritorum nauclerorum navigationes diligentèr consignatas in peritorum manibus transferri, ut inde paulò constantius de istâ latente et occultâ agitatione judicium capi posset: et naucleri, qui ista maria frequentant, commoneri, quemadmodùm æstimatas loxodromias, aut cursus in suas ephemerides referre debeant; quémque cursum pro alio substituere habeant necesse; vel potiùs, quod in talibus utique mavelim, ut visæ loxodromiæ absque nauticâ censurâ quemadmodum pyxis eas exhibuit seorsim consignarentur, et seorsim item iidem secundùm ipsorum censuram; ita enim veritas faciliùs in apertum educeretur. Ea sanè res maximi esset ulùs. Memini enim viros in hac arte peritos, mihi olim narravisse nauclerum quendam non è vulgo, qui semel atque iterùm Amsterdamiam Maderam insulam contrâ Africæ promontorium positum peteret, bis irritò cursu et re infectâ domum reversum, neque eam in medio mari positam adire potuisse. Scilicet occulta illa maris agitatio ei non erat perspecta; atque idè et ipsi et heris ipsi profectio ea fuit damnosa.

Sequitur observatio quarta ad diem vigesimum sextum 39 gr. 24 scr. ubi conjectanea latitudo datur 39 gr. 18 scr. (differentiâ minimâ inter utramque;) longitudo autem 350.0.

D d 2

Tum

Tum quinta ad diem vigesimum nonum 39 gr. 16 scr. et conjectanea 39 gr. 31 scr. et longitudo 352 gr. 36 scr.

Hinc sexta 39 gra. 24 scr. Deinde septima 39 gra. 13 scr. Post hanc octava 38 gr. 58 scr. Denique ultima ad diem 4 Augusti 38 gra. 50 scr. cum longitudine à toxodromiis derivatâ. 2 gr. 30 scr.

Atque ita primum meridianum, qui per insulas *Corvo* et *Flores* ducitur, transivimus, et concludere debemus quod *Fajal* insula, (quam hoc cursu petivimus) 2 gr. 30 scr. isto orientatior sit. Atque distantia meridianorum ab insulâ Bellâ ad hanc usque erit 22 gr. 59 scr. et latitudinis differentia 13 gr. 35 scr.

Haud aliâ ratione longitudinis et latitudinis evagationem è singulis cursibus derivant annotavimus, et cum observatâ latitudine comparavimus ab insulâ *Fajal* ad extremum summumque Hispaniæ promontorium *Caput finis terre* vulgò dictum, quæ omnia suis selidibus et laterculis ordine inscripsimus, unde differentia latitudinis datur 3 gr. 52 scr. longitudinis autem 18 gr. 1 scr. Et tota longitudinis differentia inter *Insulam Bellam* in Americano litore, atque *Caput finis terre*, Hispaniæ promontorium, datur 41 gr. 0 scr.

Illud verò obiter mihi hic notandum videtur, primum meridianum à quibusdam statui per insulas *Corvo* et *Flores*, quia illic acus magnetæ illita accuratè Septentrionem et meridiem spectat; ab aliis autem per *Terçeram* insularum Canariarum, (seu Fortunatarum, ut quibusdam videtur,) unam, in quâ est celsa et sublimis petra altitudine quinquaginta stadiorum, idque ratione perpendiculari; quæ ab intervallo quatuor graduum è meridie aut septentrione adventantibus spectatur. Cum enim altitudo poli in *Tenariiffâ* sit 28 gr. 20 scr. *Madera* autem 32 gr. 30 scr. tamen à *Madérâ* usque videtur suprâ horizontem extare altitudine templi eminè conspecti. Hunc igitur montem, *El Pico* incolis dictum, tanquam signum à naturâ positum plerique hodiè primi meridiani initium statuunt; distat autem meridianus per *El Pico* in *Tenariiffâ* eductus à meridiano per insulas *Corvo* et *Flores* gradibus quatuordecim. Quod si igitur hic primum meridianum statuas, *Fajal* erit sub meridiano 11 gr. 30 scr. Et quamvis perinde sit unde meridianorum initium derivetur ad differentiam longitudinis è cursibus investigandam; necesse tamen est ut id accuratè cognoscas, si iter tuum in ipsâ tabulâ velis consignare, ut nôris in quorumnam locorum viciniâ verferis. Quamobrem hæc ita monuisse sufficiat; et hic quoque simul secundi libri finis esto.

F I N I S.

TABULÆ

T A B U L Æ
C A N O N I C Æ
P A R A L L E L O R U M.

	0	1	2	3	4
1	1.0000	61.6031	121.027	181.0830	241.1964
2	2.0000	62.0031	122.0253	182.0811	242.1989
3	3.0000	61.0034	123.0259	183.0858	243.2013
4	4.0000	61.0036	124.0266	184.0872	244.2038
5	5.0000	65.0038	125.0272	185.0886	245.2064
6	6.0000	66.0040	126.0279	186.0901	246.2089
7	7.0000	67.0041	127.0285	187.0915	247.2115
8	8.0000	68.0043	128.0292	188.0930	248.2141
9	9.0000	69.0045	129.0299	189.0945	249.2167
10	10.0000	70.0047	130.0306	190.0960	250.2193
11	11.0000	71.0049	131.0313	191.0976	251.2220
12	12.0000	72.0051	132.0321	192.0991	252.2246
13	13.0000	73.0054	133.0328	193.1007	253.2273
14	14.0000	74.0056	134.0336	194.1022	254.2300
15	15.0000	75.0058	135.0343	195.1038	255.2328
16	16.0000	76.0061	136.0350	196.1054	256.2355
17	17.0000	77.0063	137.0359	197.1071	257.2383
18	18.0001	78.0065	138.0367	198.1087	258.2411
19	19.0001	79.0068	139.0375	199.1104	259.2439
20	20.0001	80.0071	140.0383	200.1120	260.2468
21	21.0001	81.0073	141.0391	201.1138	261.2496
22	22.0001	82.0076	142.0399	202.1155	262.2525
23	23.0002	83.0080	143.0408	203.1172	263.2554
24	24.0002	84.0082	144.0417	204.1189	264.2583
25	25.0002	85.0085	145.0426	205.1207	265.2613
26	26.0002	86.0089	146.0434	206.1225	266.2643
27	27.0002	87.0091	147.0441	207.1243	267.2673
28	28.0003	88.0094	148.0453	208.1261	268.2703
29	29.0003	89.0098	149.0462	209.1279	269.2734
30	30.0004	90.0101	150.0471	210.1298	270.2765

	0	1	2	3	4
31	31.0004	91.0101	151.0481	211.1317	271.2793
32	32.0004	92.0108	152.0491	212.1335	272.2827
33	33.0005	93.0111	153.0500	213.1354	273.2858
34	34.0005	94.0115	154.0510	214.1374	274.2890
35	35.0006	95.0119	155.0520	215.1393	275.2921
36	36.0006	96.0123	156.0530	216.1413	276.2954
37	37.0007	97.0127	157.0541	217.1432	277.2986
38	38.0007	98.0131	158.0552	218.1452	278.3018
39	39.0008	99.0135	159.0562	219.1473	279.3051
40	40.0008	100.0139	160.0572	220.1493	280.3084
41	41.0009	101.0143	161.0583	221.1513	281.3117
42	42.0010	102.0147	162.0594	222.1534	282.3151
43	43.0011	103.0152	163.0605	223.1555	283.3185
44	44.0012	104.0156	164.0617	224.1576	284.3219
45	45.0012	105.0161	165.0628	225.1597	285.3253
46	46.0013	106.0166	166.0640	226.1619	286.3287
47	47.0014	107.0170	167.0651	227.1640	287.3322
48	48.0015	108.0175	168.0663	228.1662	288.3357
49	49.0016	109.0180	169.0675	229.1684	289.3392
50	50.0017	110.0185	170.0687	230.1706	290.3428
51	51.0018	111.0190	171.0699	231.1729	291.3463
52	52.0019	112.0195	172.0712	232.1751	292.3499
53	53.0020	113.0200	173.0724	233.1774	293.3535
54	54.0021	114.0206	174.0737	234.1797	294.3572
55	55.0023	115.0212	175.0750	235.1820	295.3609
56	56.0024	116.0217	176.0763	236.1844	296.3646
57	57.0025	117.0223	177.0775	237.1868	297.3686
58	58.0027	118.0229	178.0790	238.1891	298.3720
59	59.0028	119.0235	179.0803	239.1915	299.3758
60	60.0030	120.0240	180.0816	240.1940	300.3796

	5	6	7	8	9
1	301.3831	361.6625	421.0525	482.5722	513.2167
2	302.7872	362.6681	423.0600	483.5821	514.2532
3	303.3911	363.6758	424.0676	484.5920	515.2658
4	304.3950	364.6792	425.0752	485.6019	516.2781
5	305.3989	365.6849	426.0829	486.6119	517.2910
6	306.4029	366.6905	427.0906	487.6220	518.3037
7	307.4069	367.6962	428.0983	488.6320	519.3164
8	308.4109	368.7019	429.1061	489.6421	520.3292
9	309.4149	369.7077	430.1139	490.6523	521.3421
10	310.4189	370.7135	431.1217	491.6625	522.3550
11	311.4230	371.7193	432.1296	492.6728	523.3679
12	312.4271	372.7252	433.1375	493.6830	524.3809
13	313.4313	373.7310	434.1454	494.6934	525.3939
14	314.4354	374.7370	435.1534	495.7037	526.4070
15	315.4396	375.7429	436.1615	496.7142	527.4201
16	316.4438	376.7489	437.1695	497.7246	528.4333
17	317.4481	377.7549	438.1776	498.7351	529.4465
18	318.4523	378.7609	439.1857	499.7456	530.4598
19	319.4566	379.7670	440.1939	500.7562	531.4731
20	320.4609	380.7731	441.2021	501.7669	532.4865
21	321.4653	381.7793	442.2101	502.7775	533.4999
22	322.4697	382.7854	443.2187	503.7882	534.5134
23	323.4741	383.7916	444.2270	504.7990	535.5269
24	324.4785	384.7979	445.2353	505.8098	536.5404
25	325.4829	385.8042	446.2437	506.8207	537.5541
26	326.4874	386.8105	447.2522	507.8315	538.5677
27	327.4919	387.8168	448.2607	508.8425	539.5814
28	328.4965	388.8232	449.2692	509.8534	540.5952
29	329.5011	389.8294	450.2777	510.8644	541.6090
30	330.5057	390.8360	451.2863	511.8755	542.6228

	5	6	7	8	9
31	331.5103	391.8425	452.2949	512.8866	573.6386
32	332.5149	392.8490	453.3036	513.8978	574.6507
33	333.5196	393.8555	454.3123	514.9090	575.6647
34	334.5243	394.8621	455.3211	515.9202	576.6788
35	335.5291	395.8687	456.3298	516.9315	577.6929
36	336.5338	396.8753	457.3387	517.9428	578.7070
37	337.5386	397.8820	458.3475	518.9542	579.7212
38	338.5434	398.8887	459.3564	519.9656	580.7355
39	339.5483	399.8954	460.3654	520.9771	581.7498
40	340.5532	400.9022	461.3744	521.9886	582.7641
41	341.5581	401.9090	462.3834	523.0001	583.7785
42	342.5630	402.9159	463.3924	524.0117	584.7930
43	343.5680	403.9227	464.4015	525.0234	585.8075
44	344.5730	404.9297	465.4107	526.0350	586.8220
45	345.5780	405.9366	466.4198	527.0468	587.8366
46	346.5831	406.9436	467.4291	528.0586	588.8513
47	347.5882	407.9506	468.4383	529.0704	589.8660
48	348.5933	408.9576	469.4476	530.0822	590.8807
49	349.5984	409.9647	470.4570	531.0942	591.8956
50	350.6036	410.9718	471.4663	532.1061	592.9104
51	351.6088	411.9790	472.4758	533.1181	593.9253
52	352.6140	412.9862	473.4852	534.1302	594.9403
53	353.6193	413.9934	474.4947	535.1423	595.9553
54	354.6246	415.0007	475.5043	536.1544	596.9704
55	355.6299	416.0080	476.5138	537.1666	597.9855
56	356.6353	417.0153	477.5235	538.1788	599.0007
57	357.6407	418.0227	478.5331	539.1911	600.0159
58	358.6461	419.0301	479.5428	540.2034	601.0312
59	359.6515	420.0375	480.5526	541.2158	602.0465
60	360.6570	421.0450	481.5624	542.2282	603.0618

E c

	10	11	12	13	14
1	604.0773	665.1016	726.3333	787.7929	849.5008
2	605.0927	666.1203	727.3557	788.8193	850.5315
3	606.1083	667.1392	728.3782	789.8457	851.5623
4	607.1239	668.1581	729.4007	790.8723	852.5931
5	608.1395	669.1770	730.4233	791.8988	853.6241
6	609.1552	670.1960	731.4470	792.9255	854.6550
7	610.1709	671.2151	732.4687	793.9522	855.6861
8	611.1867	672.2342	733.4915	794.9790	856.7172
9	612.2026	673.2534	734.5143	796.0058	857.7485
10	613.2185	674.2726	735.5372	797.0328	858.7797
11	614.2344	675.2919	736.5602	798.0598	859.8111
12	615.2504	676.3113	737.5832	799.0868	860.8426
13	616.2665	677.3307	738.6064	800.1140	861.8741
14	617.2826	678.3502	739.6295	801.1411	862.9057
15	618.2988	679.3697	740.6528	802.1655	863.9373
16	619.3150	680.3893	741.6761	803.1958	864.9691
17	620.3312	681.4089	742.6994	804.2232	866.0009
18	621.3476	682.4286	743.7228	805.2507	867.0328
19	622.3640	683.4484	744.7463	806.2783	868.0648
20	623.3804	684.4682	745.7699	807.3059	869.0968
21	624.3969	685.4881	746.7935	808.3336	870.1290
22	625.4134	686.5081	747.8172	809.3614	871.1612
23	626.4300	687.5281	748.8410	810.3892	872.1935
24	627.4467	688.5481	749.8648	811.4171	873.2258
25	628.4634	689.5683	750.8887	812.4451	874.2583
26	629.4801	690.5884	751.9126	813.4732	875.2908
27	630.4969	691.6087	752.9366	814.5013	876.3234
28	631.5138	692.6290	753.9607	815.5295	877.3560
29	632.5307	693.6494	754.9849	816.5578	878.3888
30	633.5477	694.6698	756.0091	817.5861	879.4216

	10	11	12	13	14
31	634.5647	695.6903	757.0334	818.6145	880.4545
32	635.5818	696.7108	758.0577	819.6430	881.4875
33	636.5989	697.7314	759.0821	820.6716	882.5205
34	637.6161	698.7521	760.1066	821.7002	883.5537
35	638.6334	699.7728	761.1311	822.7289	884.5869
36	639.6507	700.7936	762.1558	823.7577	885.6201
37	640.6681	701.8145	763.1804	824.7865	886.6535
38	641.6855	702.8351	764.2052	825.8155	887.6870
39	642.7029	703.8561	765.2300	826.8445	888.7205
40	643.7205	704.8774	766.2549	827.8735	889.7541
41	644.7381	705.8985	767.2798	828.9027	890.7878
42	645.7557	706.9196	768.3018	829.9319	891.8216
43	646.7734	707.9408	769.3299	830.9612	892.8554
44	647.7911	708.9620	770.3550	831.9905	893.8893
45	648.8089	709.9831	771.3804	833.0199	894.9233
46	649.8268	711.0048	772.4055	834.0494	895.9574
47	650.8447	712.0262	773.4309	835.0790	896.9915
48	651.8627	713.0478	774.4533	836.1087	898.0258
49	652.8807	714.0691	775.4818	837.1384	899.0601
50	653.8988	715.0910	776.5073	838.1682	900.0945
51	654.9170	716.1127	777.5330	839.1981	901.1290
52	655.9352	717.1345	778.5586	840.2280	902.1635
53	656.9534	718.1563	779.5844	841.2580	903.1981
54	657.9718	719.1782	780.6102	842.2881	904.2329
55	658.9901	720.2002	781.6361	843.3183	905.2677
56	660.0085	721.2222	782.6621	844.3485	906.3025
57	661.0270	722.2443	783.6881	845.3788	907.3375
58	662.0456	723.2665	784.7142	846.4092	908.3725
59	663.0642	724.2887	785.7404	847.4397	909.4076
60	664.0828	725.3110	786.7666	848.4702	910.4428

	15	16	17	18	19
1	911.4781	973.7461	1036.3266	1099.2422	1162.5158
2	912.5134	974.7864	1037.3724	1100.2938	1163.5735
3	913.5489	975.8269	1038.4183	1101.3454	1164.6313
4	914.5844	976.8675	1039.4642	1102.3972	1165.6893
5	915.6200	977.9081	1040.5103	1103.4491	1166.7373
6	916.6557	978.9489	1041.5565	1104.5010	1167.8055
7	917.6915	979.9897	1042.6027	1105.5531	1168.8637
8	918.7273	981.0306	1043.6491	1106.6052	1169.9221
9	919.7632	982.0716	1044.6955	1107.6575	1170.9806
10	920.7992	983.1126	1045.7420	1108.7099	1172.0392
11	921.8353	984.1539	1046.7887	1109.7623	1173.0979
12	922.8715	985.1951	1047.8354	1110.8149	1174.1567
13	923.9077	986.2365	1048.8822	1111.8676	1175.2156
14	924.9441	987.2779	1049.9291	1112.9203	1176.2746
15	925.9805	988.3194	1050.9761	1113.9732	1177.3337
16	927.0170	989.3610	1052.0232	1115.0261	1178.3929
17	928.0536	990.4027	1053.0704	1116.0792	1179.4522
18	929.0902	991.4445	1054.1177	1117.1324	1180.5117
19	930.1270	992.4864	1055.1661	1118.1856	1181.5712
20	931.1638	993.5284	1056.2126	1119.2390	1182.6309
21	932.2007	994.5704	1057.2601	1120.2925	1183.6906
22	933.2377	995.6126	1058.3078	1121.3460	1184.7505
23	934.2748	996.6548	1059.3556	1122.3997	1185.8105
24	935.3119	997.6971	1060.4034	1123.4535	1186.8706
25	936.3492	998.7395	1061.4514	1124.5071	1187.9308
26	937.3865	999.7820	1062.4994	1125.5614	1188.9911
27	938.4239	1000.8246	1063.5475	1126.6154	1190.0514
28	939.4614	1001.8673	1064.5958	1127.6696	1191.1120
29	940.4990	1002.9101	1065.6441	1128.7239	1192.1726
30	941.5366	1003.9529	1066.6926	1129.7783	1193.2336

	15	16	17	18	19
31	942.5744	1004.9959	1067.7441	1130.8328	1194.2942
32	943.6122	1006.0389	1068.7897	1131.8874	1195.3552
33	944.6501	1007.0820	1069.8385	1132.9421	1196.4163
34	945.6881	1008.1253	1070.8873	1133.9969	1197.4774
35	946.7262	1009.1685	1071.9362	1135.0518	1198.5387
36	947.7643	1010.2120	1072.9852	1136.1068	1199.6001
37	948.8026	1011.2555	1074.0343	1137.1619	1200.6616
38	949.8409	1012.2990	1075.0835	1138.2171	1201.7232
39	950.8793	1013.3427	1076.1328	1139.2724	1202.7850
40	951.9178	1014.3865	1077.1822	1140.3279	1203.8468
41	952.9564	1015.4303	1078.2317	1141.3834	1204.9087
42	953.9951	1016.4743	1079.2813	1142.4390	1205.9708
43	955.0338	1017.5183	1080.3310	1143.4947	1207.0330
44	956.0727	1018.5624	1081.3808	1144.5506	1208.0952
45	957.1116	1019.6067	1082.4307	1145.6065	1209.1576
46	958.1506	1020.6510	1083.4806	1146.6626	1210.2201
47	959.1897	1021.6954	1084.5307	1147.7187	1211.2828
48	960.2289	1022.7399	1085.5809	1148.7750	1212.3455
49	961.2682	1023.7844	1086.6312	1149.8313	1213.4083
50	962.3075	1024.8291	1087.6816	1150.8878	1214.4713
51	963.3469	1025.8739	1088.7321*	1151.9443	1215.5343
52	964.3865	1026.9187	1089.7826	1153.0010	1216.5975
53	965.4261	1027.9637	1090.8333	1154.0578	1217.6607
54	966.4658	1029.0087	1091.8841	1155.1147	1218.7241
55	967.5056	1030.0539	1092.9349	1156.1717	1219.7877
56	968.5454	1031.0991	1093.9859	1157.2287	1220.8513
57	969.5854	1032.1444	1095.0370	1158.2859	1221.9150
58	970.6254	1033.1898	1096.0881	1159.3432	1222.9788
59	971.6655	1034.2353	1097.1394	1160.4007	1224.0428
60	972.7058	1035.2809	1098.1907	1161.4582	1225.1068

	20	21	22	23	24
1	1226.1710	1290.2321	1354.7240	1419.6726	1485.1044
2	1227.2353	1291.3034	1355.8027	1420.7591	1486.1991
3	1228.2997	1292.3748	1356.8815	1421.8457	1487.2941
4	1229.3642	1293.4463	1357.9604	1422.9325	1488.3891
5	*1230.4288	1294.5179	1359.0394	1424.0194	1489.4843
6	1231.4936	1295.5896	1360.1186	1425.1064	1490.5797
7	1232.5584	1296.6615	1361.1979	1426.1936	1491.6752
8	1233.6234	1297.7335	1362.2773	1427.2809	1492.7708
9	1234.6885	1298.8056	1363.3569	1428.3683	1493.8666
10	1235.7537	1299.8778	1364.4366	1429.4559	1494.9625
11	1236.8190	1300.9502	1365.5161	1430.5435	1496.0585
12	1237.8844	1302.0226	1366.5963	1431.6311	1497.1548
13	1238.9500	1303.0952	1367.6761	1432.7194	1498.2511
14	1240.0156	1304.1679	1368.7566	1433.8075	1499.3476
15	1241.0814	1305.2408	1369.8369	1434.8958	1500.4443
16	1242.1473	1306.3137	1370.9173	1435.9842	1501.5410
17	1243.2133	1307.3868	1371.9979	1437.0726	1502.6379
18	1244.2794	1308.4600	1373.0786	1438.1613	1503.7350
19	1245.3456	1309.5333	1374.1595	1439.2501	1504.8322
20	1246.4119	1310.6067	1375.2404	1440.3391	1505.9295
21	1247.4784	1311.6803	1376.3215	1441.4281	1507.0270
22	1248.5449	1312.7540	1377.4027	1442.5173	1508.1247
23	1249.6116	1313.8278	1378.4841	1443.6067	1509.2225
24	1250.6781	1314.9017	1379.5656	1444.6962	1510.3204
25	1251.7453	1315.9758	1380.6472	1445.7858	1511.4185
26	1252.8124	1317.0499	1381.7290	1446.8755	1512.5167
27	1253.8795	1318.1242	1382.8108	1447.9654	1513.6151
28	1254.9468	1319.1986	1383.8928	1449.0554	1514.7136
29	1256.0142	1320.2732	1384.9750	1450.1456	1515.8122
30	1257.0817	1321.3478	1386.0572	1451.2359	1516.9110

	20	21	22	23	24
31	1258.1493	1322.4226	1387.1396	1452.3264	1518.0099
32	1259.2170	1323.4976	1388.2221	1453.4169	1519.1091
33	1260.2848	1324.5726	1389.3048	1454.5077	1520.2083
34	1261.3528	1325.6477	1390.3876	1455.5985	1521.3077
35	1262.4209	1326.7230	1391.4705	1456.6895	1522.4072
36	1263.4891	1327.7984	1392.5535	1457.7806	1523.5069
37	1264.5574	1328.8739	1393.6367	1458.8719	1524.6067
38	1265.6258	1329.9496	1394.7200	1459.9633	1525.7067
39	1266.6943	1331.0253	1395.8035	1461.0549	1526.8068
40	1267.7630	1332.1012	1366.8870	1462.1466	1527.9071
41	1268.8318	1333.1773	1397.9707	1463.2383	1529.0075
42	1269.9007	1334.2534	1399.0546	1464.3303	1530.1080
43	1270.9697	1335.3297	1400.1385	1465.4224	1531.2087
44	1272.0388	1336.4061	1401.2226	1466.5147	1532.3096
45	1273.1081	1337.4826	1402.3069	1467.6071	1533.4106
46	1274.1774	1338.5593	1403.3912	1468.6997	1534.5117
47	1275.2469	1339.6360	1404.4757	1469.7923	1535.6130
48	1276.3165	1340.7129	1405.5604	1470.8851	1536.7145
49	1277.3862	1341.7899	1406.6451	1471.9780	1537.8161
50	1278.4561	1342.8671	1407.7300	1473.0711	1538.9178
51	1279.5260	1343.9444	1408.8150	1474.1643	1540.0197
52	1280.5961	1345.0218	1409.9002	1475.2577	1541.1217
53	1281.6663	1346.0993	1410.9855	1476.3512	1542.2239
54	1282.7366	1347.1769	1412.0709	1477.4449	1543.3263
55	1283.8070	1348.2547	1413.1565	1478.5387	1544.4287
56	1284.8776	1349.3326	1414.2422	1479.6326	1545.5314
57	1285.9483	1350.4106	1415.3280	1480.7267	1546.6341
58	1287.0190	1351.4888	1416.4139	1481.8209	1547.7371
59	1288.0899	1352.5671	1417.5000	1482.9152	1548.8402
60	1289.1610	1353.6455	1418.5867	1484.0097	1549.9434

	25	26	27	28	29
1	1551.0468	1617.5283	1684.5785	1752.2279	1820.5082
2	1552.1503	1618.6411	1685.7010	1753.3606	1821.6518
3	1553.2540	1619.7540	1686.8237	1754.4935	1822.7955
4	1554.3578	1620.8671	1687.9465	1755.6266	1823.9394
5	1555.4618	1621.9803	1689.0695	1756.7599	1825.0835
6	1556.5659	1623.0937	1690.1926	1757.8933	1826.2278
7	1557.6702	1624.2073	1691.3160	1759.0270	1827.3722
8	1558.7746	1625.3210	1692.4395	1760.1608	1828.5169
9	1559.8792	1626.4348	1693.5631	1761.2947	1829.6617
10	1560.9839	1627.5489	1694.6869	1762.4289	1830.8067
11	1562.0888	1628.6631	1695.8109	1763.5632	1831.9591
12	1563.1938	1629.7774	1696.9351	1764.6977	1833.0973
13	1564.2990	1630.8919	1698.0591	1765.8324	1834.2429
14	1565.4044	1632.0066	1699.1839	1766.9673	1835.3887
15	1566.5099	1633.1214	1700.3086	1768.1023	1836.5346
16	1567.6155	1634.2364	1701.4334	1769.2375	1837.6808
17	1568.7213	1635.3515	1702.5585	1770.3729	1838.8271
18	1569.8272	1636.4668	1703.6836	1771.5085	1839.9736
19	1570.9333	1637.5823	1704.8098	1772.6442	1841.1203
20	1572.0396	1638.6979	1705.9345	1773.7802	1842.2672
21	1573.1460	1639.8137	1707.0602	1774.9163	1843.4142
22	1574.2525	1640.9297	1708.1860	1776.0525	1844.5615
23	1575.3592	1642.0458	1709.3120	1777.1890	1845.7090
24	1576.4661	1643.1620	1710.4382	1778.3256	1846.8566
25	1577.5731	1644.2785	1711.5646	1779.4625	1848.0044
26	1578.6803	1645.3951	1712.6911	1780.5995	1849.1524
27	1579.7876	1646.5118	1713.8178	1781.7366	1850.3006
28	1580.8950	1647.6287	1714.9447	1782.8740	1851.4490
29	1582.0027	1648.7458	1716.0717	1784.0115	1852.5976
30	1583.1104	1649.8631	1717.1989	1785.1492	1853.7464

	25	26	27	28	29
31	1584.2183	1650.9805	1718.3263	1786.2871	1854.8953
32	1585.3264	1652.0980	1719.1539	1787.4252	1856.0445
33	1586.4347	1653.2157	1720.3816	1788.5635	1857.1938
34	1587.5431	1654.3336	1721.7095	1789.7019	1858.3433
35	1588.6516	1655.4517	1722.8376	1790.8405	1859.4930
36	1589.7603	1656.5699	1723.9658	1791.9793	1860.6429
37	1590.8692	1657.6883	1725.0942	1793.1183	1861.7930
38	1591.9782	1658.8068	1726.2228	1794.2574	1862.9433
39	1593.0873	1659.9255	1727.3515	1795.3968	1864.0938
40	1594.1967	1661.0444	1728.4805	1796.5363	1865.2444
41	1595.3061	1662.1634	1729.6096	1797.6760	1866.3953
42	1596.4158	1663.2826	1730.7388	1798.8159	1867.5463
43	1597.5255	1664.4019	1731.8683	1799.9559	1868.6976
44	1598.6355	1665.5215	1732.9979	1801.0961	1869.8490
45	1599.7456	1666.6412	1734.1277	1802.2366	1871.0006
46	1600.8558	1667.7610	1735.2576	1803.3772	1872.1524
47	1601.9662	1668.8810	1736.3878	1804.5180	1873.3044
48	1603.0768	1670.0012	1737.5181	1805.6590	1874.4566
49	1604.1875	1671.1215	1738.6485	1806.8001	1875.6090
50	1605.2983	1672.2420	1739.7792	1807.9414	1876.7616
51	1606.4091	1673.3627	1740.9100	1809.0830	1877.9144
52	1607.5206	1674.4835	1742.0410	1810.2247	1879.0673
53	1608.6319	1675.6045	1743.1722	1811.3666	1880.2205
54	1609.7434	1676.7257	1744.3035	1812.5086	1881.3738
55	1610.8551	1677.8470	1745.4351	1813.6509	1882.5274
56	1611.9669	1678.9685	1746.5668	1814.7933	1883.6811
57	1613.0789	1680.0902	1747.6986	1815.9359	1884.8350
58	1614.1910	1681.2120	1748.8307	1817.0787	1885.9891
59	1615.3033	1682.3340	1749.9629	1818.2217	1887.1434
60	1616.4157	1683.4562	1751.0953	1819.3649	1888.2980

Vol. IV.

Ff

	30	31	32	33	34
1	1889.4527	1959.0956	2029.4732	2100.6227	2172.5836
2	1890.6075	1960.2625	2030.6526	2101.8153	2173.7901
3	1891.7626	1961.4295	2031.8322	2103.0081	2174.9968
4	1892.9179	1962.5968	2033.0120	2104.2011	2176.2037
5	1894.0734	1963.7642	2034.1920	2105.3944	2177.4109
6	1895.2291	1964.9319	2035.3723	2106.5880	2178.6183
7	1896.3850	1966.0997	2036.5527	2107.7816	2179.8260
8	1897.5410	1967.2678	2037.7334	2108.9756	2181.0338
9	1898.6973	1968.4361	2038.9143	2110.1697	2182.2420
10	1899.8537	1969.6045	2040.0954	2111.3641	2183.4503
11	1901.0104	1970.7732	2041.2768	2112.5587	2184.6589
12	1902.1672	1971.9421	2042.4583	2113.7536	2185.8677
13	1903.3243	1973.1112	2043.6401	2114.9487	2187.0768
14	1904.4815	1974.2805	2044.8221	2116.1440	2188.2861
15	1905.6389	1975.4500	2046.0042	2117.3395	2189.4956
16	1906.7966	1976.6197	2047.1867	2118.5353	2190.7054
17	1907.9544	1977.7896	2048.3693	2119.7312	2191.9154
18	1909.1124	1978.9597	2049.5521	2120.9275	2193.1257
19	1910.2706	1980.1301	2050.7352	2122.1239	2194.3362
20	1911.4290	1981.3006	2051.9185	2123.3206	2195.5470
21	1912.5876	1982.4714	2053.1020	2124.5175	2196.7580
22	1913.7465	1983.6423	2054.2857	2125.7147	2197.9692
23	1914.9055	1984.8135	2055.4697	2126.9121	2199.1807
24	1916.0647	1985.9848	2056.6538	2128.1111	2200.3924
25	1917.2241	1987.1564	2057.8382	2129.3075	2201.6013
26	1918.3837	1988.3281	2059.0228	2130.5055	2202.8165
27	1919.5435	1989.5002	2060.2076	2131.7038	2204.0290
28	1920.7035	1990.6724	2061.3926	2132.9023	2205.2417
29	1921.8637	1991.8448	2062.5779	2134.1011	2206.4516
30	1923.0240	1993.0174	2063.7633	2135.3000	2207.6677

	30	31	32	33	34
31	1924.1816	1991.1903	2061.9490	2136.4992	2208.8811
32	1925.3454	1995.3633	2066.1349	2137.6987	2210.0948
33	1926.5064	1996.5365	2067.3210	2138.8984	2211.3087
34	1927.6676	1997.7100	2068.5071	2140.0983	2212.5228
35	1928.8290	1998.8837	2069.6939	2141.2981	2213.7372
36	1929.9906	2000.0575	2070.8807	2142.4987	2214.9518
37	1931.1524	2001.2316	2072.0677	2143.6993	2216.1667
38	1932.3144	2002.4059	2073.2550	2144.9002	2217.3818
39	1933.4766	2003.5804	2074.4424	2146.1012	2218.5972
40	1934.6389	2004.7551	2075.6301	2147.3025	2219.8127
41	1935.8015	2005.9301	2076.8180	2148.5040	2221.0286
42	1936.9643	2007.1052	2078.0061	2149.7058	2222.2447
43	1938.1273	2008.2806	2079.1945	2150.9078	2223.4610
44	1939.2905	2009.4561	2080.3830	2152.1010	2224.6776
45	1940.4539	2010.6319	2081.5718	2153.3125	2225.8944
46	1941.6175	2011.8079	2082.7608	2154.5152	2227.1115
47	1942.7813	2012.9841	2083.9500	2155.7181	2228.3288
48	1943.9453	2014.1605	2085.1395	2156.9212	2229.5463
49	1945.1095	2015.3371	2086.3292	2158.1246	2230.7642
50	1946.2739	2016.5140	2087.5190	2159.3283	2231.9822
51	1947.4385	2017.6910	2088.7092	2160.5321	2233.2005
52	1948.6033	2018.8683	2089.8995	2161.7362	2234.4190
53	1949.7683	2020.0457	2091.0901	2162.9405	2235.6378
54	1950.9335	2021.2234	2092.2809	2164.1451	2236.8569
55	1952.0989	2022.4013	2093.4719	2165.3499	2238.0762
56	1953.2645	2023.5794	2094.6631	2166.5549	2239.2957
57	1954.4303	2024.7577	2095.8546	2167.7602	2240.5153
58	1955.5963	2025.9363	2097.0463	2168.9657	2241.7356
59	1956.7626	2027.1150	2098.2382	2170.1715	2242.9558
60	1957.9290	2028.2940	2099.4303	2171.3775	2244.1764

	35	36	37	38	39
1	2245.3971	2319.1062	2377.7061	2469.3953	2546.0731
2	2246.6182	2320.3125	2378.9124	2470.6646	2547.3602
3	2247.8394	2321.5192	2380.2616	2471.9342	2548.6475
4	2249.0609	2322.8160	2397.5145	2473.2041	2549.9352
5	2250.2827	2324.0531	2398.7677	2474.4743	2551.2232
6	2251.5046	2325.2905	2400.0213	2475.7448	2552.5115
7	2252.7270	2326.5281	2401.2750	2477.0156	2553.8001
8	2253.9495	2327.7660	2402.5291	2478.2865	2555.0889
9	2255.1723	2329.0043	2403.7834	2479.5578	2556.3781
10	2256.3953	2330.2426	2405.0381	2480.8295	2557.6676
11	2257.6186	2331.4813	2406.2930	2482.1014	2558.9574
12	2258.8421	2332.7203	2407.5481	2483.3736	2560.2475
13	2260.0659	2333.9595	2408.8036	2484.6461	2561.5380
14	2261.2899	2335.1990	2410.0593	2485.9189	2562.8287
15	2262.5142	2336.4387	2411.3153	2487.1920	2564.1197
16	2263.7387	2337.6787	2412.5716	2488.4653	2565.4110
17	2264.9635	2338.9190	2413.8281	2489.7390	2566.7027
18	2266.1885	2340.1596	2415.0849	2491.0129	2567.9946
19	2267.4138	2341.4004	2416.3420	2492.2872	2569.2869
20	2268.6393	2342.6414	2417.5994	2493.5617	2570.5795
21	2269.8651	2343.8828	2418.8571	2494.8366	2571.8723
22	2271.0912	2345.1244	2420.1150	2496.1117	2573.1655
23	2272.3175	2346.3662	2421.3732	2497.3871	2574.4590
24	2273.5440	2347.6084	2422.6318	2498.6628	2575.7528
25	2274.7708	2348.8508	2423.8906	2499.9388	2577.0469
26	2275.9979	2350.0934	2425.1497	2501.2151	2578.3413
27	2277.2252	2351.3361	2426.4090	2502.4917	2579.6360
28	2278.4527	2352.5796	2427.6687	2503.7686	2580.9311
29	2279.6806	2353.8230	2428.9286	2505.0458	2582.2264
30	2280.9086	2355.0668	2430.1888	2506.3233	2583.5221

	35	36	37	38	39
31	2282.1370	2356.3108	2431.4492	2507.6011	2581.8180
32	2283.3655	2357.5550	2432.7100	2508.8792	2586.1143
33	2284.5944	2358.7996	2433.9710	2510.1575	2587.4409
34	2285.8235	2360.0444	2435.2323	2511.4362	2588.7078
35	2287.0528	2361.2895	2436.4939	2512.7152	2590.0050
36	2288.2824	2362.5348	2437.7558	2513.9944	2591.3025
37	2289.5123	2363.7804	2439.0180	2515.2740	2592.6001
38	2290.7424	2365.0263	2440.2804	2516.5538	2593.8985
39	2291.9728	2366.2725	2441.5432	2517.8340	2595.1970
40	2293.2034	2367.5189	2442.8062	2519.1144	2596.4958
41	2294.4343	2368.7656	2444.0694	2520.3952	2597.7948
42	2295.6651	2370.0125	2445.3331	2521.6762	2599.0943
43	2296.8968	2371.2598	2446.5969	2522.9576	2600.3940
44	2298.1285	2372.5073	2447.8611	2524.2392	2601.6940
45	2299.3604	2373.7550	2449.1255	2525.5212	2602.9943
46	2300.5926	2375.0031	2450.3902	2526.8034	2604.2950
47	2301.8249	2376.2514	2451.6552	2528.0859	2605.5960
48	2303.0577	2377.5000	2452.9205	2529.3688	2606.8973
49	2304.2906	2378.7488	2454.1861	2530.6519	2608.1989
50	2305.5238	2379.9980	2455.4519	2531.9351	2609.5008
51	2306.7573	2381.2474	2456.7181	2533.2191	2610.8030
52	2307.9910	2382.4970	2457.9845	2534.5031	2612.1056
53	2309.2250	2383.7470	2459.2512	2535.7875	2613.4084
54	2310.4592	2384.9972	2460.5182	2537.0722	2614.7116
55	2311.6937	2386.2477	2461.7855	2538.3571	2616.0151
56	2312.9285	2387.4985	2463.0531	2539.6423	2617.3189
57	2314.1635	2388.7495	2464.3210	2540.9279	2618.6231
58	2315.3988	2390.0008	2465.5891	2542.2137	2619.9275
59	2316.6341	2391.2524	2466.8576	2543.4999	2621.2323
60	2317.8702	2392.5043	2468.1263	2544.7864	2622.5374

	40	41	42	43	44
1	2623.8128	2702.7603	2782.8847	2864.2789	2947.0091
2	2625.1485	2704.0856	2784.2307	2865.6166	2948.3997
3	2626.4546	2705.4113	2785.5771	2867.0146	2949.7906
4	2627.7609	2706.7373	2786.9238	2868.3831	2951.1820
5	2629.0676	2708.0637	2788.2708	2869.7519	2952.5737
6	2630.3746	2709.3904	2789.6182	2871.1211	2953.9658
7	2631.6819	2710.7174	2790.9660	2872.4906	2955.3583
8	2632.9896	2712.0448	2792.3141	2873.8606	2956.7512
9	2634.2976	2713.3725	2793.6625	2875.2309	2958.1445
10	2635.6058	2714.7003	2795.0113	2876.6015	2959.5383
11	2636.9145	2716.0289	2796.3605	2877.9726	2960.9323
12	2638.2231	2717.3576	2797.7100	2879.3440	2962.3268
13	2639.5326	2718.6867	2799.0599	2880.7158	2963.7217
14	2640.8422	2720.0161	2800.4102	2882.0880	2965.1169
15	2642.1521	2721.3458	2801.7608	2883.4606	2966.5126
16	2643.4623	2722.6759	2803.1117	2884.8335	2967.9087
17	2644.7729	2724.0063	2804.4630	2886.2068	2969.3051
18	2646.0837	2725.3370	2805.8147	2887.5805	2970.7020
19	2647.3949	2726.6681	2807.1667	2888.9545	2972.0992
20	2648.7064	2727.9995	2808.5191	2890.3289	2973.4969
21	2650.0182	2729.3313	2809.8718	2891.7038	2974.8949
22	2651.3304	2730.6634	2811.2249	2893.0801	2976.2933
23	2652.6429	2731.9959	2812.5784	2894.4515	2977.6922
24	2653.9557	2733.3287	2813.9322	2895.8304	2979.0914
25	2655.2688	2734.6618	2815.2864	2897.2068	2980.4910
26	2656.5823	2735.9953	2816.6409	2898.5835	2981.8911
27	2657.8960	2737.3291	2817.9958	2899.9605	2983.2915
28	2659.2102	2738.6632	2819.3511	2901.3380	2984.6923
29	2660.5246	2739.9977	2820.7067	2902.7158	2986.0936
30	2661.8394	2741.3326	2822.0627	2904.0941	2987.4952

	40	41	42	43	44
31	2663.1544	2742.6678	2823.4190	2905.4727	2988.8972
32	2664.4699	2741.0033	2824.7757	2906.8516	2990.2997
33	2665.7856	2745.3392	2826.1328	2908.2310	2991.7025
34	2667.1017	2746.6754	2827.4902	2909.6107	2993.1057
35	2668.4181	2748.0120	2828.8480	2910.9909	2994.5094
36	2669.7348	2749.3489	2830.2062	2912.3714	2995.9134
37	2671.0518	2750.6862	2831.5647	2913.7523	2997.3179
38	2672.3692	2752.0238	2832.9236	2915.1335	2998.7227
39	2673.6869	2753.3617	2834.2828	2916.5152	3000.1279
40	2675.0050	2754.7000	2835.6124	2917.8972	3001.5336
41	2676.3233	2756.0387	2837.0024	2919.2796	3002.9397
42	2677.6420	2757.3777	2838.3627	2920.6624	3004.3461
43	2678.9611	2758.7170	2839.7234	2922.0456	3005.7530
44	2680.2804	2760.0567	2841.0845	2923.4292	3007.1603
45	2681.6001	2761.3967	2842.4459	2924.8132	3008.5679
46	2682.9201	2762.7371	2843.8077	2926.1975	3009.9760
47	2684.2405	2764.0778	2845.1699	2927.5822	3011.3845
48	2685.5612	2765.4189	2846.5324	2928.9673	3012.7934
49	2686.8822	2766.7603	2847.8953	2930.3529	3014.2027
50	2688.2035	2768.1021	2849.2587	2931.7387	3015.6124
51	2689.5252	2769.4442	2850.6222	2933.1250	3017.0225
52	2690.8472	2770.7867	2851.9862	2934.5117	3018.4331
53	2692.1695	2772.1295	2853.3506	2935.8987	3019.8440
54	2693.4922	2773.4727	2854.7153	2937.2862	3021.2553
55	2694.8152	2774.8162	2856.0804	2938.6740	3022.6671
56	2696.1386	2776.1601	2857.4459	2940.0622	3024.0792
57	2697.4622	2777.5043	2858.8118	2941.4508	3025.4918
58	2698.7862	2778.8489	2860.1780	2942.8398	3026.9048
59	2700.1106	2780.1938	2861.5446	2944.2292	3028.3182
60	2701.4353	2781.5391	2862.9115	2945.6190	3029.7320

	45	46	47	48	49
1	3031.1462	3116.7653	3203.9166	3292.7757	3383.3443
2	3032.5608	3118.2053	3205.4133	3294.2707	3384.8691
3	3033.9759	3119.6457	3206.8905	3295.7661	3386.3944
4	3035.3913	3121.0866	3208.3182	3297.2621	3387.9202
5	3036.8072	3122.5279	3209.8163	3298.7585	3389.4465
6	3038.2235	3123.9656	3211.2849	3300.2554	3390.9733
7	3039.6401	3125.4118	3212.7539	3301.7518	3392.5006
8	3041.0572	3126.8544	3214.2234	3303.2506	3394.0284
9	3042.4748	3128.2974	3215.6933	3304.7490	3395.5568
10	3043.8927	3129.7409	3217.1638	3306.2478	3397.0856
11	3045.3110	3131.1848	3218.6346	3307.7471	3398.6150
12	3046.7298	3132.6291	3220.1060	3309.2470	3400.1449
13	3048.1490	3134.0739	3221.5778	3310.7473	3401.6753
14	3049.5686	3135.5192	3223.0500	3312.2480	3403.2062
15	3050.9886	3136.9648	3224.5228	3313.7493	3404.7377
16	3052.4090	3138.4109	3225.9959	3315.2511	3406.2696
17	3053.8298	3139.8575	3227.4696	3316.7534	3407.8021
18	3055.2511	3141.3045	3228.9437	3318.2561	3409.3351
19	3056.6728	3142.7519	3230.4183	3319.7593	3410.8686
20	3058.0949	3144.1997	3231.8933	3321.2631	3412.4027
21	3059.5174	3145.6481	3233.3688	3322.7673	3413.9372
22	3060.9403	3147.0968	3234.8448	3324.2720	3415.4723
23	3062.3637	3148.5460	3236.3213	3325.7772	3417.0079
24	3063.7874	3149.9956	3237.7982	3327.2829	3418.5440
25	3065.2116	3151.4457	3239.2755	3328.7891	3420.0806
26	3066.6362	3152.8962	3240.7534	3330.2958	3421.6178
27	3068.0613	3154.3472	3242.2317	3331.8030	3423.1554
28	3069.4867	3155.7986	3243.7105	3333.3106	3424.6936
29	3070.9126	3157.2504	3245.1807	3334.8188	3426.2324
30	3072.3389	3158.7027	3246.6694	3336.3275	3427.7716

	45	46	47	48	49
31	3073.7656	3160.1555	3218.1496	3337.8366	3429.3114
32	3075.1928	3161.6087	3249.6303	3339.3463	3430.8517
33	3076.6303	3163.0623	3251.1115	3340.5864	3432.3924
34	3078.0483	3164.5164	3252.5930	3342.3671	3433.9438
35	3079.4767	3165.9709	3254.0751	3343.8782	3435.4757
36	3080.9056	3167.4259	3255.5576	3345.3899	3437.0181
37	3082.3348	3168.8813	3257.0406	3346.9021	3438.5610
38	3083.7645	3170.3371	3258.5241	3348.4147	3440.1045
39	3085.1946	3171.7935	3260.0081	3349.9278	3441.6481
40	3086.6251	3173.2502	3261.4925	3351.4415	3443.1930
41	3088.0561	3174.7074	3262.9774	3352.9556	3444.7380
42	3089.4875	3176.1651	3264.4628	3354.4703	3446.2846
43	3090.9198	3177.6232	3265.9486	3355.9854	3447.8297
44	3092.3515	3179.0818	3267.4350	3357.5011	3449.3763
45	3093.7842	3180.5408	3268.9218	3359.0172	3450.9234
46	3095.2173	3182.0002	3270.4091	3360.5339	3452.4711
47	3096.6508	3183.4602	3271.8968	3362.0510	3454.0194
48	3098.0848	3184.9205	3273.3851	3363.5687	3455.5681
49	3099.5192	3186.3813	3274.8738	3365.0869	3457.1174
50	3100.9540	3187.8436	3276.3630	3366.6055	3458.6672
51	3102.3892	3189.3043	3277.8526	3368.1247	3460.2175
52	3103.8248	3190.7665	3279.3428	3369.6444	3461.7685
53	3105.2609	3192.2292	3280.8334	3371.1646	3463.3199
54	3106.6975	3193.6992	3282.3245	3372.6853	3464.8719
55	3108.1345	3195.1558	3283.8161	3374.2065	3466.4244
56	3109.5719	3196.6198	3285.3082	3375.7282	3467.9774
57	3111.0097	3198.0842	3286.8007	3377.2501	3469.5310
58	3112.4479	3199.5491	3288.2938	3378.7731	3471.0851
59	3113.8866	3201.0145	3289.7873	3380.2964	3472.6397
60	3115.3257	3202.4803	3291.2813	3381.8201	3474.1949

VOL. IV.

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	50	51	52	53	54
1	3475.7506	3570.1000	3666.5000	3765.0910	3865.9871
2	3477.3069	3571.6896	3668.1309	3766.7533	3867.6891
3	3478.8637	3573.2798	3669.7564	3768.4162	3869.3918
4	3480.4210	3574.8705	3671.3825	3770.0798	3871.0951
5	3481.9789	3576.4618	3673.0091	3771.7440	3872.7992
6	3483.5373	3578.0537	3674.6364	3773.4088	3874.5039
7	3485.0963	3579.6461	3676.2644	3775.0743	3876.2093
8	3486.6558	3581.2391	3677.8929	3776.7405	3877.9154
9	3488.2158	3582.8328	3679.5220	3778.4073	3879.6221
10	3489.7765	3584.4269	3681.1517	3780.0747	3881.3296
11	3491.3376	3586.0217	3682.7821	3781.7428	3883.0378
12	3492.8993	3587.6170	3684.4130	3783.4115	3884.7466
13	3494.4615	3589.2129	3686.0446	3785.0809	3886.4561
14	3496.0243	3590.8094	3687.6768	3786.7509	3888.1663
15	3497.5876	3592.4064	3689.3096	3788.3216	3889.8772
16	3499.1515	3594.0041	3690.9430	3790.0929	3891.5888
17	3500.7159	3595.6023	3692.5770	3791.7649	3893.3011
18	3502.2809	3597.2011	3694.2116	3793.4376	3895.0141
19	3503.8464	3598.8005	3695.8469	3795.1109	3896.7278
20	3505.4125	3600.4004	3697.4828	3796.7848	3898.4422
21	3506.9791	3602.0010	3699.1192	3798.4594	3900.1572
22	3508.5463	3603.6021	3700.7563	3800.1346	3901.8730
23	3510.1140	3605.2038	3702.3941	3801.8106	3903.5894
24	3511.6822	3606.8061	3704.0324	3803.4871	3905.3066
25	3513.2510	3608.4090	3705.6713	3805.1643	3907.0244
26	3514.8204	3610.0124	3707.3109	3806.8422	3908.7430
27	3516.3903	3611.6165	3708.9511	3808.5207	3910.4622
28	3517.9608	3613.2211	3710.5919	3810.1999	3912.1822
29	3519.5318	3614.8263	3712.2334	3811.8798	3913.9028
30	3521.1034	3616.4321	3713.8755	3813.5603	3915.6242

	50	51	52	53	54
31	3522.6755	3618.0385	3715.5181	3815.2415	3917.3162
32	3524.2182	3619.6455	3717.1614	3816.9233	3919.0690
33	3525.8215	3621.2530	3718.8053	3818.6058	3920.7921
34	3527.3953	3622.8612	3720.4499	3820.2890	3922.5166
35	3528.9696	3624.4699	3722.0954	3821.9728	3924.2415
36	3530.5445	3626.0793	3723.7409	3823.6573	3925.9670
37	3532.1200	3627.6892	3725.3873	3825.3424	3927.6933
38	3533.6960	3629.2997	3727.0343	3827.0282	3929.4204
39	3535.2726	3630.9108	3728.6820	3828.7147	3931.1480
40	3536.8498	3632.5225	3730.3303	3830.4019	3932.8764
41	3538.4275	3634.1348	3731.9792	3832.0897	3934.6055
42	3540.0038	3635.7477	3733.6288	3833.7782	3936.3353
43	3541.5816	3637.3612	3735.2790	3835.4672	3938.0658
44	3543.1610	3638.9752	3736.9299	3837.1571	3939.7971
45	3544.7439	3640.5899	3738.5813	3838.8476	3941.5290
46	3546.3241	3642.2052	3740.2334	3840.5388	3943.2617
47	3547.9055	3643.8210	3741.8861	3842.2306	3944.9951
48	3549.4872	3645.4375	3743.5395	3843.9231	3946.7292
49	3551.0694	3647.0545	3745.1935	3845.6163	3948.4640
50	3552.6521	3648.6722	3746.8481	3847.3102	3950.1995
51	3554.2355	3650.2904	3748.5034	3849.0047	3951.9357
52	3555.8194	3651.9093	3750.1592	3850.6999	3953.6727
53	3557.4038	3653.5287	3751.8158	3852.3958	3955.4104
54	3558.9889	3655.1488	3753.4729	3854.0923	3957.1488
55	3560.5745	3656.7694	3755.1307	3855.7895	3958.8879
56	3562.1606	3658.3907	3756.7892	3857.4874	3960.6277
57	3563.7474	3660.0126	3758.4483	3859.1860	3962.3683
58	3565.3347	3661.6350	3760.1080	3860.8853	3964.1096
59	3566.9226	3663.2581	3761.7683	3862.5852	3965.8516
60	3568.5110	3664.8817	3763.4293	3864.2858	3967.5943

	55	56	57	58	59
1	3969.3377	4075.2982	4184.0379	4295.7413	4410.6107
2	3971.0819	4077.0873	4185.8748	4297.6998	4412.5532
3	3972.8268	4078.8771	4187.7125	4299.5181	4414.4967
4	3974.5724	4080.6677	4189.5511	4301.4078	4416.4411
5	3976.3188	4082.4591	4191.3904	4303.2984	4418.3865
6	3978.0659	4084.2513	4193.2306	4305.1900	4420.3328
7	3979.8137	4086.0442	4195.0717	4307.0823	4422.2801
8	3981.5622	4087.8379	4196.9135	4308.9755	4424.2283
9	3983.3115	4089.6324	4198.7562	4310.8697	4426.1774
10	3985.0615	4091.4277	4200.5997	4312.7647	4428.1276
11	3986.8122	4093.2237	4202.4441	4314.6606	4430.0786
12	3988.5636	4095.0205	4204.2893	4316.5574	4432.0306
13	3990.3158	4096.8181	4206.1353	4318.4551	4433.9836
14	3992.0688	4098.6165	4207.9821	4320.3537	4435.9375
15	3993.8224	4100.4157	4209.8298	4322.2532	4437.8924
16	3995.5768	4102.2157	4211.6783	4324.1536	4439.8482
17	3997.3320	4104.0164	4213.5276	4326.0548	4441.8050
18	3999.0878	4105.8179	4215.3778	4327.9570	4442.7627
19	4000.8444	4107.6202	4217.2289	4329.8600	4445.7214
20	4002.6018	4109.4233	4219.0807	4331.7640	4447.6811
21	4004.3599	4111.2272	4220.9334	4333.6688	4449.6417
22	4006.1187	4113.0319	4222.7870	4335.5746	4451.6033
23	4007.8783	4114.8373	4224.6414	4337.4812	4453.5658
24	4009.6386	4116.6436	4226.4966	4339.3887	4455.5298
25	4011.3996	4118.4506	4228.3527	4341.2972	4457.4938
26	4013.1614	4120.2584	4230.2096	4343.2066	4459.4593
27	4014.9239	4122.0671	4232.0674	4345.1168	4461.4267
28	4016.6872	4123.8765	4233.9260	4347.0280	4463.3930
29	4018.4512	4125.6867	4235.7855	4348.9400	4465.3614
30	4020.2160	4127.4977	4237.6458	4350.8530	4467.3307

	55	56	57	58	59
31	4021.9815	4129.3095	4239.5069	4352.7669	4469.3010
32	4023.7178	4131.1221	4241.3689	4354.6817	4471.2723
33	4025.5148	4182.9355	4248.2318	4356.5974	4473.2445
34	4027.2825	4184.7497	4245.0955	4358.5140	4475.2177
35	4029.0512	4136.5647	4246.9601	4360.4315	4477.1919
36	4030.8203	4138.8805	4248.8255	4362.3500	4479.1671
37	4032.5903	4140.1970	4250.6918	4364.2693	4481.1433
38	4034.3611	4142.0145	4252.5589	4366.1896	4483.1205
39	4036.1326	4143.8327	4254.4269	4368.1107	4485.0985
40	4037.9049	4145.6517	4256.2957	4370.0328	4487.0776
41	4039.6779	4147.4715	4258.1654	4371.9559	4489.0577
42	4041.4517	4149.2921	4260.0360	4373.8798	4491.0387
43	4043.2263	4151.1135	4261.9074	4375.8047	4493.0208
44	4045.0016	4152.9357	4263.7797	4377.7304	4495.0038
45	4046.7776	4154.7588	4265.6529	4379.6571	4496.9879
46	4048.5544	4156.5826	4267.5269	4381.5848	4498.9729
47	4050.3320	4158.4073	4269.4017	4383.5133	4500.9589
48	4052.1104	4160.2327	4271.2775	4385.4428	4502.9459
49	4053.8894	4162.0590	4273.1541	4387.3732	4504.9339
50	4055.6693	4163.8861	4275.0316	4389.3045	4506.9230
51	4057.4499	4165.7140	4276.9099	4391.2368	4508.9130
52	4059.2313	4167.5427	4278.7891	4393.1699	4510.9039
53	4061.0134	4169.3722	4280.6692	4395.1041	4512.8959
54	4062.7964	4171.2026	4282.5502	4397.0391	4514.8889
55	4064.5800	4173.0337	4284.4320	4398.9751	4516.8829
56	4066.3645	4174.8657	4286.3147	4400.9120	4518.8778
57	4068.1497	4176.6995	4288.1983	4402.8499	4520.8738
58	4069.9357	4178.5321	4290.0827	4404.7887	4522.8708
59	4071.7224	4180.3666	4291.9680	4406.7284	4524.8688
60	4073.5099	4182.2018	4293.8542	4408.6690	4526.8678

	60	61	62	63	64
1	4528.8677	4650.7566	4776.5470	4906.5380	5041.0622
2	4530.8687	4652.8203	4778.6782	4908.7419	5043.3147
3	4532.8707	4654.8851	4780.8106	4910.9471	5045.6286
4	4534.8737	4656.9511	4782.9442	4913.2536	5047.9139
5	4536.8778	4659.0181	4785.0789	4915.3613	5050.2003
6	4538.8828	4661.0862	4787.2148	4917.5703	5052.4885
7	4540.8889	4663.1552	4789.3519	4919.7806	5054.7779
8	4542.8960	4665.2256	4791.4901	4921.9921	5057.0686
9	4544.9041	4667.2970	4793.6295	4924.2049	5059.3607
10	4546.9120	4669.3694	4795.7701	4926.4190	5061.6542
11	4548.9234	4671.4430	4797.9119	4928.6344	5063.9491
12	4550.9345	4673.5176	4800.0549	4930.8510	5066.2453
13	4552.9467	4675.5934	4802.1990	4933.0689	5068.5430
14	4554.9599	4677.6703	4804.3443	4935.2880	5070.8420
15	4556.9731	4679.7482	4806.4909	4937.5085	5073.1424
16	4558.9893	4681.8272	4808.6386	4939.7302	5075.4441
17	4561.0066	4683.9074	4810.7875	4941.9532	5077.7473
18	4563.0229	4685.9887	4812.9375	4944.1776	5080.0519
19	4565.0413	4688.0710	4815.0888	4946.4032	5082.3578
20	4567.0606	4690.1545	4817.2413	4948.6300	5084.6652
21	4569.0810	4692.2391	4819.3949	4950.8582	5086.9739
22	4571.1024	4694.3248	4821.5498	4953.0877	5089.2841
23	4573.1249	4696.4116	4823.7058	4955.3184	5091.5956
24	4575.1484	4698.4995	4825.8631	4957.5505	5093.9086
25	4577.1729	4700.5885	4828.0215	4959.7838	5096.2230
26	4579.1985	4702.6786	4830.1811	4962.0184	5098.5387
27	4581.2251	4704.7699	4832.3420	4964.2544	5100.8559
28	4583.2527	4706.8623	4834.5041	4966.4916	5103.1745
29	4585.2814	4708.9558	4836.6673	4968.7302	5105.4945
30	4587.3111	4711.0504	4838.8318	4970.9700	5107.8159

	60	61	62	63	64
31	4589.3419	4713.1461	4840.9975	4973.2111	5110.1387
32	4597.3737	4715.2430	4843.1613	4975.4539	5112.4629
33	4593.4066	4717.3410	4845.3324	4977.6974	5114.7886
34	4595.4405	4719.4401	4847.5017	4979.9425	5117.1157
35	4597.4755	4721.5403	4849.6723	4982.1889	5119.4441
36	4599.5115	4723.6417	4851.8441	4984.4366	5121.7741
37	4601.5485	4725.7442	4851.0170	4986.6857	5121.1054
38	4603.5866	4727.8478	4856.1912	4988.9360	5126.4382
39	4605.6258	4729.9526	4858.3666	4991.1877	5128.7724
40	4607.6660	4732.0585	4860.5433	4993.4407	5131.1081
41	4609.7073	4734.1655	4862.7211	4995.6950	5133.4452
42	4611.7496	4736.2737	4864.9002	4997.9507	5135.7837
43	4613.7930	4738.3830	4867.0805	5000.2076	5138.1237
44	4615.8375	4740.4935	4869.2621	5002.4659	5140.4650
45	4617.8830	4742.6051	4871.4448	5004.7256	5142.8079
46	4619.9296	4744.7178	4873.6288	5006.9866	5145.1522
47	4621.9772	4746.8317	4875.8141	5009.2488	5147.4979
48	4624.0259	4748.9467	4878.0006	5011.5125	5149.8451
49	4626.0757	4751.0629	4880.1883	5013.7775	5152.1937
50	4628.1265	4753.1802	4882.3772	5016.0438	5154.5438
51	4630.1784	4755.2987	4884.5674	5018.3114	5156.8954
52	4632.2314	4757.4183	4886.7589	5020.5801	5159.2484
53	4634.2855	4759.5391	4888.9515	5022.8508	5161.6028
54	4636.3406	4761.6610	4891.1455	5025.1225	5163.9587
55	4638.3968	4763.7841	4893.3406	5027.3955	5166.3161
56	4640.4540	4765.9084	4895.5371	5029.6699	5168.6750
57	4642.5124	4768.0338	4897.7347	5031.9456	5171.0353
58	4644.5718	4770.1603	4899.9336	5034.2227	5173.3971
59	4646.6323	4772.2881	4902.1338	5036.5012	5175.7603
60	4648.6939	4774.4169	4904.3353	5038.7810	5178.1250

	65	66	67	68	69
1	5180.1913	5325.2122	5475.7853	5612.6531	5796.4527
2	5182.8589	5327.7021	5478.3161	5635.3245	5799.2453
3	5185.2281	5330.1613	5480.9012	5637.9978	5802.0399
4	5187.5987	5332.6277	5483.4738	5640.6730	5804.8367
5	5189.9708	5335.0927	5486.0401	5643.3502	5807.6356
6	5192.3444	5337.5591	5488.6082	5646.0293	5810.4367
7	5194.7196	5340.0276	5491.1781	5648.7104	5813.2399
8	5197.0961	5342.4975	5493.7497	5651.3984	5816.0452
9	5199.4742	5344.9690	5496.3231	5654.0783	5818.8526
10	5201.8538	5347.4422	5498.8983	5656.7652	5821.6629
11	5204.2348	5349.9170	5501.4753	5659.4510	5824.4740
12	5206.6174	5352.3934	5504.0541	5662.1418	5827.2879
13	5209.0015	5354.8714	5506.6346	5664.8376	5830.1039
14	5211.3870	5357.3511	5509.2169	5667.5323	5832.9221
15	5213.7741	5359.8314	5511.8011	5670.2290	5835.7425
16	5216.1627	5362.3153	5514.3870	5672.9276	5838.5661
17	5218.5528	5364.7999	5516.9747	5675.6282	5841.3898
18	5220.9443	5367.2862	5519.5642	5678.3308	5844.2166
19	5223.3375	5369.7741	5522.1555	5681.0353	5847.0457
20	5225.7321	5372.2636	5524.7486	5683.7419	5849.8769
21	5228.1282	5374.7548	5527.3435	5686.4504	5852.7104
22	5230.5259	5377.2476	5529.9402	5689.1609	5855.5460
23	5232.9250	5379.7421	5532.5388	5691.8734	5858.3838
24	5235.3257	5382.2383	5535.1391	5694.5878	5861.2237
25	5237.7280	5384.7361	5537.7413	5697.3043	5864.0659
26	5240.1317	5387.2356	5540.3453	5700.0228	5866.9108
27	5242.5370	5389.7367	5542.9511	5702.7433	5869.7569
28	5244.9438	5392.2396	5545.5587	5705.4657	5872.6057
29	5247.3522	5394.7441	5548.1682	5708.1902	5875.4567
30	5249.7620	5397.2502	5550.7795	5710.9167	5878.3100

	65	66	67	68	69
31	5252.1735	5399.7581	5533.3926	5713.6453	5881.1654
32	5254.3864	5402.9676	5556.0075	5716.3758	5884.0231
33	5257.0009	5404.7788	5558.6243	5719.1033	5886.8830
34	5259.6170	5407.2917	5561.2430	5721.8429	5889.7451
35	5261.8346	5409.8062	5563.8635	5724.5795	5892.6095
36	5264.2537	5412.3225	5566.4858	5727.3181	5895.4761
37	5266.6744	5414.8405	5569.1100	5730.0587	5898.3449
38	5269.0967	5417.3061	5571.7360	5732.8014	5901.2160
39	5271.5205	5419.8815	5574.3639	5735.5462	5904.0894
40	5273.9458	5422.4045	5576.9937	5738.2929	5906.9650
41	5276.3728	5424.9292	5579.6251	5741.0417	5909.8428
42	5278.8012	5427.4557	5582.2588	5743.7926	5912.7229
43	5281.2313	5429.9838	5584.8941	5746.5455	5915.6053
44	5283.6629	5432.5137	5587.5314	5749.3005	5918.4900
45	5286.0961	5435.0453	5590.1705	5752.0576	5921.3769
46	5288.5308	5437.5786	5592.8114	5754.8166	5924.2661
47	5290.9672	5440.1136	5595.4543	5757.5778	5927.1576
48	5293.4051	5442.6503	5598.0990	5760.3410	5930.0513
49	5295.8446	5445.1887	5600.7456	5763.1063	5932.9474
50	5298.2856	5447.7289	5603.3941	5765.8737	5935.8457
51	5300.7283	5450.2708	5606.0445	5768.6431	5938.7463
52	5303.1725	5452.8144	5608.6968	5771.4147	5941.6493
53	5305.6183	5455.3598	5611.3510	5774.1883	5944.5545
54	5308.0657	5457.9069	5614.0071	5776.9640	5947.4621
55	5310.5147	5460.4557	5616.6631	5779.7418	5950.3719
56	5312.9653	5463.0063	5619.3225	5782.5217	5953.2841
57	5315.4175	5465.5586	5621.9868	5785.3037	5956.1986
58	5317.8713	5468.1126	5624.6505	5788.0878	5959.1154
59	5320.3267	5470.6684	5627.3161	5790.8740	5962.0345
60	5322.7837	5473.2260	5629.9837	5793.6623	5964.9560

Vol. IV.

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C A N O N E S
L O X O D R O M I C I

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Gradus.	Miliaria loxodromica.	Miliaria meodynamica.	Scrupula.	Miliaria loxodromica.	Miliaria meodynamica.	Scrupula.	Miliaria loxodromica.	Miliaria meodynamica.
1	15.0180	7.3690	1	2403	1228	31	7.7593	3807
2	30.0362	14.7380	2	5006	2456	32	8.0096	3930
3	45.0543	22.1070	3	7509	3684	33	8.2599	4052
4	60.0724	29.4760	4	1.0012	4912	34	8.5102	4175
5	75.0905	36.8451	5	1.2515	6140	35	8.7605	4298
6	90.1085	44.2141	6	1.5018	7369	36	9.0108	4421
7	105.1266	51.5831	7	1.7521	8597	37	9.2611	4544
8	120.1447	58.9521	8	2.0024	9825	38	9.5114	4666
9	135.1628	66.3211	9	2.2527	1105	39	9.7617	4789
10	150.1809	73.6902	10	2.5030	1228	40	10.0120	4912
20	300.3618	147.3804	11	2.7533	1350	41	10.2623	5035
30	450.5427	221.0702	12	3.0036	1473	42	10.5126	5158
40	600.7236	294.7604	13	3.2539	1596	43	10.7629	5281
50	750.9045	368.4506	14	3.5042	1719	44	11.0132	5403
60	901.0854	442.1408	15	3.7545	1842	45	11.2635	5526
70	1051.2663	515.8310	16	4.0048	1965	46	11.5138	5649
80	1201.4472	589.5212	17	4.2551	2087	47	11.7641	5772
			18	4.5054	2210	48	12.0144	5895
			19	4.7557	2333	49	12.2647	6017
			20	5.0060	2456	50	12.5150	6140
			21	5.2563	2579	51	12.7653	6263
			22	5.5066	2701	52	13.0156	6386
			23	5.7569	2824	53	13.2659	6509
			24	6.0072	2947	54	13.5162	6632
			25	6.2575	3070	55	13.7665	6754
			26	6.5078	3193	56	14.0168	6877
			27	6.7581	3315	57	14.2671	7000
			28	7.0084	3438	58	14.5174	7123
			29	7.2587	3561	59	14.7677	7246
			30	7.5090	3684	60	15.0180	7369

$$\frac{1}{\epsilon}$$

Gradus.	Milliaria loxodromica.	Milliaria mecodynamica.	Ser- pula.	Milliaria loxodrom.	Milliaria mecodyna.	Ser- pula.	Milliaria loxodrom.	Milliaria mecodyna.
1	15.0725	1.4774	1	2512	246	31	7.7874	7630
2	30.1451	2.9547	2	5024	492	32	8.0387	7876
3	45.2177	4.4321	3	7536	739	33	8.2899	8123
4	60.2901	5.9095	4	1.0048	987	34	8.5411	8369
5	75.3628	7.3867	5	1.2560	1231	35	8.7924	8615
6	90.4354	8.8642	6	1.5072	1417	36	9.0435	8862
7	105.5080	10.3416	7	1.7584	1724	37	9.2947	9109
8	120.5806	11.8189	8	2.0096	1970	38	9.5459	9354
9	135.6532	13.2963	9	2.2609	2216	39	9.7971	9600
10	150.7258	14.7737	10	2.5120	2462	40	10.0483	9849
20	301.4515	29.5473	11	2.7633	2708	41	10.2995	1.0096
30	452.1773	44.3210	12	3.0145	2954	42	10.5508	1.0341
40	602.9030	59.0947	13	3.2657	3200	43	10.8020	1.0587
50	753.6288	73.8684	14	3.5169	3447	44	11.0532	1.0833
60	904.3545	88.6421	15	3.7681	3693	45	11.3044	1.1080
70	1055.0803	103.4158	16	4.0193	3939	46	11.5556	1.1326
80	1205.8060	118.1894	17	4.2705	4185	47	11.8068	1.1572
			18	4.5217	4431	48	12.0580	1.1818
			19	4.7729	4678	49	12.3092	1.2064
			20	5.0241	4924	50	12.5604	1.2311
			21	5.2754	5170	51	12.8117	1.2557
			22	5.5266	5416	52	13.0629	1.2803
			23	5.7778	5662	53	13.3141	1.3049
			24	6.0290	5909	54	13.5653	1.3295
			25	6.2802	6155	55	13.8165	1.3542
			26	6.5314	6399	56	14.0677	1.3788
			27	6.7826	6645	57	14.3189	1.4034
			28	7.0338	6891	58	14.5701	1.4281
			29	7.2850	7138	59	14.8213	1.4527
			30	7.5362	7384	60	15.0726	1.4773

Gradus.	Miliaria loxodromica.	Miliaria meodynamica.	Scru- pula.	Miliaria loxodrom.	Miliaria meodyna.	Scru- pula.	Miliaria loxodrom.	Miliaria meodyna.
1	15.1641	2.2230	1	25.27	370	31	7.8347	1.1496
2	30.3283	4.4500	2	50.54	741	32	8.0875	1.1866
3	45.4924	6.6750	3	75.82	1112	33	8.3402	1.2237
4	60.6565	8.9000	4	101.09	1483	34	8.5930	1.2608
5	75.8206	11.1250	5	126.36	1854	35	8.8457	1.2979
6	90.9848	13.3500	6	151.64	2225	36	9.0984	1.3350
7	106.1489	15.5750	7	176.91	2595	37	9.3512	1.3721
8	121.3130	17.8000	8	202.18	2966	38	9.6039	1.4091
9	136.4772	20.0250	9	227.46	3337	39	9.8566	1.4462
10	151.6413	22.2500	10	252.73	3708	40	10.1094	1.4833
20	303.2826	44.5010	11	278.00	4079	41	10.3621	1.5204
30	454.9239	66.7510	12	303.28	4450	42	10.6148	1.5575
40	606.5652	89.0020	13	328.55	4820	43	10.8676	1.5946
50	758.2065	111.2520	14	353.82	5191	44	11.1203	1.6317
60	909.8478	133.5020	15	379.10	5562	45	11.3730	1.6687
70	1061.4891	155.7530	16	404.37	5933	46	11.6258	1.7058
80	1213.1304	178.0030	17	429.65	6304	47	11.8785	1.7429
			18	454.92	6675	48	12.1313	1.7800
			19	480.19	7045	49	12.3840	1.8171
			20	505.47	7416	50	12.6367	1.8542
			21	530.74	7787	51	12.8895	1.8912
			22	556.01	8158	52	13.1422	1.9283
			23	581.29	8529	53	13.3949	1.9654
			24	606.56	8900	54	13.6477	2.0025
			25	631.83	9271	55	13.9004	2.0396
			26	657.11	9641	56	14.1531	2.0767
			27	682.38	10012	57	14.4059	2.1137
			28	707.65	10383	58	14.6586	2.1508
			29	732.93	10754	59	14.9113	2.1879
			30	758.20	11125	60	15.1641	2.2250

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Gr- du	Miliaria loxodromica	Miliaria meridynamica	Scru- pula	Miliaria loxodrom.	Miliaria meridyna.	Scru- pula	Miliaria loxodrom.	Miliaria meridyna.
1	15.2937	2.9837	1	2549	497	31	7.9017	1.5415
2	30.5873	5.9674	2	5098	995	32	8.1566	1.5913
3	45.8810	8.9511	3	7647	1492	33	8.4115	1.6410
4	61.1747	11.9347	4	1.0196	1989	34	8.6663	1.6907
5	76.4683	14.9184	5	1.2744	2486	35	8.9212	1.7404
6	91.7620	17.9021	6	1.5291	2984	36	9.1761	1.7902
7	107.0556	20.8858	7	1.7843	3481	37	9.4310	1.8399
8	122.3493	23.8695	8	2.0392	3978	38	9.6859	1.8896
9	137.6430	26.8532	9	2.2940	4476	39	9.9408	1.9394
10	152.9366	29.8367	10	2.5487	4972	40	10.1958	1.9894
20	305.8732	59.6737	11	2.8038	5470	41	10.4506	2.0388
30	458.8099	89.5106	12	3.0587	5967	42	10.7055	2.0885
40	611.7466	119.3474	13	3.3136	6464	43	10.9604	2.1383
50	764.6832	149.1843	14	3.5685	6962	44	11.2153	2.1880
60	917.6198	179.0211	15	3.8233	7459	45	11.4702	2.2377
70	1070.5565	208.8580	16	4.0782	7956	46	11.7251	2.2875
80	1223.4931	238.6942	17	4.3331	8453	47	11.9800	2.3372
			18	4.5880	8951	48	12.2349	2.3869
			19	4.8429	9448	49	12.4897	2.4366
			20	5.0978	9945	50	12.7446	2.4864
			21	5.3527	1.0442	51	12.9995	2.5361
			22	5.6076	1.0940	52	13.2544	2.5858
			23	5.8625	1.1437	53	13.5093	2.6355
			24	6.1174	1.1934	54	13.7642	2.6853
			25	6.3723	1.2432	55	14.0191	2.7350
			26	6.6272	1.2929	56	14.2740	2.7847
			27	6.8821	1.3426	57	14.5289	2.8345
			28	7.1370	1.3928	58	14.7838	2.8842
			29	7.3919	1.4421	59	15.0386	2.9339
			30	7.6468	1.4918	60	15.2935	2.9837

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Gradus.	Milliaria loxodromica.	Milliaria meodynamica.	Scrupula.	Milliaria loxodrom.	Milliaria meodyn.	Scrupula.	Milliaria loxodrom.	Milliaria meodyn.
1	15.4634	3.7573	1	2577	626	31	7.9894	1.9412
2	30.9268	7.5146	2	5154	1252	32	8.2471	2.0038
3	46.3902	11.2719	3	7731	1878	33	8.5048	2.0665
4	61.8537	15.0292	4	1.0308	2504	34	8.7626	2.1291
5	77.3171	18.7865	5	1.2886	3131	35	9.0203	2.1917
6	92.7805	22.5438	6	1.5463	3757	36	9.2780	2.2543
7	108.2438	26.3011	7	1.8040	4383	37	9.5357	2.3169
8	123.7072	30.0584	8	2.0617	5009	38	9.7935	2.3796
9	139.1707	33.8157	9	2.3195	5635	39	10.0512	2.4422
10	154.6341	37.5773	10	2.5772	6262	40	10.3089	2.5048
20	309.2683	75.1460	11	2.8349	6888	41	10.5666	2.5674
30	463.9024	112.7190	12	3.0926	7514	42	10.8243	2.6301
40	618.3366	150.2921	13	3.3504	8140	43	11.0821	2.6927
50	773.1707	187.8651	14	3.6081	8767	44	11.3398	2.7553
60	927.8049	225.4381	15	3.8658	9393	45	11.5975	2.8179
70	1082.4391	263.0111	16	4.1235	1.0019	46	11.8552	2.8805
80	1237.0732	300.5842	17	4.3812	1.0645	47	12.1130	2.9432
			18	4.6390	1.1271	48	12.3707	3.0058
			19	4.8967	1.1898	49	12.6284	3.0684
			20	5.1544	1.2524	50	12.8861	3.1310
			21	5.4121	1.3150	51	13.1439	3.1937
			22	5.6699	1.3776	52	13.4016	3.2563
			23	5.9276	1.4402	53	13.6593	3.3189
			24	6.1853	1.5029	54	13.9170	3.3815
			25	6.4430	1.5655	55	14.1747	3.4441
			26	6.7008	1.6281	56	14.4325	3.5068
			27	6.9585	1.6907	57	14.6902	3.5694
			28	7.2162	1.7533	58	14.9479	3.6321
			29	7.4739	1.8160	59	15.2056	3.6946
			30	7.7317	1.8786	60	15.4634	3.7573

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Grav. dist.	Milliaria loxodromica.	Milliaria meodysamica.	Scro- pala.	Milliaria loxodrom.	Milliaria meodyu.	Scro- pala.	Milliaria loxodrom.	Milliaria meodyu.
1	15.6749	4.5502	1	26.12	7.384	31	8.0987	2.3059
2	31.3499	9.1004	2	52.25	15.27	32	8.3599	2.4267
3	47.0249	13.6506	3	78.37	22.75	33	8.6212	2.5026
4	62.6998	18.2008	4	1.0450	30.33	34	8.8821	2.5781
5	78.3748	22.7510	5	1.3062	37.92	35	9.1437	2.6543
6	94.0497	27.3012	6	1.5675	45.50	36	9.4049	2.7301
7	109.7247	31.8514	7	1.8287	53.08	37	9.6662	2.8059
8	125.3997	36.4016	8	2.0900	60.67	38	9.9274	2.8818
9	141.0746	40.9518	9	2.3512	68.25	39	10.1887	2.9576
10	156.7496	45.5020	10	2.6124	75.83	40	10.4499	3.0334
20	313.4992	91.0039	11	2.8737	83.42	41	10.7112	3.1093
30	470.2487	136.5059	12	3.1349	91.00	42	10.9724	3.1851
40	626.9983	182.0078	13	3.3962	98.58	43	11.2337	3.2609
50	783.7479	227.5098	14	3.6574	1.0617	44	11.4949	3.3368
60	940.4975	273.0118	15	3.9187	1.1376	45	11.7562	3.4126
70	1097.2471	318.5137	16	4.1799	1.2134	46	12.0174	3.4885
80	1253.9966	364.0157	17	4.4412	1.2892	47	12.2787	3.5643
			18	4.7024	1.3650	48	12.5399	3.6401
			19	4.9637	1.4409	49	12.8012	3.7160
			20	5.2249	1.5167	50	13.0624	3.7918
			21	5.4862	1.5925	51	13.3237	3.8677
			22	5.7474	1.6684	52	13.5849	3.9435
			23	6.0087	1.7442	53	13.8462	4.0193
			24	6.2699	1.8200	54	14.1074	4.0951
			25	6.5312	1.8959	55	14.3687	4.1710
			26	6.7924	1.9717	56	14.6299	4.2468
			27	7.0537	2.0476	57	14.8912	4.3227
			28	7.3149	2.1234	58	15.1524	4.3985
			29	7.5762	2.1992	59	15.4137	4.4744
			30	7.8374	2.2751	60	15.6749	4.5502

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Gradus	Millaria loxodromica.	Millaria meodysynora.	Ser- pula.	Millaria loxodrom.	Millaria meodyn.	Ser- pula.	Millaria loxodrom.	Millaria meodyn.
1	15.9313	5.3671	1	2655	894	31	8.2311	2.7729
2	31.8626	10.7342	2	5310	1789	32	8.4966	2.8664
3	47.7938	16.1013	3	7965	2683	33	8.7692	2.9518
4	63.7251	21.4683	4	1.0620	3578	34	9.0277	3.0413
5	79.6564	26.8354	5	1.3276	4472	35	9.2932	3.1307
6	95.5877	32.2025	6	1.5931	5367	36	9.5587	3.2202
7	111.5189	37.5696	7	1.8586	6261	37	9.8242	3.3097
8	127.4502	42.9367	8	2.1241	7156	38	10.0898	3.3991
9	143.3815	48.3038	9	2.3896	8050	39	10.3552	3.4886
10	159.3128	53.6708	10	2.6552	8945	40	10.6208	3.5780
20	318.6256	107.3417	11	2.9207	9839	41	10.8863	3.6675
30	477.9383	161.0125	12	3.1862	1.0734	42	11.1518	3.7569
40	637.2511	214.6834	13	3.4517	1.1628	43	11.4174	3.8464
50	796.5639	268.3542	14	3.7172	1.2523	44	11.6829	3.9358
60	955.8768	322.0250	15	3.9828	1.3417	45	11.9484	4.0253
70	1115.1895	375.6959	16	4.2483	1.4312	46	12.2139	4.1147
80	1274.5022	429.3667	17	4.5138	1.5206	47	12.4795	4.2042
			18	4.7793	1.6101	48	12.7450	4.2936
			19	5.0449	1.6995	49	13.0105	4.3831
			20	5.3104	1.7890	50	13.2760	4.4725
			21	5.5759	1.8784	51	13.5415	4.5620
			22	5.8414	1.9670	52	13.8071	4.6514
			23	6.1079	2.0573	53	14.0726	4.7409
			24	6.3725	2.1468	54	14.3381	4.8303
			25	6.6380	2.2362	55	14.6036	4.9198
			26	6.9035	2.3257	56	14.8691	5.0092
			27	7.1690	2.4151	57	15.1347	5.0987
			28	7.4345	2.5036	58	15.4002	5.1881
			29	7.7001	2.5930	59	15.6657	5.2776
			30	7.9656	2.6825	60	15.9312	5.3670

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Grada.	Milliaria loxodromica.	Milliaria mecodynamica.	Serupala.	Milliaria loxodrom.	Milliaria mecodyn.	Serupala.	Milliaria loxodrom.	Milliaria mecodyn.
1	16.2359	6.2132	1	2706	1035	31	8.3885	3.2101
2	32.4717	12.4264	2	5411	2071	32	8.6591	3.3137
3	48.7076	18.6396	3	8117	3107	33	8.9297	3.4172
4	64.9435	24.8528	4	1.0239	4142	34	9.2003	3.5208
5	81.1794	31.0660	5	1.3230	5178	35	9.4409	3.6243
6	97.4153	37.2792	6	1.6236	6213	36	9.7415	3.7279
7	113.6516	43.4924	7	1.8942	7249	37	10.0121	3.8314
8	129.8870	49.7056	8	2.1648	8284	38	10.2827	3.9350
9	146.1229	55.9188	9	2.4354	9320	39	10.5533	4.0385
10	162.3588	62.1320	10	2.7059	1.0355	40	10.8239	4.1421
20	324.7176	124.2640	11	2.9765	1.1390	41	11.0945	4.2456
30	487.0764	186.3959	12	3.2471	1.2426	42	11.3651	4.3492
40	649.4352	248.5279	13	3.5177	1.3461	43	11.6357	4.4527
50	811.7940	310.6599	14	3.7883	1.4497	44	11.9063	4.5563
60	974.1528	372.7919	15	4.0589	1.5532	45	12.1769	4.6598
70	1136.5116	434.9239	16	4.3295	1.6568	46	12.4475	4.7634
80	1298.8704	497.0558	17	4.6001	1.7603	47	12.7181	4.8669
			18	4.8707	1.8639	48	12.9887	4.9705
			19	5.1413	1.9674	49	13.2593	5.0740
			20	5.4119	2.0710	50	13.5299	5.1776
			21	5.6825	2.1745	51	13.8005	5.2811
			22	5.9531	2.2781	52	14.0711	5.3847
			23	6.2237	2.3816	53	14.3417	5.4882
			24	6.4943	2.4851	54	14.6123	5.5918
			25	6.7649	2.5887	55	14.8829	5.6953
			26	7.0355	2.6923	56	15.1535	5.7989
			27	7.3061	2.7958	57	15.4241	5.9024
			28	7.5767	2.8993	58	15.6947	6.0060
			29	7.8473	3.0029	59	15.9653	6.1095
			30	8.1179	3.1066	60	16.2359	6.2131

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Gr. duc.	Milliaria loxodromica.	Milliaria mec-dynamica.	Scru- pula.	Milliaria loxodrom.	Milliaria mecodyn.	Scru- pula.	Milliaria loxodrom.	Milliaria mecodyn.
1	16.5925	7.0715	1	2765	1182	31	8.5727	3.6654
2	33.1810	14.1889	2	5530	2365	32	8.8493	3.7837
3	49.7775	21.2834	3	8296	3547	33	9.1258	3.9019
4	66.3700	28.3779	4	1.1061	4729	34	9.4024	4.0202
5	82.9626	35.4724	5	1.3827	5912	35	9.6789	4.1384
6	99.5551	42.5668	6	1.6592	7094	36	9.9555	4.2566
7	116.1476	49.6613	7	1.9357	8276	37	10.2320	4.3749
8	132.7401	56.7558	8	2.2123	9459	38	10.5085	4.4931
9	149.3326	63.8502	9	2.4888	1.0641	39	10.7851	4.6114
10	165.9251	70.9447	10	2.7654	1.1824	40	11.0616	4.7296
20	331.8502	141.8894	11	3.0419	1.3006	41	11.3382	4.8478
30	497.7754	212.8341	12	3.3185	1.4188	42	11.6147	4.9661
40	663.7006	283.7789	13	3.5950	1.5371	43	11.8913	5.0843
50	829.6257	354.7236	14	3.8715	1.6553	44	12.1678	5.2026
60	995.5508	425.6683	15	4.1481	1.7736	45	12.4443	5.3208
70	1161.4760	496.6130	16	4.4246	1.8918	46	12.7209	5.4390
80	1327.4011	567.5578	17	4.7012	2.0101	47	12.9974	5.5573
			18	4.9777	2.1283	48	13.2740	5.6755
			19	5.2542	2.2465	49	13.5505	5.7938
			20	5.5308	2.3648	50	13.8270	5.9120
			21	5.8073	2.4830	51	14.1036	6.0303
			22	6.0839	2.6013	52	14.3781	6.1485
			23	6.3604	2.7195	53	14.6547	6.2667
			24	6.6370	2.8377	54	14.9312	6.3850
			25	6.9135	2.9560	55	15.2078	6.5032
			26	7.1900	3.0742	56	15.4843	6.6215
			27	7.4666	3.1925	57	15.7608	6.7397
			28	7.7431	3.3107	58	16.0374	6.8579
			29	8.0187	3.4289	59	16.3139	6.9762
			30	8.2962	3.5472	60	16.5905	7.0944

24

Gradus.	Miliaria loxodromica.	Miliaria meodynamica.	Scrupula.	Miliaria loxodrom.	Miliaria meodyna.	Scrupula.	Miliaria loxodrom.	Miliaria meodyna.
1	17.0083	8.0177	1	2835	1336	31	8.7876	4.1424
2	34.0166	16.0353	2	5569	2627	32	9.0711	4.2761
3	51.0249	24.0530	3	8504	4009	33	9.3545	4.4097
4	68.0333	32.0706	4	1.1339	5345	34	9.6380	4.5433
5	85.0416	40.0883	5	1.4174	6681	35	9.9215	4.6770
6	102.0499	48.1060	6	1.7008	8018	36	10.2049	4.8106
7	119.0582	56.1236	7	1.9843	9354	37	10.4884	4.9442
8	136.0666	64.1413	8	2.2678	1.0690	38	10.7719	5.0778
9	153.0749	72.1589	9	2.5513	1.2026	39	11.0554	5.2115
10	170.0832	80.1766	10	2.8347	1.3362	40	11.3389	5.3451
20	340.1664	160.3532	11	3.1182	1.4699	41	11.6223	5.4787
30	510.2496	240.5299	12	3.4016	1.6035	42	11.9058	5.6123
40	680.3328	320.7065	13	3.6851	1.7371	43	12.1892	5.7460
50	850.4160	400.8831	14	3.9686	1.8708	44	12.4727	5.8796
60	1020.4992	481.0597	15	4.2520	2.0044	45	12.7562	6.0133
70	1190.5824	561.2363	16	4.5355	2.1380	46	13.0397	6.1469
80	1360.6656	641.4130	17	4.8190	2.2718	47	13.3231	6.2805
			18	5.1024	2.4053	48	13.6066	6.4141
			19	5.3859	2.5389	49	13.8901	6.5477
			20	5.6694	2.6725	50	14.1735	6.6814
			21	5.9529	2.8061	51	14.4570	6.8150
			22	6.2363	2.9398	52	14.7405	6.9486
			23	6.5198	3.0734	53	15.0239	7.0822
			24	6.8033	3.2070	54	15.3073	7.2159
			25	7.0867	3.3407	55	15.5908	7.3495
			26	7.3702	3.4743	56	15.8743	7.4831
			27	7.6537	3.6079	57	16.1577	7.6168
			28	7.9372	3.7415	58	16.4412	7.7504
			29	8.2206	3.8752	59	16.7247	7.8840
			30	8.5041	4.0088	60	17.0083	8.0176

2 $\frac{1}{2}$

Gradus.	Militaria loxodromica.	Militaria meodynamica.	Scrupula.	Militaria loxodrom.	Militaria meodyna.	Scrupula.	Militaria loxodrom.	Militaria meodyna.
1	17.4880	8.9966	1	2914	1498	31	9.0354	4.6151
2	34.9761	17.9813	2	5829	2996	32	9.3269	4.7950
3	52.4641	26.9719	3	8744	4495	33	9.6184	4.9448
4	69.9522	35.9626	4	1.1658	5993	34	9.9098	5.0947
5	87.4402	44.9533	5	1.4573	7492	35	10.2013	5.2445
6	104.9283	53.9439	6	1.7488	8990	36	10.4828	5.3943
7	122.4163	62.9346	7	2.0402	1.0489	37	10.7842	5.5442
8	139.9044	71.9252	8	2.3317	1.1987	38	11.0757	5.6940
9	157.3924	80.9159	9	2.6232	1.3485	39	11.3672	5.8439
10	174.8805	89.9065	10	2.9146	1.4984	40	11.6587	5.9937
20	349.7610	179.8130	11	3.2061	1.6482	41	11.9501	6.1436
30	524.6415	269.7195	12	3.4976	1.7981	42	12.2416	6.2934
40	699.5220	359.6261	13	3.7890	1.9479	43	12.5331	6.4433
50	874.4025	449.5326	14	4.0805	2.0978	44	12.8245	6.5931
60	1049.2830	539.4391	15	4.3720	2.2476	45	13.1160	6.7429
70	1224.1635	629.3456	16	4.6634	2.3975	46	13.4075	6.8929
80	1399.0440	719.2522	17	4.9549	2.5473	47	13.6989	7.0426
			18	5.2464	2.6971	48	13.9904	7.1925
			19	5.5378	2.8470	49	14.2819	7.3423
			20	5.8293	2.9968	50	14.5733	7.4922
			21	6.1208	3.1467	51	14.8648	7.6420
			22	6.4122	3.2965	52	15.1563	7.7919
			23	6.7037	3.4464	53	15.4478	7.9417
			24	6.9952	3.5962	54	15.7392	8.0915
			25	7.2866	3.7461	55	16.0307	8.2414
			26	7.5781	3.8959	56	16.3221	8.3912
			27	7.8696	4.0457	57	16.6136	8.5411
			28	8.1610	4.1956	58	16.9051	8.6909
			29	8.4525	4.3454	59	17.1965	8.8408
			30	8.7440	4.4953	60	17.4880	8.9906

3

Gr- dus.	Milliaria loxodromica.	Milliaria meodynamica.	Scru- pula.	Milliaria loxodrom.	Milliaria meodyna.	Scru- pula.	Milliaria loxodrom.	Milliaria meodyna.
1	18.0403	10.0227	1	3007	1670	31	9.308	5.1783
2	36.0806	20.0454	2	6013	3341	32	9.6215	5.3454
3	54.1209	30.0680	3	9020	5011	33	9.9221	5.5124
4	72.1612	40.0907	4	1.2027	6682	34	10.2228	5.6795
5	90.2015	50.1134	5	1.5034	8352	35	10.5235	5.8465
6	108.2418	60.1361	6	1.8040	1.0025	36	10.8241	6.0135
7	126.2822	70.1587	7	2.1047	1.1693	37	11.1248	6.1806
8	144.3225	80.1814	8	2.4054	1.3364	38	11.4255	6.3476
9	162.3628	90.2041	9	2.7060	1.5034	39	11.7261	6.5147
10	180.4031	100.2268	10	3.0067	1.6704	40	12.0268	6.6817
20	360.8063	200.4535	11	3.3073	1.8374	41	12.3275	6.8488
30	541.2094	300.6803	12	3.6080	2.0045	42	12.6282	7.0158
40	721.6126	400.9070	13	3.9087	2.1715	43	12.9288	7.1829
50	902.0157	501.1338	14	4.2294	2.3386	44	13.2295	7.3499
60	1082.4188	601.3606	15	4.5100	2.5056	45	13.5301	7.5169
70	1262.8220	701.5873	16	4.8107	2.6726	46	13.8308	7.6840
80	1443.2251	801.8141	17	5.1114	2.8397	47	14.1315	7.8510
			18	5.4120	3.0067	48	14.4322	8.0181
			19	5.7127	3.1738	49	14.7328	8.1851
			20	6.0134	3.3408	50	15.0336	8.3522
			21	6.3141	3.5079	51	15.3343	8.5192
			22	6.6147	3.6749	52	15.6349	8.6863
			23	6.9154	3.8420	53	15.9356	8.8533
			24	7.2161	4.0090	54	16.2363	9.0203
			25	7.5167	4.1760	55	16.5369	9.1871
			26	7.8174	4.3431	56	16.8376	9.3541
			27	8.1181	4.5101	57	17.1383	9.5215
			28	8.4188	4.6772	58	17.4390	9.6885
			29	8.7194	4.8442	59	17.7396	9.8555
			30	9.0201	5.0113	60	18.0403	10.0226

3¹.

Gradus.	Milliaria loxodromica.	Milliaria meodynamica.	Scru- pula.	Milliaria loxodrom.	Milliaria meodyna.	Scru- pula.	Milliaria loxodrom.	Milliaria meodyna.
1	18.6751	11.1217	1	3112	1834	31	9.6488	5.7477
2	37.3501	22.2495	2	6225	3708	32	9.9600	5.9332
3	56.0254	33.3742	3	9837	5562	33	10.2713	6.1186
4	74.7005	44.4990	4	1.2450	7416	34	10.5825	6.3040
5	93.3756	55.6238	5	1.5562	9270	35	10.8938	6.4894
6	112.0507	66.7485	6	1.8675	1.1124	36	11.2050	6.6748
7	130.7258	77.8733	7	2.1787	1.2978	37	11.5363	6.8602
8	149.4009	88.9980	8	2.4900	1.4833	38	11.8275	7.0456
9	168.0761	100.1228	9	2.8012	1.6687	39	12.1388	7.2310
10	186.7512	111.2475	10	3.1125	1.8541	40	12.4500	7.4165
20	373.5024	222.4951	11	3.4237	2.0395	41	12.7613	7.7019
30	560.2536	333.7427	12	3.7350	2.2249	42	13.0725	7.7873
40	747.0048	444.9902	13	4.0462	2.4103	43	13.3838	7.9727
50	933.7560	556.2378	14	4.3575	2.5957	44	13.6950	8.1581
60	1120.5072	667.4853	15	4.6687	2.7811	45	14.0063	8.3435
70	1307.2584	778.7329	16	4.9800	2.9666	46	14.3175	8.5289
80	1494.0096	889.9805	17	5.2912	3.1520	47	14.6288	8.7143
			18	5.6025	3.3374	48	14.9400	8.8998
			19	5.9137	3.5228	49	15.2513	9.0852
			20	6.2250	3.7082	50	15.5626	9.2706
			21	6.5362	3.8936	51	15.8738	9.4560
			22	6.8475	4.0790	52	16.1851	9.6414
			23	7.1587	4.2644	53	16.4963	9.8268
			24	7.4700	4.4499	54	16.8076	10.0122
			25	7.7813	4.6353	55	17.1188	10.1976
			26	8.0925	4.8207	56	17.4301	10.3831
			27	8.4038	5.0061	57	17.7413	10.5685
			28	8.7150	5.1915	58	18.0526	10.7539
			29	9.0263	5.3769	59	18.3638	10.9393
			30	9.3375	5.5623	60	18.6751	11.1247

3¹

Gradu.	Milliaria loxodromica.	Milliaria meodysarica.	Scru- pula.	Milliaria loxodrom.	Milliaria meodys.	Scru- pula.	Milliaria loxodrom.	Milliaria meodys.
1	19.4046	12.3102	1	3234	2052	31	10.0257	6.3602
2	38.8093	24.6203	2	6468	4103	32	10.3491	6.5654
3	58.2139	36.9305	3	9702	6155	33	10.6725	6.7706
4	77.6186	49.2407	4	1.2936	8207	34	10.9959	6.9757
5	97.0233	61.5509	5	1.6170	1.0258	35	11.3193	7.1809
6	116.4279	73.8610	6	1.9405	1.2310	36	11.6427	7.3861
7	135.8326	86.1713	7	2.2639	1.4362	37	11.9662	7.5912
8	155.2372	98.4814	8	2.5873	1.6414	38	12.2896	7.7964
9	174.6419	110.7916	9	2.9107	1.8466	39	12.6130	8.0016
10	194.0465	123.1018	10	3.2341	2.0517	40	12.9364	8.2068
20	388.0931	246.2036	11	3.5575	2.2568	41	13.2598	8.4119
30	582.1396	369.3055	12	3.8809	2.4620	42	13.5836	8.6171
40	776.1862	492.4072	13	4.2043	2.6672	43	13.9066	8.8223
50	970.2327	615.5091	14	4.5277	2.8723	44	14.2300	9.0274
60	1164.2792	738.6109	15	4.8511	3.0775	45	14.5534	9.2326
70	1358.3258	861.7127	16	5.1745	3.2827	46	14.8769	9.4378
80	1552.3723	984.8145	17	5.4979	3.4878	47	15.2003	9.6429
			18	5.8213	3.6930	48	15.5237	9.8481
			19	6.1448	3.8982	49	15.8471	10.0533
			20	6.4682	4.1034	50	16.1705	10.2584
			21	6.7916	4.3085	51	16.4939	10.4636
			22	7.1150	4.5137	52	16.8173	10.6688
			23	7.4384	4.7189	53	17.1407	10.8739
			24	7.7618	4.9240	54	17.4621	11.0791
			25	8.0852	5.1292	55	17.7875	11.2843
			26	8.4086	5.3344	56	18.1110	11.4895
			27	8.7320	5.5395	57	18.4344	11.6946
			28	9.0555	5.7447	58	18.7578	11.8998
			29	9.3789	5.9499	59	19.0812	12.1050
			30	9.7023	6.1551	60	19.4046	12.3101

Vol. IV.

K k

Gradus	Milliaria loxodromica.	Milliaria mecodynamica.	Serupula.	Milliaria loxodrom.	Milliaria mecodyn.	Serupula.	Milliaria loxodrom.	Milliaria mecodyn.
1	20.2442	13.5952	1	3374	2265	31	1.0459	7.0241
2	40.4885	27.1904	2	6748	4531	32	10.7969	7.2507
3	60.7327	40.7856	3	1.0122	6797	33	11.1343	7.4773
4	80.9770	54.3808	4	1.3496	9063	34	11.4717	7.7039
5	101.2212	67.9760	5	1.6870	1.1329	35	11.8091	7.9305
6	121.4655	81.5712	6	2.0244	1.3595	36	12.1465	8.1571
7	141.7097	95.1664	7	2.3618	1.5861	37	12.4839	8.3837
8	161.9540	108.7616	8	2.6992	1.8126	38	12.8213	8.6102
9	182.1982	122.3569	9	3.0366	2.0392	39	13.1587	8.8368
10	202.4425	135.9521	10	3.3740	2.2658	40	13.4961	9.0634
20	404.8849	271.9041	11	3.7114	2.4924	41	13.8335	9.2900
30	607.3274	407.8562	12	4.0488	2.7190	42	14.1709	9.5166
40	809.7698	543.8083	13	4.3862	2.9456	43	14.5083	9.7432
50	1012.2123	679.7604	14	4.7236	3.1722	44	14.8457	9.9698
60	1214.6548	815.7121	15	5.0610	3.3988	45	15.1831	10.1964
70	1417.0972	951.6646	16	5.3984	3.6253	46	15.5205	10.4229
80	1619.5397	1087.6166	17	5.7358	3.8519	47	15.8579	10.6495
			18	6.0732	4.0785	48	16.1953	10.8761
			19	6.4106	4.3051	49	16.5328	11.1027
			20	6.7480	4.5317	50	16.8702	11.3293
			21	7.0854	4.7583	51	17.2076	11.5559
			22	7.4228	4.9849	52	17.5450	11.7825
			23	7.7602	5.2114	53	17.8824	12.0091
			24	8.0976	5.4380	54	18.2198	12.2357
			25	8.4351	5.6646	55	18.5572	12.4623
			26	8.7725	5.8912	56	18.8946	12.6888
			27	9.1099	6.1178	57	19.2320	12.9154
			28	9.4473	6.3444	58	19.5694	13.1420
			29	9.7847	6.5710	59	19.9068	13.3686
			30	10.1221	6.7976	60	20.2442	13.5952

4

Grados.	Miliaria loxodromica.	Miliaria mecdynamica.	Serapala.	Miliaria loxodrom.	Miliaria mecdyn.	Serapala.	Miliaria loxodrom.	Miliaria mecdyn.
1	21.2132	15.0000	1	8535	2500	31	10.9600	7.7500
2	42.4264	30.0000	2	7071	5000	32	11.3136	8.0000
3	63.6396	45.0000	3	1.0607	7500	33	11.6671	8.2500
4	84.8528	60.0000	4	1.4142	1.0000	34	12.0207	8.5000
5	106.0660	75.0000	5	1.7678	1.2500	35	12.3742	8.7500
6	127.2792	90.0000	6	2.1213	1.5000	36	12.7278	9.0000
7	148.4924	105.0000	7	2.4749	1.7500	37	13.0813	9.2500
8	169.7056	120.0000	8	2.8284	2.0000	38	13.4349	9.5000
9	190.9188	135.0000	9	3.1820	2.2500	39	13.7884	9.7500
10	212.1320	150.0000	10	3.5355	2.5000	40	14.1420	10.0000
20	424.2641	300.0000	11	3.8890	2.7500	41	14.4955	10.2500
30	636.3961	450.0000	12	4.2426	3.0000	42	14.8491	10.5000
40	848.5282	600.0000	13	4.5961	3.2500	43	15.2026	10.7500
50	1060.6602	750.0000	14	4.9497	3.5000	44	15.5562	11.0000
60	1272.7922	900.0000	15	5.3032	3.7500	45	15.9097	11.2500
70	1484.9243	1050.0000	16	5.6568	4.0000	46	16.2633	11.5000
80	1697.0563	1200.0000	17	6.0103	4.2500	47	16.6168	11.7500
			18	6.3639	4.5000	48	16.9704	12.0000
			19	6.7174	4.7500	49	17.3239	12.2500
			20	7.0710	5.0000	50	17.6776	12.5000
			21	7.4245	5.2500	51	18.0312	12.7500
			22	7.7781	5.5000	52	18.3847	13.0000
			23	8.1316	5.7500	53	18.7383	13.2500
			24	8.4852	6.0000	54	19.0918	13.5000
			25	8.8387	6.2500	55	19.4454	13.7500
			26	9.1923	6.5000	56	19.7994	14.0000
			27	9.5458	6.7500	57	20.1530	14.2500
			28	9.8994	7.0000	58	20.5065	14.5000
			29	10.2529	7.2500	59	20.8601	14.7500
			30	10.6065	7.5000	60	21.2132	15.0000

K k 2

4¹.

Gradus.	Millaria loxodromica.	Millaria mecodynamica.	Scrupula.	Millaria loxodrom.	Millaria mecodyn.	Scrupula.	Millaria loxodrom.	Millaria mecodyn.
1	22.9361	16.5500	1	37.23	2758	31	11.5403	8.5508
2	44.6721	33.0999	2	74.15	5517	32	11.9126	8.8266
3	67.0082	49.6498	3	1.1168	8275	33	12.2848	9.1024
4	89.3443	66.1998	4	1.4890	1.1033	34	12.6571	9.3783
5	111.6804	82.7497	5	1.8618	1.3792	35	13.0293	9.6541
6	134.0165	99.2997	6	2.2336	1.6530	36	13.4016	9.9300
7	156.3526	115.8496	7	2.6059	1.9308	37	13.7739	10.2058
8	178.6887	132.3996	8	2.9781	2.2067	38	14.1462	10.4816
9	201.0248	148.9495	9	3.3504	2.4825	39	14.5184	10.7575
10	223.361	165.4995	10	3.7226	2.7583	40	14.8907	11.0333
20	446.7237	330.9990	11	4.0949	3.0341	41	15.2630	11.3091
30	670.0826	496.4985	12	4.4672	3.3100	42	15.6352	11.5849
40	893.4434	661.9980	13	4.8395	3.5858	43	16.0075	11.8608
50	1116.8043	827.4975	14	5.2117	3.8617	44	16.3798	12.1366
60	1340.1651	992.9970	15	5.5840	4.1375	45	16.7520	12.4124
70	1563.5260	1158.4965	16	5.9563	4.4133	46	17.1243	12.6883
80	1786.8868	1323.9960	17	6.3285	4.6891	47	17.4966	12.9641
			18	6.7008	4.9650	48	17.8688	13.2400
			19	7.0731	5.2408	49	18.2411	13.5158
			20	7.4453	5.5167	50	18.6134	13.7916
			21	7.8176	5.7925	51	18.9857	14.0674
			22	8.1899	6.0683	52	19.3579	14.3433
			23	8.5622	6.3441	53	19.7302	14.6191
			24	8.9344	6.6200	54	20.1024	14.8950
			25	9.3067	6.8958	55	20.4747	15.1708
			26	9.6790	7.1716	56	20.8470	15.4466
			27	10.0512	7.4475	57	21.2193	15.7224
			28	10.4235	7.7233	58	21.5915	15.9983
			29	10.7958	7.9991	59	21.9638	16.2741
			30	11.1680	8.2750	60	22.3361	16.5499

4⁺

Gradus.	Milliaria loxodromica.	Milliaria meodynamica.	Scrupula	Milliaria loxodrom.	Milliaria meodyna.	Scrupula	Milliaria loxodrom.	Milliaria meodyna.
1	23.6416	18.2775	1	3941	3046	31	12.2163	9.4834
2	47.2893	36.5551	2	7881	6092	32	12.6103	9.7480
3	70.9339	54.8327	3	1.1822	9139	33	13.0044	10.0527
4	94.5785	73.1102	4	1.5763	1.2185	34	13.3985	10.3573
5	118.2232	91.3878	5	1.9704	1.5231	35	13.7925	10.6519
6	141.8678	109.6653	6	2.3644	1.8277	36	14.1866	10.9665
7	165.5124	127.9429	7	2.7585	2.1323	37	14.5807	11.2712
8	189.1570	146.2204	8	3.1526	2.4369	38	14.9748	11.5758
9	212.8017	164.4980	9	3.5467	2.7415	39	15.3688	11.8801
10	236.4463	182.7755	10	3.9407	3.0462	40	15.7630	12.1851
20	472.8926	365.5511	11	4.3348	3.3509	41	16.1570	12.4896
30	709.3390	548.3266	12	4.7289	3.6555	42	16.5511	12.7943
40	945.7853	731.1022	13	5.1229	3.9601	43	16.9451	13.0989
50	1182.2316	913.8777	14	5.5170	4.2648	44	17.3392	13.5035
60	1418.6779	1096.6532	15	5.9111	4.5694	45	17.7333	13.7082
70	1655.1242	1279.4288	16	6.3051	4.8740	46	18.1274	14.0128
80	1891.5710	1462.2043	17	6.6992	5.1787	47	18.5214	14.3174
			18	7.0933	5.4833	48	18.9155	14.6221
			19	7.4833	5.7879	49	19.3096	14.9267
			20	7.8814	6.0925	50	19.7037	15.2313
			21	8.2755	6.3971	51	20.0978	15.5359
			22	8.6696	6.7018	52	20.4918	15.8405
			23	9.0636	7.0064	53	20.8859	16.1452
			24	9.4577	7.3110	54	21.2800	16.4498
			25	9.8518	7.6157	55	21.6740	16.7544
			26	10.2458	7.9203	56	22.0681	17.0591
			27	10.6399	8.2249	57	22.4622	17.3637
			28	11.0331	8.5296	58	22.8562	17.6683
			29	11.4280	8.8342	59	23.2503	17.9729
			30	12.8222	9.1388	60	23.6446	18.3776

4 $\frac{1}{2}$

Gradus.	Miliaria loxodromica.	Miliaria meodynamica.	Scru- pula.	Miliaria loxodrom.	Miliaria meodyna.	Scru- pula.	Miliaria loxodrom.	Miliaria meodyna.
1	25.1801	20.2252	1	4196	3370	31	13.0099	10.4496
2	50.3610	40.4503	2	8393	6741	32	13.4295	10.7867
3	75.5414	60.6755	3	1.2590	1.0112	33	13.8492	11.1238
4	100.7220	80.9006	4	1.6786	1.3483	34	14.2689	11.4609
5	125.9024	101.1258	5	2.0983	1.6834	35	14.6886	11.7980
6	151.0829	121.3509	6	2.5180	2.0225	36	15.1082	12.1350
7	176.2634	141.5761	7	2.9377	2.3596	37	15.5279	12.4721
8	201.4439	161.8013	8	3.3573	2.6966	38	15.9476	12.8092
9	226.6244	182.0264	9	3.7770	3.0337	39	16.3673	13.1463
10	251.8049	202.2516	10	4.1967	3.3708	40	16.7869	13.4834
20	503.6097	404.5032	11	4.6164	3.7079	41	17.2066	13.8205
30	755.4146	606.7548	12	5.0360	4.0450	42	17.6263	14.1576
40	1007.2195	809.0064	13	5.4557	4.3821	43	18.0460	14.4946
50	1259.0244	1011.2580	14	5.8754	4.7192	44	18.4656	14.8317
60	1510.8293	1213.5096	15	6.2951	5.0562	45	18.8853	15.1688
70	1762.6341	1415.7612	16	6.7147	5.3933	46	19.3050	15.5059
80	2014.4390	1618.0128	17	7.1344	5.7304	47	19.7247	15.8430
			18	7.5541	6.0675	48	20.1443	16.1801
			19	7.9738	6.4046	49	20.5640	16.5172
			20	8.3934	6.7417	50	20.9837	16.8543
			21	8.8131	7.0788	51	21.4034	17.1913
			22	9.2328	7.4158	52	21.8230	17.5284
			23	9.6525	7.7529	53	22.2427	17.8655
			24	10.0721	8.0900	54	22.6624	18.2026
			25	10.4918	8.4271	55	23.0821	18.5397
			26	10.9115	8.7642	56	23.5017	18.8768
			27	11.3312	9.1013	57	23.9214	19.2139
			28	11.7508	9.4384	58	24.3411	19.5509
			29	12.1705	9.7754	59	24.7608	19.8880
			30	12.5902	10.1125	60	25.1804	20.2251

5

Grados.	Milliaria loxodromica.	Milliaria mecodynamica.	Scrupula.	Milliaria loxodrom.	Milliaria mecodyna.	Scrupula.	Milliaria loxodrom.	Milliaria mecodyna.
1	26.9993	22.4191	1	4500	3741	31	13.9196	11.5986
2	58.9987	44.8982	2	9000	7483	32	14.3996	11.9728
3	80.9978	67.3473	3	1.3500	1.1224	33	14.8196	12.3469
4	107.9971	89.7963	4	1.8000	1.4966	34	15.2996	12.7411
5	134.9964	112.2454	5	2.2499	1.8708	35	15.7495	13.0952
6	161.9957	134.6945	6	2.6999	2.2449	36	16.1995	13.4694
7	188.9950	157.1436	7	3.1499	2.6190	37	16.6495	13.8435
8	215.9943	179.5927	8	3.5999	2.9932	38	17.0995	14.2177
9	242.9936	202.0417	9	4.0499	3.3673	39	17.5495	14.5919
10	269.9927	224.4908	10	4.4998	3.7415	40	17.9995	14.9660
20	539.9857	448.9817	11	4.9498	4.1156	41	18.4495	15.3402
30	809.9786	673.4725	12	5.3998	4.4898	42	18.8995	15.7143
40	1079.9714	897.9633	13	5.8198	4.8639	43	19.3494	16.0885
50	1349.9643	1122.4542	14	6.2998	5.2381	44	19.7994	16.4626
60	1619.9571	1346.9450	15	6.7498	5.6122	45	20.2494	16.8368
70	1889.9500	1571.4359	16	7.1998	5.9864	46	20.6994	17.2109
80	2159.9429	1795.9267	17	7.6498	6.3605	47	21.1494	17.5851
			18	8.0998	6.7347	48	21.5994	17.9592
			19	8.5497	7.1088	49	22.0494	18.3334
			20	8.9997	7.4830	50	22.4994	18.7075
			21	9.4497	7.8571	51	22.9494	19.0817
			22	9.8997	8.2313	52	23.3994	19.4558
			23	10.3497	8.6054	53	23.8493	19.8300
			24	10.7997	8.9796	54	24.2993	20.2041
			25	11.2497	9.3537	55	24.7493	20.5783
			26	11.6997	9.7279	56	25.1993	20.9524
			27	12.1496	10.1020	57	25.6493	21.3266
			28	12.5996	10.4762	58	26.0993	21.7007
			29	13.0496	10.8503	59	26.5493	22.0749
			30	13.4996	11.2245	60	26.9993	22.4490

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Gradus.	Miliaria loxodromica.	Miliaria meridionica.	Scru- pula.	Miliaria loxodrom.	Miliaria meridyna.	Scru- pula.	Miliaria loxodrom.	Miliaria meridyna.
1	29.1720	25.0260	1	4862	4170	31	15.0748	12.9300
2	58.3511	50.0529	2	9725	8341	32	15.5610	13.3471
3	87.5312	75.0780	3	14588	12513	33	16.0473	13.7642
4	116.7082	100.1039	4	19451	16683	34	16.5336	14.1813
5	145.8852	125.1299	5	24314	20854	35	17.0199	14.5984
6	175.0623	150.1560	6	29177	25025	36	17.5062	15.0155
7	204.2393	175.1819	7	34039	29196	37	17.9925	15.4326
8	233.4164	200.2079	8	38902	33367	38	18.4787	15.8497
9	262.5934	225.2339	9	43765	37538	39	18.9650	16.2668
10	291.7705	250.2599	10	48628	41709	40	19.4513	16.6839
20	583.5409	500.5198	11	53491	45880	41	19.9376	17.1014
30	875.3114	750.7796	12	58354	50051	42	20.4239	17.5181
40	1167.0818	1001.0395	13	63216	54222	43	20.9102	17.9352
50	1458.8523	1251.2994	14	68079	58393	44	21.3965	18.3523
60	1750.6628	1501.5593	15	72942	62564	45	21.8827	18.7694
70	2042.3932	1751.8192	16	77805	66735	46	22.3690	19.1865
80	2334.1637	2002.0790	17	82668	70906	47	22.8553	19.6036
			18	87531	75077	48	23.3416	20.0207
			19	92394	79248	49	23.8279	20.4378
			20	97256	83419	50	24.3142	20.8549
			21	102119	87590	51	24.8004	21.2720
			22	106982	91761	52	25.2867	21.6891
			23	111845	95932	53	25.7730	22.1062
			24	116708	100103	54	26.2593	22.5233
			25	121571	104274	55	26.7456	22.9404
			26	126433	108445	56	27.2319	23.3575
			27	131296	112616	57	27.7181	23.7746
			28	136159	116787	58	28.2044	24.1917
			29	141022	120958	59	28.6907	24.6088
			30	145885	125129	60	29.1770	25.0259

5¹

Grad.	Miliaria loxodrom.	Miliaria merid. nautica.	Scru- pula	Miliaria loxodrom.	Miliaria merid. nautica.	Scru- pula	Miliaria loxodrom.	Miliaria merid. nautica.
1	31.8203	28.0830	1	5303	4680	31	16.1405	14.5095
2	63.6407	56.1660	2	10206	9361	32	16.9718	14.9776
3	95.4610	84.2490	3	15910	14011	33	17.5011	15.4456
4	127.2813	112.3320	4	21213	18722	34	18.0315	15.9137
5	159.1017	140.4150	5	26517	23402	35	18.5613	16.3817
6	190.9220	168.4980	6	31820	28083	36	19.0922	16.8498
7	222.7424	196.5811	7	37123	32763	37	19.6225	17.3178
8	254.5627	224.6641	8	42427	37444	38	20.1528	17.7859
9	286.3830	252.7471	9	47730	42114	39	20.6832	18.2539
10	318.2034	280.8301	10	53033	46805	40	21.2135	18.7220
20	636.4068	561.6602	11	58337	51485	41	21.7438	19.1900
30	954.6102	842.4903	12	63640	56660	42	22.2742	19.6581
40	1272.8136	1123.3204	13	68944	60846	43	22.8045	20.1261
50	1591.0170	1404.1505	14	74247	65527	44	23.3349	20.5942
60	1909.2203	1684.9806	15	79550	70207	45	23.8652	21.0622
70	2227.4238	1965.8107	16	84853	74888	46	24.3955	21.5303
80	2545.6272	2246.6408	17	90157	79568	47	24.9259	21.9983
			18	95461	84249	48	25.4562	22.4664
			19	100764	88929	49	25.9866	22.9344
			20	106067	94610	50	26.5169	23.4025
			21	111371	98490	51	27.0472	23.8705
			22	116674	102971	52	27.5776	24.3386
			23	121978	107651	53	28.1079	24.8066
			24	127281	112332	54	28.6383	25.2747
			25	132585	117012	55	29.1686	25.7427
			26	137888	121693	56	29.6989	26.2108
			27	143191	126373	57	30.2293	26.6788
			28	148497	131054	58	30.7596	27.1469
			29	153800	135734	59	31.2900	27.6149
			30	159101	140415	60	31.8203	28.0830

Vol. IV.

L 1

Gradus	Milliaria loxodromica.	Milliaria mecdynamica.	Scrupula.	Milliaria loxodrom.	Milliaria mecdyn.	Scrupula.	Milliaria loxodrom.	Milliaria mecdyn.
1	35.0832	31.7148	1	5817	5286	31	18.1263	16.3860
2	70.1664	63.4297	2	1.1694	1.0572	32	18.7110	16.9146
3	105.2496	95.1445	3	1.7541	1.5857	33	19.2957	17.4431
4	140.3328	126.8593	4	2.3389	2.1143	34	19.8804	17.9717
5	175.4120	158.5742	5	2.9236	2.6129	35	20.4652	18.5003
6	210.4992	190.2890	6	3.5083	3.1715	36	21.0499	19.0289
7	245.5824	222.0038	7	4.0930	3.7000	37	21.6346	19.5575
8	280.6656	253.7187	8	4.6777	4.2286	38	22.2193	20.0860
9	315.7488	285.4335	9	5.2624	4.7572	39	22.8041	20.6146
10	350.8319	317.1483	10	5.8472	5.2858	40	23.3888	21.1432
20	701.6639	634.2967	11	6.4319	5.8144	41	23.9735	21.6718
30	1052.4958	951.4451	12	7.0166	6.3429	42	24.5582	22.2004
40	1403.3277	1268.5934	13	7.6014	6.8715	43	25.1429	22.7289
50	1754.1597	1585.7418	14	8.1861	7.4001	44	25.7277	23.2575
60	2104.9916	1902.8901	15	8.7708	7.9287	45	26.3124	23.7861
70	2455.8236	2220.0385	16	9.3555	8.4573	46	26.8971	24.3147
80	2806.6555	2537.1869	17	9.9402	8.9859	47	27.4818	24.8433
			18	10.5249	9.5144	48	28.0665	25.3719
			19	11.1097	10.0434	49	28.6512	25.9004
			20	11.6944	10.5716	50	29.2360	26.4290
			21	12.2791	11.1002	51	29.8207	26.9576
			22	12.8638	11.6238	52	30.4054	27.4862
			23	13.4485	12.1573	53	30.9901	28.0148
			24	14.0333	12.6859	54	31.5749	28.5433
			25	14.6180	13.2145	55	32.1596	29.0719
			26	15.2027	13.7431	56	32.7443	29.6005
			27	15.7874	14.2712	57	33.3290	30.1290
			28	16.3721	14.8003	58	33.9137	30.6576
			29	16.9569	15.3288	59	34.4985	31.1862
			30	17.5416	15.8574	60	35.0832	31.7148

6

Gr- du.	Millaria loxodromic.	Millaria meodynamica.	Scru- pula.	Millaria loxodrom.	Millaria meodyn.	Scru- pula.	Millaria loxodrom.	Millaria meodyn.
1	39.1979	36.2131	1	6533	6035	31	20.2517	18.7101
2	78.3958	72.4263	2	1.3065	1.2071	32	20.9050	19.3137
3	117.5936	108.6395	3	1.9598	1.8106	33	21.5583	19.9172
4	156.7915	144.8527	4	2.6131	2.4142	34	22.2116	20.5208
5	195.9894	181.0658	5	3.2665	3.0178	35	22.8648	21.1243
6	235.1873	217.2790	6	3.9198	3.6213	36	23.5181	21.7279
7	274.3852	253.4922	7	4.5731	4.2248	37	24.1714	22.3314
8	313.5830	289.7053	8	5.2264	4.8284	38	24.8247	22.9350
9	352.7809	325.9185	9	5.8797	5.4318	39	25.4780	23.5385
10	391.9788	362.1317	10	6.5328	6.0355	40	26.1312	24.1420
20	783.9576	724.2634	11	7.1861	6.6391	41	26.7845	24.7455
30	1175.9364	1086.3951	12	7.8394	7.2426	42	27.4378	25.3491
40	1567.9152	1448.5268	13	8.4926	7.8462	43	28.0911	25.9526
50	1959.8940	1810.6585	14	9.1459	8.4497	44	28.7444	26.5562
60	2351.8728	2172.7902	15	9.7992	9.0533	45	29.3987	27.1597
70	2743.8516	2534.9219	16	10.4525	9.6568	46	30.0519	27.7633
80	3135.8304	2897.0536	17	11.1058	10.2603	47	30.7052	28.3668
			18	11.7590	10.8639	48	31.3585	28.9704
			19	12.4123	11.4675	49	32.0118	29.5739
			20	13.0656	12.0711	50	32.6651	30.1777
			21	13.7189	12.6746	51	33.3184	30.7812
			22	14.3722	13.2782	52	33.9716	31.3848
			23	15.0254	13.8817	53	34.6249	31.9883
			24	15.6787	14.4853	54	35.2782	32.5919
			25	16.3320	15.0888	55	35.9315	33.1954
			26	16.9853	15.6924	56	36.5847	33.7990
			27	17.6386	16.2959	57	37.2380	34.4025
			28	18.2919	16.8995	58	37.8918	35.0060
			29	18.9451	17.5030	59	38.5446	35.6096
			30	19.5984	18.1066	60	39.1978	36.2132

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Gra- des.	Milliaria loxodromica.	Milliaria meodysarmica.	Ser- pola.	Milliaria loxodrom.	Milliaria meodyn.	Ser- pola.	Milliaria loxodrom.	Milliaria meodyn.
1	44.5249	41.9222	1	7.121	6.987	31	23.0045	21.6398
2	89.0499	83.8444	2	1.4842	1.3974	32	23.7466	22.3585
3	133.5748	125.7666	3	2.2262	2.0961	33	24.4887	23.0572
4	178.0997	167.6888	4	2.9683	2.7948	34	25.2308	23.7559
5	222.6247	209.6110	5	3.7104	3.4935	35	25.9729	24.4546
6	267.1496	251.5331	6	4.4525	4.1922	36	26.7150	25.1533
7	311.6746	293.4553	7	5.1946	4.8909	37	27.4570	25.8520
8	356.1995	335.3775	8	5.9366	5.5896	38	28.1991	26.5507
9	400.7244	377.2997	9	6.6787	6.2883	39	28.9412	27.2494
10	445.2493	419.2219	10	7.4208	6.9870	40	29.6833	27.9481
20	890.4987	838.4438	11	8.1629	7.6857	41	30.4254	28.6468
30	1335.7481	1257.6658	12	8.9050	8.3844	42	31.1675	29.3455
40	1780.9975	1676.8877	13	9.6471	9.0831	43	31.9095	30.0442
50	2226.2469	2096.1096	14	10.3891	9.7818	44	32.6516	30.7429
60	2671.4963	2515.3315	15	11.1312	10.4805	45	33.3937	31.4416
70	3116.7456	2934.5534	16	11.8733	11.1792	46	34.1358	32.1403
80	3561.9950	3353.7754	17	12.6154	11.8779	47	34.8779	32.8390
			18	13.3575	12.5766	48	35.6200	33.5377
			19	14.0996	13.2754	49	36.3620	34.2364
			20	14.8416	13.9741	50	37.1041	34.9352
			21	15.5837	14.6727	51	37.8462	35.6339
			22	16.3258	15.3715	52	38.5883	36.3326
			23	17.0679	16.0702	53	39.3304	37.0313
			24	17.8100	16.7689	54	40.0724	37.7300
			25	18.5520	17.4676	55	40.8145	38.4287
			26	19.2941	18.1663	56	41.5566	39.1274
			27	20.0362	18.8650	57	42.2987	39.8261
			28	20.7783	19.5637	58	43.0408	40.5248
			29	21.5204	20.2624	59	43.7828	41.2235
			30	22.2625	20.9611	60	44.5249	41.9222

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Gradus.	Miliaria loxodromica.	Miliaria meodynamica.	Scru- pula	Miliaria loxodrom.	Miliaria meodyna.	Scru- pula	Miliaria loxodrom.	Miliaria meodyna.
1	51.6734	47.4484	1	8.012	8241	31	26.6978	25.5483
2	103.3468	98.8967	2	17.224	1.6483	32	27.5590	26.3725
3	155.0202	148.3451	3	25.837	2.4724	33	28.4203	27.1966
4	206.6936	197.7935	4	34.149	3.2965	34	29.2815	28.0207
5	258.3670	247.2418	5	43.062	4.1207	35	30.1427	28.8440
6	310.0405	296.6902	6	51.673	4.9418	36	31.0039	29.6690
7	361.7139	346.1386	7	60.285	5.7690	37	31.8651	30.4931
8	413.3873	395.5870	8	68.898	6.5931	38	32.7264	31.3173
9	465.0607	445.0353	9	77.510	7.4172	39	33.5876	32.1414
10	516.7341	494.4837	10	86.122	8.2413	40	34.4489	32.9655
20	1033.4682	988.9674	11	94.734	9.0655	41	35.3101	33.7897
30	1550.2023	1483.4511	12	103.346	9.8896	42	36.1713	34.6138
40	2066.9364	1977.9348	13	111.959	10.7138	43	37.0326	35.4380
50	2583.6705	2472.4185	14	120.571	11.5379	44	37.8938	36.2621
60	3100.4046	2966.9022	15	129.183	12.3620	45	38.7550	37.0862
70	3617.1387	3461.3859	16	137.795	13.1862	46	39.6162	37.9104
80	4133.8728	3955.8696	17	146.407	14.0103	47	40.4774	38.7346
			18	155.020	14.8345	48	41.3387	39.5587
			19	163.632	15.6586	49	42.1999	40.3828
			20	172.244	16.4827	50	43.0611	41.2069
			21	180.856	17.3069	51	43.9223	42.0311
			22	189.468	18.1310	52	44.7835	42.8552
			23	198.081	18.9552	53	45.6447	43.6794
			24	206.693	19.7793	54	46.5060	44.5035
			25	215.305	20.6034	55	47.3672	45.3276
			26	223.917	21.4276	56	48.2284	46.1518
			27	232.529	22.2517	57	49.0896	46.9759
			28	241.142	23.0759	58	49.9508	47.8001
			29	249.754	23.9000	59	50.8121	48.6242
			30	258.366	24.7241	60	51.6733	49.4483

Gradus.	Miliaria loxodromica.	Miliaria meccodynamica.	Scru- pula.	Miliaria loxodrom.	Miliaria meccodyna.	Scru- pula.	Miliaria loxodrom.	Miliaria meccodyna.
1	61.7334	59.8833	1	1.0289	9980	31	31.8956	30.9397
2	123.4668	119.7667	2	2.0578	1.9961	32	32.9245	31.9378
3	185.2003	159.6889	3	3.0867	2.9942	33	33.9534	32.9358
4	246.9337	239.5334	4	4.1156	3.9922	34	34.9823	33.9339
5	308.6672	299.4168	5	5.1441	4.9903	35	36.0112	34.9320
6	370.4006	359.3001	6	6.1733	5.9883	36	37.0400	35.9300
7	432.1340	419.1835	7	7.2022	6.9864	37	38.0689	36.9281
8	493.8674	479.0668	8	8.2311	7.9844	38	39.0978	37.9261
9	555.6009	538.9502	9	9.2600	8.9825	39	40.1267	38.9242
10	617.3343	598.8335	10	10.2889	9.9805	40	41.1556	39.9222
20	1234.6686	1197.6671	11	11.3178	10.9786	41	42.1845	40.9203
30	1852.0029	1596.8891	12	12.3467	11.9767	42	43.2134	41.9183
40	2469.3372	2395.3341	13	13.3756	12.9727	43	44.2423	42.9164
50	3086.6715	2994.1677	14	14.4045	13.9728	44	45.2712	43.9145
60	3704.0058	3593.0012	15	15.4333	14.9708	45	46.3001	44.9125
70	4321.3401	4191.8348	16	16.4622	15.9689	46	47.3290	45.9106
80	4938.6744	4790.6683	17	17.4911	16.9669	47	48.3578	46.9086
			18	18.5200	17.9650	48	49.3867	47.9067
			19	19.5489	18.9630	49	50.4156	48.9047
			20	20.5778	19.9611	50	51.4445	49.9028
			21	21.6067	20.9592	51	52.4734	50.9008
			22	22.6356	21.9572	52	53.5023	51.8989
			23	23.6645	22.9553	53	54.5312	52.8970
			24	24.6934	23.9533	54	55.5601	53.8950
			25	25.7223	24.9514	55	56.5890	54.8931
			26	26.7511	25.9494	56	57.6179	55.8911
			27	27.7800	26.9475	57	58.6467	56.8892
			28	28.8089	27.9456	58	59.6756	57.8872
			29	29.8378	28.9436	59	60.7045	58.8853
			30	30.8667	29.9417	60	61.7334	59.8833

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Ordus	Miliaria loxodromica.	Miliaria meodysamica	Seru- pala.	Miliaria loxodrom.	Miliaria meodyna.	Seru- pala.	Miliaria loxodrom.	Miliaria meodyna.
1	76.8875	75.4101	1	1.2814	1.2568	31	39.7252	38.9619
2	153.7749	150.8202	2	2.5629	2.5137	32	41.0066	40.2187
3	230.6624	226.2302	3	3.8444	3.7705	33	42.2881	41.4755
4	307.5498	301.6403	4	5.1258	5.0273	34	43.5695	42.7323
5	384.4373	377.0505	5	6.4073	6.2842	35	44.8510	43.9892
6	461.3248	452.4608	6	7.6887	7.5410	36	46.1325	45.2460
7	538.2122	527.8706	7	8.9702	8.7978	37	47.4139	46.5029
8	615.0997	603.2807	8	10.2517	10.0547	38	48.6954	47.7597
9	693.9871	678.6908	9	11.5331	11.3115	39	49.9768	49.0165
10	770.8746	754.1009	10	12.8145	12.5683	40	51.2583	50.2734
20	1537.7492	1508.2019	11	14.0960	13.8251	41	52.5398	51.5302
30	2306.6238	2262.3028	12	15.3775	15.0820	42	53.8212	52.7870
40	3075.4985	3016.4037	13	16.6589	16.3388	43	55.1027	54.0439
50	3844.3731	3770.5047	14	17.9404	17.5956	44	56.3841	55.3007
60	4613.2477	4524.6056	15	19.2218	18.8525	45	57.6656	56.5576
70	5382.1223	5278.7066	16	20.5033	20.1093	46	58.9471	57.8144
80	6150.9970	6032.8075	17	21.7848	21.3660	47	60.2285	59.0712
			18	23.0662	22.6230	48	61.5100	60.3280
			19	24.3477	23.8798	49	62.7914	61.5849
			20	25.6291	25.1367	50	64.0728	62.8417
			21	26.9106	26.3935	51	65.3543	64.0986
			22	28.1921	27.6504	52	66.6358	65.3554
			23	29.4735	28.9072	53	67.9173	66.6122
			24	30.7550	30.1640	54	69.1987	67.8691
			25	32.0364	31.4208	55	70.4801	69.1259
			26	33.3179	32.6777	56	71.7616	70.3827
			27	34.5993	33.9345	57	73.0431	71.6396
			28	35.8808	35.1913	58	74.3245	72.8964
			29	37.1622	36.4482	59	75.6060	74.1532
			30	38.4437	37.7050	60	76.8874	75.4101

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Gra- Juz.	Miliaria loxodromica.	Miliaria meodromica	Seru- pula.	Miliaria loxodrom.	Miliaria meodroma.	Seru- pula.	Miliaria loxodrom.	Miliaria meodroma.
1	1022.282	101.1218	1	1.7038	1.6853	31	52.8179	52.2162
2	201.1565	202.2436	2	3.4076	3.3707	32	51.5217	53.9316
3	306.6817	301.3653	3	5.1114	5.0561	33	56.2255	55.6170
4	408.9130	401.1871	4	6.8152	6.7415	34	57.9293	57.3023
5	511.1412	505.6089	5	8.5190	8.4268	35	59.6331	58.9877
6	613.3695	606.7307	6	10.2228	10.1122	36	61.3369	60.6731
7	715.5977	707.8525	7	11.9266	11.7975	37	63.0407	62.3581
8	817.8260	808.9743	8	13.6304	13.4829	38	64.7446	64.0488
9	920.0543	910.0962	9	15.3342	15.1683	39	66.4484	65.7292
10	1022.2825	1011.2179	10	17.0380	16.8536	40	68.1521	67.4145
20	2044.5650	2022.4357	11	18.7418	18.5390	41	69.8560	69.0999
30	3066.8475	3033.6536	12	20.4456	20.2244	42	71.5598	70.7852
40	4089.1301	4044.8714	13	22.1494	21.9097	43	73.2636	72.4706
50	5111.4126	5056.0893	14	23.8532	23.5951	44	74.9674	74.1560
60	6133.6951	6067.3072	15	25.5571	25.2804	45	76.6712	75.8413
70	7155.9776	7078.5250	16	27.2609	26.9658	46	78.3750	77.5267
80	8178.2603	8089.7430	17	28.9647	28.6512	47	80.0788	79.2121
			18	30.6685	30.3365	48	81.7826	80.8974
			19	32.3722	32.0219	49	83.4864	81.5828
			20	34.0760	33.7073	50	85.1902	84.2682
			21	35.7799	35.3926	51	86.8940	85.9535
			22	37.4837	37.0780	52	88.5978	87.6389
			23	39.1875	38.7633	53	90.3016	89.3242
			24	40.8913	40.4487	54	92.0054	91.0096
			25	42.5951	42.1341	55	93.7092	92.6950
			26	44.2989	43.8194	56	95.4130	94.3803
			27	46.0027	45.5048	57	97.1168	96.0657
			28	47.7065	47.1902	58	98.8206	97.7511
			29	49.4103	48.8755	59	100.5244	99.4364
			30	51.1141	50.5609	60	102.2282	101.1218

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Gradus.	Milliaria loxodromica.	Milliaria meodynamica.	Serupula.	Milliaria loxodrom.	Milliaria meodyn.	Serupula.	Milliaria loxodrom.	Milliaria meodyn.
1	153.0344	152.2975	1	2.5505	2.5383	31	79.0678	78.6870
2	806.0689	304.5950	2	5.1011	5.0766	32	81.6183	81.2253
8	459.1033	456.8926	3	7.6517	7.6149	33	84.1689	83.7636
4	612.1378	609.1901	4	10.2023	10.1582	34	86.7195	86.3019
5	765.1723	761.0877	5	12.7529	12.6914	35	89.2700	88.8402
6	918.2067	913.7852	6	15.3034	15.2297	36	91.8206	91.3785
7	1071.2412	1066.0828	7	17.1540	17.7680	37	94.3712	93.9168
8	1224.2756	1218.3803	8	20.4046	20.3063	38	96.9217	96.4550
9	1377.3101	1370.6778	9	22.9552	22.8447	39	99.4723	98.9933
10	1530.3446	1522.9753	10	25.5057	25.3829	40	102.0229	101.5316
20	8060.6891	8045.9507	11	28.0563	27.9212	41	104.5735	104.0699
30	4591.0337	4568.9261	12	30.6068	30.4895	42	107.1241	106.6082
40	6121.3783	6091.9015	13	33.1574	32.9977	43	109.6746	109.1465
50	7651.7229	7614.8796	14	35.7080	35.5360	44	112.2252	111.6848
60	9182.0675	9137.8523	15	38.2585	38.0743	45	114.7758	114.2231
70	10712.4121	10660.8276	16	40.8091	40.6126	46	117.3263	116.7614
80	12242.7566	12183.8030	17	43.3597	43.1509	47	119.8769	119.2997
			18	45.9103	45.6892	48	122.4275	121.8379
			19	48.4608	48.2275	49	124.9781	124.3762
			20	51.0114	50.7658	50	127.5287	126.9145
			21	53.5620	53.3041	51	130.0792	129.4528
			22	56.1126	55.8320	52	132.6298	131.9911
			23	58.6632	58.3807	53	135.1804	134.5294
			24	61.2137	60.9190	54	137.7309	137.0677
			25	63.7643	63.4573	55	140.2815	139.6060
			26	66.3149	65.9955	56	142.8321	142.1443
			27	68.8654	68.5338	57	145.3827	144.6826
			28	71.4160	71.0721	58	147.9332	147.2208
			29	73.9666	73.6104	59	150.4838	149.7591
			30	76.5172	76.1487	60	153.0344	152.2974

Vol. IV.

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Gradus	Miliaria loxodromica.	Miliaria meridiana.	Scru- pula.	Miliaria loxodrom.	Miliaria meridiana.	Scru- pula.	Miliaria loxodrom.	Miliaria meridiana.
1	153.0314	152.2975	1	2.5506	2.5382	31	79.0678	78.6871
2	306.0687	301.5951	2	5.1011	5.0766	32	81.6181	81.2253
3	459.1091	456.8926	3	7.6517	7.6149	33	84.1689	83.7636
4	612.1378	609.1901	4	10.2023	10.1532	34	86.7195	86.3019
5	765.1723	761.1877	5	12.7529	12.6915	35	89.2701	88.8402
6	918.2067	913.7825	6	15.3034	15.2297	36	91.8207	91.3785
7	1072.2412	1066.0828	7	17.8540	17.7680	37	94.3712	93.9168
8	1224.2756	1218.3803	8	20.4046	20.3061	38	96.9218	96.4551
9	1377.3101	1370.6778	9	22.9552	22.8446	39	99.4724	98.9934
10	1530.3447	1522.9751	10	25.5057	25.3829	40	102.0230	101.5317
20	3060.6891	3045.9507	11	28.0563	27.9212	41	104.5735	104.0700
30	4591.0337	4568.9261	12	30.6069	30.4595	42	107.1241	106.6083
40	6121.3783	6091.9015	13	33.1575	32.9978	43	109.6747	109.1466
50	7151.7229	7614.8769	14	35.7080	35.5361	44	112.2253	111.6849
60	9182.0675	9137.8523	15	38.2586	38.0744	45	114.7758	114.2231
70	10712.4120	10660.8277	16	40.8092	40.6127	46	117.3264	116.7614
80	12242.7566	12183.8030	17	43.3598	43.1510	47	119.8770	119.2997
			18	45.9103	45.6893	48	122.4276	121.8380
			19	48.4609	48.2275	49	124.9781	124.3763
			20	51.0115	50.7658	50	127.5287	126.9146
			21	53.5621	53.3041	51	130.0793	129.4529
			22	56.1126	55.8424	52	132.6299	131.9912
			23	58.6632	58.3807	53	135.1804	134.5295
			24	61.2138	60.9190	54	137.7310	137.0678
			25	63.7643	63.4573	55	140.2816	139.6061
			26	66.3149	65.9956	56	142.8322	142.1444
			27	68.8655	68.5339	57	145.3827	144.6827
			28	71.4161	71.0722	58	147.9333	147.2209
			29	73.9666	73.6105	59	150.4839	149.7592
			30	76.5172	76.1488	60	153.0345	152.2975

CLAUDIANUS DE MAGNETE.

QUISQUIS sollicitâ mundum ratione secutus
 Semina rimatur rerum, quo Luna laborat
 Defectu, quæ causa jubet pallefcere Solem,
 Unde rubefcentes ferali crine cometæ,
 Unde fluant venti, trepidæ quis viscera terræ
 Commoveat motus, quæ fulgura ducat origo,
 Unde tonent nubes, quo lumine floreât arcus :
 Hoc mihi quærenti, si quid deprendere veri
 Mens valet, expediat. Lapis est, cognomine Magnes,
 Decolor, obscurus, vilis ; non ille repexam
 Cæsariem Regum, nec candida virginis ornat
 Colla, nec insigni splendet per cingula morfu :
 Sed nova si nigri videas miracula saxi,
 Tunc superat pulchros cultus, et quidquid Eois
 Indus littoribus rubrà scrutatur in algâ.
 Ex ferro meruit vitam, ferrique rigore
 Vescitur : has dulces epulas, hæc pabula novit.
 Hinc proprias renovat vires : hinc fusa per artus
 Aspera secretum servant alimenta vigorem.
 Hoc absente perit ; tristi morientia torpent
 Membra fame, venâsque sitis consumit apertas.
 Mavors, sanguineâ qui cuspide verberat urbes,
 Et Venus, humanas quæ laxat in otia curas,
 Aurati delubra tenent communia templi,
 Effigies non una Deis : sed ferrea Martis
 Forma nitet, Venerem magnetica gemma figurat.
 Illis connubium celebrat de more sacerdos :
 Ducit flamma choros : festâ frondentia myrto
 Limina cinguntur, roseisque cubilia surgunt
 Floribus, et thalamos dotalis purpura velat.
 Hic mirum confurgit opus. Cytherea maritum
 Sponte rapit, cælique toros imitata priores
 Pectora lascivo statu Mavortia nectit,
 Et tantum suspendit onus, galeæque lacertos
 Implicat, et vivis totum complexibus ambit.

M m 2

Eile

Ille laceffitus longo spiraminis actu
 Arcanis trahitur gemmâ de conjuge nodis,
 Pronuba fit natura Deis, ferrûmque maritat
 Aura tenax, subitis sociantur Numina furis.
 Quis calor infundit geminis alterna metallis
 Fœdeta? quæ duras jungit concordia mentes?
 Flagrat anhela fîlex; et amicam faucia sentit
 Materiem, placidôsq; chalybs cognoscit amores.
 Sic Venus horrificum belli compescere Regem,
 Et vultum mollire solet, cum sanguine præceps
 Æstuat, et stridis mucronibus asperat iras,
 Sola feris occurrit equis, mollitque tumorem
 Pectoris, et blando præcordia temperat igne.
 Pax animo tranquilla datur, pugnaque calentes
 Deferit, et rutilas declinat in oscula cristas.
 Quæ tibi, sæve puer, non est permîssa potestas?
 Tu magnum superas fulmen, cælôque relicto
 Fluctibus in mediis cogis mugire Tonantem.
 Jam gelidas rupes, vivoque carentia sensu
 Membra feris: jam saxa tuis obnoxia telis,
 Et lapides suos ardor agit; ferrûmque tenetur
 Illecebris: rigido regnant in marmore flammæ.

F I N I S.

NOTA

NOTA ad hæc verba in paginâ 172, lineis 6tâ et 7timâ, scilicet, *Aut vel nunc jam per regulam falsarum positionum [licebit] ex inventis ad minutum usque propemodum definire, hoc modo.*

AUCTORIS ratiocinium in hac applicatione regulæ falsarum positionum ad inventionem numeri 0.60143 ex numeris 0.56817 et 0.61877, quæsitio numero ex utraq[ue] parte vicinis, seu quorum unus est minor, alter verò major, numero quæsitio, explicatione aliquâ mihi videtur indigere. Poterit verò explicari hoc modo.

Ponatur littera *a* pro numero 0.56817, quem scimus esse quæsitio minorem; et littera *b* pro numero 0.61877, quem scimus esse quæsitio majorem; et littera *x* pro ipso quæsitio. Erit igitur loxodromiæ, quæ ipsi *a*, seu 0.56817, respondet, longitudo milliarium $804 + \frac{7862}{10,000}$, seu 804.7862; et loxodromiæ numero quæsitio *x* respondentis longitudo milliarium 800; et loxodromiæ numero *b*, seu 0.61877, respondentis, longitudo milliarium $797 + \frac{5042}{10,000}$, seu 797.5042. Harum loxodromiarum prima excedit secundam 4.7862 milliariis, secunda autem tertiam milliariis 2.4959. Prior excessus vocetur *c*, secundus *d*.

His ita positis, jam, secundum auctoris præcepta, debemus multiplicare primum numerum 0.56817, seu *a*, in numerum quartum 2.4959, seu *d*, et multiplicare numerum secundum 0.61877, seu *b*, in numerum tertium 4.7862, seu *c*; unde fiunt, ex priore multiplicatione productum 1.418,095,503, seu *ad*, et ex secundâ multiplicatione productum 2.961,556,974, seu *bc*. Horum autem productorum secundum, scilicet, 2.961,556,975, seu *bc*, est addendum priori, scilicet, 1.418,095,503, seu *ad*; unde fit summa 4.379,652,477, sive *ad + bc*. Jam verò excessus 2.4959, seu *d*, est addendus excessui 4.7862, seu *c*; unde fit summa 7.2821, seu *c + d*. Et postremò, summa prior 4.379,652,477, seu *ad + bc*, per summam posteriorem 7.2821, seu *c + d*, est dividenda; unde fit quotiens 0.60143, seu fractio $\frac{ad + bc}{c + d}$, seu $\frac{ad + bc}{d + c}$; quæ, ut auctor asserit, erit, quàm-proximè, æqualis ipsi *x*, seu numero quæsitio.

Hujus autem conclusionis veritas, scilicet, "quod hæc fractio $\frac{ad + bc}{d + c}$ accedet propius ad quæsitum numerum *x* quàm una saltèm, si non quàm alterutra, priorum quantitatum *a* et *b* (inter quas quantitas *x* sita est,)" demonstrari potest hoc modo.

Quoniam

Quoniam quantitas prima, 0.56817, five a , est minor secundâ 0.61877, five b , erit quantitas $\frac{ad+ac}{d+c}$ minor ipsâ $\frac{ad+bc}{d+c}$, seu prædictâ quotiente, quam invenimus. Sed quantitas $\frac{ad+ac}{d+c}$ est æqualis ipsi a . Ergò quantitas a erit minor ipsâ $\frac{ad+bc}{d+c}$, quam invenimus; ideòque, si quantitas $\frac{ad+bc}{d+c}$ (quam invenimus,) est minor quæsitâ quantitate x , differentia inter ipsam et quantitatem x erit minor quàm differentia inter a et x ; id est, dicta quantitas $\frac{ad+bc}{d+c}$, quam invenimus, accedet propiùs quam a ad quantitatem quæsitam x .

Porò, quoniam quantitas a est minor ipsâ b , erit quantitas $\frac{ad+bc}{d+c}$ minor ipsâ $\frac{bd+bc}{d+c}$. Sed quantitas $\frac{bd+bc}{d+c}$ est æqualis ipsi b . Ergò quantitas $\frac{ad+bc}{d+c}$ (quam invenimus,) minor erit ipsâ b ; ideòque, si hæc quantitas $\frac{ad+bc}{d+c}$ (quam invenimus,) est major quæsitâ quantitate x , differentia inter ipsam et quantitatem x erit minor quàm differentia inter b et x ; id est, dicta quantitas $\frac{ad+bc}{d+c}$, quam invenimus, accedet propiùs quàm b ad quantitatem quæsitam x .

Neceffe est autem ut dicta quantitas $\frac{ad+bc}{d+c}$ sit aut ipsi quantitati quæsitæ x omninò æqualis, aut major ipsâ, aut minor. Si sit ipsi x omninò æqualis, habemus id quod quærimus, scilicet, veram magnitudinem ipsius x . Si verò dicta quantitas sit major ipsâ x , accedet propiùs quàm ipsa b ad eam; et, si dicta quantitas sit minor ipsâ x , accedet propiùs quàm ipsa a ad eam. Q. E. D.

Et plerumque dicta quantitas $\frac{ad+bc}{d+c}$, quam invenimus, accedet propiùs quam alterutra duarum quantitatum a et b ad quæsitam quantitatem x , licet id non in omnibus casibus necessarium esse videatur, aut demonstrari possit. Et in hoc ipso exemplo, (in quo a et b sunt æquales numeris 0.56817 et 0.61877, et loxodromiæ quæ correspondent ipsi numeris a , x , et b , seu 0.56817, x , et 0.61877, sunt 804.7862, 800, et 797.5042,) hæc quantitas $\frac{ad+bc}{d+c}$, seu 0.60143, tam propè ad veram magnitudinem quæsitæ numeri x accedit quòd longitudo loxodromiæ quæ ipsi respondet, seu ex ipsâ derivatur, invenitur esse 799.56 milliarius, quæ deficit à datæ loxodromiæ longitudine, scilicet, 800 milliariibus, tantùm 0.56, seu $\frac{56}{100}$ unius milliarii, seu parte paulo majore quàm dimidium unius milliarii, quæ in tantâ longitudine ferè pro nihilo haberi potest. Unde, è converso, concludi potest, differentiam inter x (quæ datæ loxodromiæ longitudini,

longitudini, nempe milliaribus 800, responder,) et quantitatem $\frac{ad+bc}{d+c}$, seu 0.60143, (quæ loxodromiæ, cujus longitudo est milliarius 799.56 responder,) esse etiam partem valde parvam quantitatis x , et proinde fere pro nihilo haberi posse.

Hæc autem quantitas $\frac{ad+bc}{d+c}$ breviter et faciliter derivari poterit ex hæc suppositione, nempe, quod, si sint tres quantitates a , x , et b , quæ inter se non multum differunt, prima autem sit aliquanto minor secundâ, et secunda minor tertiâ; et si sint tres aliæ quantitates e , f , et g , quæ à tribus prioribus a , x , et b , certo aliquo modo derivantur, et iisdem respectivè respondent, scilicet, quantitas e quantitati a , quantitas f quantitati x , et quantitas g quantitati b ; in his autem quantitativibus prima e sit aliquanto major secundâ f , et secunda f paulo major tertiâ, contrâ quàm sit in prioribus tribus a , x , et b ; hæc autem quantitates (sicuti et tres priores a , x , b ,) paulùm admodùm inter se differant; erit differentia primæ et secundæ trium priorum quantitatum, scilicet, $x - a$, ad differentiam secundæ et tertiæ earundem, scilicet, $b - x$, quam proximè in eadem ratione ac differentia primæ et secundæ trium posteriorum quantitatum, scilicet, $e - f$, ad differentiam secundæ et tertiæ earundem, scilicet, $f - g$.

Nam, si hæc suppositio sit vera, aut veritati proxima, ponamus e pro $e - f$, et d pro $f - g$, ut suprà in demonstratione præcedente. Et erit $x - a$ ad $b - x$ ut e ad d . Ergò $x - a \times d$ erit $= b - x \times e$, hoc est, $dx - ad$ erit $= be - ex$, et (addendo utrinque ex) erit $dx + ex - ad = be$, et (addendo utrinque ad) erit $dx + ex = ad + be$; hoc est, $x \times d + e$ erit $= ad + be$. Et proinde x erit $= \frac{ad+be}{d+e}$. Q. E. I.

A
C O M P A R I S O N
OF THE
DIFFERENT METHODS
OF
SOLVING THE PROBLEM IN NAVIGATION,
COMMONLY CALLED,
DOCTOR HALLEY'S PROBLEM;
WITH
A NEW METHOD FOR THE SAME PURPOSE:

BY
A N D R E W M A C K A Y,
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DOCTOR OF LAWS, AND FELLOW OF THE ROYAL SOCIETY OF EDINBURGH.

DOCTOR MACKAY'S *Comparison of the different Methods of Solving the Problem, commonly known by the Name of DOCTOR HALLEY'S Problem, with a Method drawn up by Himself for that Purpose.*

INTRODUCTION.

THIS Problem has exercised the skill of many eminent Mathematicians; and all the methods of Solution that have been applied to it, except one, are founded upon indirect principles. If a rhumb-line, or loxodromick curve, co incided with a great circle on the sphere, the solution would be both easy and accurate:—but the case is otherwise; and the difficulty of the Problem is so great, that no direct solution has ever been given of it, except that at the beginning of this Volume. And that Solution is so long and intricate, that few mariners will be found both able and willing to go through all the steps of it. But many indirect solutions have been given of this Problem, the greater part of which may be easily understood by the generality of masters of ships, and other persons acquainted with the practice of Navigation.

1. This Problem is very difficult to be solved by a direct method;

2. but the indirect methods may be easily understood.

It has been already observed, in page 114th of this Volume, that this Problem appears to have been first proposed in a book intituled, *A Learned Treatise of Globes, both Celestial and Terrestrial; with their several Uses. Written first in Latine, by Mr. ROBERT HUES; and by him so published. Afterward illustrated with Notes, by JOANNES ISAACUS PONTANUS; and now lately made English, for the benefit of the unlearned, by JOHN CHILMEAD, M. A. Christ-Church in Oxon. London, 1639.* This Problem is expressed in the above-mentioned book, page 181, as follows. *The difference of Longitude and Distance being given; how to find the Rumbe, and difference of Latitude.* After which the proposer adds the following words: *There is not any thing in all this Art more difficult and hard to be found, than the Rumbe out of the distance and difference of LONGITUDE given. Neither can it be done upon the GLOBE, without long and tedious practice, and many repetitions and menfurations.*

3. It was first proposed by Robert Hues.

The next who appears to have mentioned this Problem, is WILLEBRORDUS SNELLIUS, Professor of Mathematicks at Leyden, in his *Tiphys Batavus*, pages 50, 51, 52, and 53, where it constitutes the 34th and last proposition of book first of that learned work. SNELLIUS quotes the words of ROBERT HUES; but it must have been from an older edition than that above-mentioned, as his *Tiphys Batavus* was printed in the year 1624. SNELLIUS has solved this Problem in an indirect manner; and it seems probable that his solution was the first that ever was given of it. This Solution of SNELLIUS, together with all the rest of his *Tiphys Batavus*, has been reprinted in the foregoing part of this Volume.

4. Again proposed by Snellius,

5. and solved by him.

6.
Proposed the
third time by
Dr. Halley.

This Problem was afterwards proposed to the learned world by the celebrated Dr. EDMUND HALLEY, in the year 1695-6, in the nineteenth volume of the Philosophical Transactions, Number 219, in a very learned and ingenious Tract, intitled, *An easy Demonstration of the Analogy of the Logarithmick Tangent's to the Meridian Line, or Sum of the Secants; with various Methods for computing the sume to the utmost exactness*; which Tract was afterwards published a second time in the second volume of a Collection of Mathematical and Philosophical Tracts, intitled *Miscellanea Curiosa*, that was printed at London in the year 1708; and has within these few years been published a third time in the second volume of this present Collection of Tracts, called *Scriptores Logarithmici*. And from the manner in which Dr. HALLEY there speaks of this Problem, it seems probable that he either had not seen the aforesaid books of HUES and SNELLIUS, or had forgot that this Problem was mentioned in them. For, in the 35th page of the second volume of the *Miscellanea Curiosa*, he proposes it in the following words: *A Ship sails from a given Latitude, and, having run a certain number of Leagues, has altered her Longitude by a given angle; it is required to find the course steered.* And he then adds — *The Solution thereof would be very acceptable, if not to the Publick, at least to the Author of this Tract, being likely to open some further light into the Mysteries of Geometry.*

7.
Since that
time solved
in an indirect
manner by se-
veral authors.

8.
And lately
in a direct
manner by
Baron Maseres
and Mr. At-
wood.

Since that time, this Problem has been solved in an indirect manner, by several writers on Navigation, and others:—As Monsieur BOUGUER in his *Nouveau Traité de Navigation*; Mr. ROBERTSON, in the second volume of his *Elements of Navigation*; Mr. EMERSON, in his *Theory of Navigation*, which accompanies his *Mathematical Principles of Geography*; Mr. ISRAEL LYONS, in the *Nautical Almanack* for 1772; and Monsieur BEZOUT, in his *Traité de Navigation*; and lately, Baron MASERES, with the assistance of Mr. ATWOOD, has given a direct Solution of this Problem, being the first direct Solution that has been given of it, and which is at the beginning of this Volume. In the present paper, I propose to collect together the several Solutions that have been given of this Problem, and to compare them with another Solution of it lately drawn up by myself, and which I will, therefore, now proceed to exhibit to the reader.

P R O B L E M.

Given the Distance sailed upon a direct Course, the Difference of Longitude, and either the Latitude sailed-from, or that come-to; to find the Course steered, and the other Latitude.

S O L U T I O N.

I. When the given place is on the Equator.

9.
The Solution
with which
the different
examples are
to be com-
pared.

To the logarithmick secant of the difference of longitude, add the log. cosine of the distance expressed in degrees. And the sum, rejecting radius, will be the log.

log. line of the required latitude nearly. But, because the distance between any two places, when measured on the arch of a great circle, is always less than the distance between the same two places, when reckoned on a loxodromick curve; therefore, the arch thus found will always exceed the true latitude. Now, from the logarithm of this latitude reduced to miles, its index being increased by 10, subtract the log. of the distance: and the remainder will be the log. cosine of an arch, to the log. tangent of which, if we add the log. of the meridional parts of the above found latitude, the sum, rejecting the radius, will be the log. of the first approximated difference of longitude; the difference between which and the difference of longitude that is given, will be the first error.

From the logarithm of the difference of longitude, its index being increased by 10, subtract the log. of the distance, and the remainder is the log. sine of an arch, to the log. cosine of which if we add the log. of the distance, the sum will be the log. of the approximated difference of latitude, which is always less than the truth. To the log. tangent of the above arch, add the log. of the meridional parts answering to this new latitude: and the sum, rejecting 10 from the index, is the log. of the second approximated difference of longitude; the difference between which and the given difference of longitude, is the second error.

Then, As the difference, or the sum, of the errors; according as they are of the same, or a contrary, denomination:

Is to the first error:

So is the difference of the two approximated latitudes,

To the first correction of latitude.

Now this first correction of latitude, being applied to the first approximated latitude, will give a latitude near the truth.

Reduce this latitude to minutes, and increase the index of its log. by 10; and then subtract from the said logarithm the log. of the distance: and the remainder will be the log. cosine of the course nearly; to the log. tangent of which if we add the log. of the meridional parts of the corrected latitude, the sum, rejecting 10 from the index, will be the log. of the difference of longitude; which, if it agrees with the given difference of longitude, will authorize us to conclude that the corrected latitude will be also the true latitude. But, if it does not agree with the given difference of longitude, we must then assume another latitude, either greater or less, according as the last-computed difference of longitude is greater or less than that given:—with which, and the given distance, we must recompute the course and difference of longitude. If this also disagrees with that given, then, with the difference between the two last-found differences of longitude, the difference between the first of them and that given, and the difference between the two last latitudes, find the correction of latitude as before;—also, with the difference between the two computed courses, find the correction of the course: and by these means, the true latitude and course will be obtained.

Q₃ E. 1.

II. When

II. When the given place is not on the Equator.

To the log. cosine of the difference of longitude, add the log. co-tangent of the given latitude: and the sum, rejecting radius, will be the log. co-tangent of arch first. And to the log. cosine of the given distance, add the log. cosecant of the latitude, and the log. sine of arch first; the sum, rejecting 20 from the index, will be the log. cosine of arch second. The sum or difference of arches first and second, according as the ship has been receding from, or approaching towards, the Equator, will be the latitude of the place come-to nearly, especially if the given quantities be small; which latitude we will call *the computed latitude*. If the ship sails towards the South from a place in North latitude, or towards the North from a place in South latitude; then, if arch second is greater than arch first, the ship will have crossed the Equator, and the difference between these arches will be the computed latitude of the place of arrival, of a contrary denomination to that failed-from.

. From the log. of the difference between the given and computed latitudes, reduced to minutes, its index being increased by 10, subtract the log. of the distance: the remainder is the log. cosine of the first approximated course. To the log. tangent of which if we add the log. of the meridional difference of latitude, the sum, rejecting radius, will be the log. of the first approximated difference of longitude; the difference between which and that given, will be the first error.

If the given latitude be small, and the direction of the ship be towards the Equator, another limit, or error, is to be found in the same manner as directed in the second article of the preceding rule. But, if the given latitude be considerable, (as the first-found error, in most cases, will be small, especially when the distance is not very great;) instead of finding another limit for the latitude come-to, let a new latitude be assumed according to circumstances. And then, with the given and assumed latitudes, find the meridional difference of latitude; and, with the difference between these latitudes and the distance, find a second approximated course; with which, and the meridional difference of latitude, find the difference of longitude as in the last article. The difference between this difference of longitude, and the given difference of longitude, will be the second error.

Now, with these errors, and the difference between the approximated and assumed latitudes, find the correction of latitude, as at the end of the former part of this Solution. In like manner find the correction of the course. Hence the true latitude come-to, and the course steered, will be obtained; and the truth of the operation may be verified by the agreement of a new computed distance and difference of longitude from these data, with those given.

Q. E. I.

REMARKS.

R E M A R K S.

1. If any of the errors be considerable, it will be necessary to repeat the operation a third time; and the less the two last errors are, and the nearer to an equality, the more exact will the course and latitude be obtained: another repetition of the process will, however, do away any small error that may remain.

2. The distance cannot be less than an arch whose sine is equal to the sine of the difference of longitude multiplied by the cosine of the given latitude.

3. When the distance is less than twice an arch whose sine is equal to the sine of half the difference of longitude multiplied by the cosine of the given latitude, the question will admit of two solutions; that is, two different courses in the same quarter of the compass, and two latitudes may be found from the same data, each corresponding pair of which will give an answer to the question. In order to determine which of the latitudes thus found is that which is wanted, it will be necessary to know either the latitude come-to, or the course steered, within certain limits. This is the case in the example given by M. BEZOUT.

4. According to the above limit for the distance, the course cannot be towards the Equator. But, if the distance be greater than twice the above arch, the course may be in any quarter of the compass, while the distance does not exceed the distance of the given place from the pole towards which the ship is sailing, multiplied by the secant of the course, the radius being supposed unity.

5. The limitations in the 2d, and 3d articles, and in the first part of the fourth article, have been given upon the supposition that the distance between the place of departure and the place of arrival is a portion of a great circle of the Earth; which is always less than the arch of a loxodromick curve lying between the same places. But, when the distance of the two places is not great, the difference between the said portion of a great circle and the said arch of a loxodromick curve will be but a small quantity; and therefore, the above limitations will, in those cases, answer nearly in the loxodromick hypothesis.

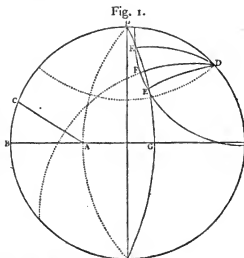
DEMON-

DEMONSTRATION

OF THE PRECEEDING RULES.

10.
And its De-
monstration.

Let AB, fig. 1, be a portion of the Equator equal to the given difference of longitude, A the place sailed from, and PA, PB, two meridians meeting in the pole P. Let the hypotenuse AC, of the right-angled spherical triangle ABC, be equal to the given distance; hence C will be near the place come-to, and BC its latitude, nearly: to find which, we have the following analogy.



As cosine AB : cosine AC :: R : cosine BC.

Hence cosine lat. = $\frac{\text{secant diff. of long.} \times \text{cosine distance}}{R}$.

In this determination of the measure of BC, the distance AC has been supposed to be a portion of a great circle of the sphere, which (as has been already observed,) is the shortest distance between any two places on its surface. But the distance, as given in the Problem, is an arch of a loxodromick curve; and consequently (since the distance between any two places, reckoned on a rhumb-line, is greater than the portion of a great circle contained between the same two places,) the latitude thus found will exceed the truth.

Again, let the given place be not situated upon the Equator, but at any other point, as D; and let the angle DPG be equal to the given difference of longitude. Now, DE being the given distance, GE will be the latitude required, nearly. Let DF be a perpendicular from the point D, to the meridian PG. Then, in the right-angled spherical triangle DFP,

R : cosine DPG :: tangent DP : tangent PF.

That is, R : cosine diff. of long. :: cotan. given lat. : cotang. GF.

Hence cotang. GF = $\frac{\text{cosine diff. of long.} \times \text{cotang. given lat.}}{R}$.

Again,

Again, in the spherical triangles DPF, DEF,
 Cofine DP : cofine DE :: cofine FP : cofine FE.
 That is, fine lat. : cofine dist. :: fine GF : cofine FE.
 Whence cofine FE = $\frac{\text{cofecant lat.} \times \text{cofine dist.} \times \text{fine cs.}}{A^2}$.

Let fig. 2d represent a figure in MERCATOR'S sailing, in which AC is the given distance, AB the difference between the given and computed latitudes, and the angle BAC the course; to find which we have the following analogy :

AC : AB :: R : cofine A.

Hence, cofine course = $\frac{\text{Diff. of lat.} \times \text{radius}}{\text{Distance}}$.

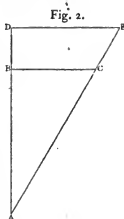
Also, AD is the difference, or the sum, of the meridional parts of the given and computed, or approximated, latitudes, according as these latitudes are of the same, or of a contrary, denomination; and DE is the difference of longitude.

Now R : tangent A :: AD : DE.

Hence, Diff. of longitude = $\frac{\text{Mer. diff. of lat.} \times \text{tan. course}}{R}$.

Q. E. D.

The simplest method has been followed in the above demonstration. We now proceed to solve, by the preceding Rules, the different Examples that have been given to this Problem.



I.

*The EXAMPLE given by WILLEBRORDUS SNELLIUS in his
 TIPHYBATAVUS.*

11.
 Snell's Ex-
 ample.

LET a Ship set out from a point in the Equator, and proceed on some loxodromick line for a distance of 800 sea miles, and then have varied her longitude by $50^{\circ} 19'$: It is required to find the course she has followed, and her latitude of arrival.

THE Solution given by SNELL to this Example is already re-printed in this already given Volume, pages 170, 171, and 172, and therefore it is unnecessary to repeat it in this Volume. IV. O o here.

here. We shall therefore proceed to solve it by the preceeding rules. It will, however, be proper to premise, that the distance given in this Example is expressed in Dutch miles, fifteen of which are equal to one degree; and, therefore, the distance in nautical or geographical miles is 3200, or in degrees $53^{\circ} 20'$.

SOLUTION OF SNELLIUS'S EXAMPLE.

12.
This Ex-
ample solved
by the pre-
ceeding rules.

Difference of longitude	-	-	$50^{\circ} 19'$	-	-	secant	-	-	0.1948092
Distance	-	-	53	20	-	-	cosine	-	9.7760897
Computed latitude	-	-	20	44	$29''\frac{1}{2}$	-	cosine	-	9.9708989

Hence, the difference of latitude is 1244.487, and the meridional difference of latitude 1272.6.

Diff. of lat.	-	-	1244.487	-	3.0949906
Distance	-	-	3200	-	3.5051500

Approx. course	$67^{\circ} 6' 49''\frac{1}{2}$	-	cosine	9.5898406	-	-	tangent	0.3745505
Meridional difference of latitude	-	-	-	1272.6	-	-	-	3.1046919
First approx. diff. of longitude	-	-	-	3014.62	-	-	-	3.4794424
Given difference of longitude	-	-	-	3019.	-	-	-	

First error	-	-	-	-	-	4.38
-------------	---	---	---	---	---	------

Diff. of longitude	-	3019	-	3.4798631
Distance	-	3200	-	3.5051500

2d appr. course	$70^{\circ} 38' 13''\frac{1}{2}$	fine	9.9747131	cosine	9.5205498	tangent	0.4541633.
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Approx. diff. of latitude	1060.962	-	-	-	3.0256998	
Meridional diff. of latitude	-	-	1078.22	-	-	3.0327074
Second approx. diff. of longitude	-	3068.11	-	-	-	3.4868707
Given difference of longitude	-	3019.				

Second error	-	-	-	49.11
--------------	---	---	---	-------

First error	-	-	-	4.38
-------------	---	---	---	------

Sum	-	-	-	53.49
-----	---	---	---	-------

Now, As the sum of the errors	-	-	53.49	-	-	1.7282726
-------------------------------	---	---	-------	---	---	-----------

Is to the first error	-	-	4.38	-	-	0.6414741
-----------------------	---	---	------	---	---	-----------

So is the diff. of the comp. lat.	-	-	183.525	-	-	2.2636952
-----------------------------------	---	---	---------	---	---	-----------

To the first correction of lat.	-	-	15.028	-	-	1.1768967
---------------------------------	---	---	--------	---	---	-----------

First computed latitude	-	-	-	-	$20^{\circ} 44' 29''\frac{1}{2}$
-------------------------	---	---	---	---	----------------------------------

Correction of latitude	-	-	-	-	$15' 1\frac{1}{2}$
------------------------	---	---	---	---	--------------------

Latitude come to nearly	-	-	-	-	$20^{\circ} 29' 27''\frac{1}{2}$
-------------------------	---	---	---	---	----------------------------------

Since

Since one of the errors is very considerable, the latitude thus found will not be exact; and, therefore, in order to obtain accuracy, it will be necessary to repeat the operation, using the above-found latitude as a part of the *data*. Hence, the difference of latitude is 1229.458, and the meridional difference of latitude 1256.54.

Diff. of latitude	-	-	1229.458	-	3.0897139		
Distance	-	-	3200.	-	3.5051500		
3d approx. course	67° 24' 20"	-	cofine	9.5845639	-	tangent	0.3807542
Meridional difference of latitude	-	-	-	1256.54	-	-	3.0991763
3d approx. difference of longitude	-	-	-	3019.47	-	-	3.4799305
Given difference of longitude	-	-	-	3019.			
Third error	-	-	-	-	-	+	.47
Second error	-	-	-	-	-	-	4.38
Sum	-	-	-	-	-	-	4.85
As	4.85	:	.47	::	15' 1 $\frac{1}{2}$	:	1' 27 $\frac{1}{2}$ nearly
Latitude	-	-	-	-	20 29	:	27 $\frac{1}{2}$
True latitude very near	-	-	-	-	20 30	:	55
And	4.85	:	.47	::	17' 30 $\frac{1}{2}$.4	:	1' 42 $\frac{1}{2}$ nearly
Third approx. course	-	-	-	-	67 24	:	20
True course very near	-	-	-	-	67 22	:	38

If the operation be again repeated, the error of the above results will scarcely exceed 3 seconds. In order to confirm the truth of the above determination of the course and latitude come-to, we shall, from these data, compute the distance and difference of longitude; and, if they agree with the given distance and difference of longitude, it will be evident that the quantities above found must be exact.

Course	-	67° 22' 38"	-	secant	0.4149204	-	-	tangent	0.3801491
Diff. of lat.	-	1230.92	-	3.0902299	-	Mer. diff. of lat.	1250.1	-	3.0997152
Distance	-	3200.	-	3.5051503	-	Diff. of long.	3019.	-	3.4798643

Now, since these agree with the given distance and difference of longitude, the course and latitude come-to are therefore determined with sufficient exactness. SNELLIVS finds the course to be 67° 26', and the latitude come-to to be 20° 27' 40". See above, page 172.

II.

MR. BOUGUER'S METHOD OF SOLUTION.

13.
Bouguer's
method of
Solution.

THIS Problem is BOUGUER's sixth general Problem of Navigation, and is expressed as follows, in page 292 of his *Traité de Navigation*, 8vo. edit. by M. DE LA CAILLE, printed at Paris in the year 1781.

On connoit la différence en longitude et les lieues de distance : on veut découvrir le rumb de vent qu'on a suivi, et trouver la latitude d'arrivée.

EXEMPLE. On est parti de $40^{\circ} 45'$ de latitude Nord, et de 15 deg. de longitude. On a couru 60 lieues entre le Nord et l'Est, et on est arrivé par $17^{\circ} 15'$ de longitude. On demande le rumb de vent, et la latitude d'arrivée.

Nous ne pouvons résoudre ce Probleme que par approximation. La différence en longitude est de $2^{\circ} 15'$. Je suppose qu'on est arrivé par 42 deg. de latitude, on aura $41^{\circ} 22'$ pour le premier moyen parallèle supposé ; et, réduisant les $2^{\circ} 15'$ en lieues Est, on en trouvera $33\frac{1}{2}$, qu'il faut faire convenir avec les 60 lieues de distance, et il viendra $49\frac{1}{2}$ lieues au Nord, valeur de $2^{\circ} 28\frac{1}{2}'$ de différence en latitude. On aura donc $43^{\circ} 13\frac{1}{4}'$ pour la latitude d'arrivée ; et comme elle diffère de celle que nous avons supposée, il faut faire une seconde tentative.

Nous prendrons $43^{\circ} 13'$ pour la latitude d'arrivée : nous aurons $41^{\circ} 59'$ pour second moyen parallèle supposé. Nous réduirons les $2^{\circ} 15'$ de différence en longitude en lieues Est, et il nous viendra $33\frac{1}{2}$ lieues, que nous ferons convenir avec les 60 de distance. Nous trouverons 50 lieues au Nord, ou $2^{\circ} 30'$ de différence en latitude, ce qui nous donnera $43^{\circ} 15'$ pour nouvelle latitude d'arrivée. Mais comme elle nous feroit trouver un troisième moyen parallèle supposé qui ne différeroit pas du second, nous devons regarder $43^{\circ} 15'$ comme la vraie latitude d'arrivée, et le rumb de vent sera le N E $\frac{1}{4}$ N.

As BOUGUER performs this Problem by repeated suppositions for the latitude come-to, the operation must, therefore, be very tedious ; and, since he resolves it upon the principles of middle-latitude sailing, it is evident, his final result will not be accurate unless the given quantities be very small.

SOLUTION OF BOUGUER'S EXAMPLE.

14.
His example
solved by the
preceding
rules.

As the place sailed-from is not in the Equator, this Example must therefore be performed by the second part of the rule.

Diff.

Diff. of long.	$2^{\circ} 15'$	-	cofine	9.9996650	-	Diff. $3^{\circ} 0'$	-	cofine	9.9994044
Latitude left	$40^{\circ} 45'$	-	cotan.	0.0646565	-	-	-	cosecant	0.1852466
Arch first	$40^{\circ} 46' 18''.7$	-	cotan.	0.0643315	-	-	-	sine	9.8149456
Arch second	$2^{\circ} 28' 9''$	-	-	-	-	-	-	cofine	9.9995566
Compr. lat.	$43^{\circ} 14' 27.7$	-	Mer. parts	-	2882.91				
Latitude left	$40^{\circ} 45'$	-	Mer. parts	-	2681.76				
Diff. of lat.	$\left\{ \begin{array}{l} 2^{\circ} 29' 27.7 \\ \text{or } 149' 27''.7 \end{array} \right.$	-	Mer. diff. lat.	-	201.15			2.3035200	
Distance	180.	-	-	-	2.1745298				
		-	-	-	2.2552725				
First approx. course	$33^{\circ} 51' 58''$	-	cof.	9.9192573	-	tangent	9.8267962		
Approximate difference of longitude		-	-	134.995	-		2.1303162		

Now, since the computed difference of longitude agrees with that given, within one two-hundredth part of a minute, it may therefore be said to be exact; and this example may be said to have had nearly a direct solution. Hence the true latitude of the place arrived at, in round numbers, is $43^{\circ} 14' \frac{1}{2}$ N. and the course N. $33^{\circ} 52'$ E. or N. E. by N. nearly. This latitude agrees within half a minute with BOVEVER's; but as his course is N. E. $\frac{1}{4}$ N. it is therefore three-fourths of a point greater than that in the above solution.

III.

MR. ROBERTSON'S METHOD OF SOLUTION.

[As given in his *Elements of Navigation*, vol. II, page 172, fourth edition.]

P R O B L E M.

Given One Latitude, Distance, and Difference of Longitude---Required the Course, and the present Latitude.

S O L U T I O N.

1. Find the meridional distance to use as a first departure. When the course is towards the Equator, augment this meridional distance for a second departure; if towards the Pole, diminish it for a second departure. Let the first departure be altered about a fifth, or sixth, of itself, for a second departure.

^{15.}Robertson's
method of
Solution.

2. With

2. With the given distance, and these departures, taken separately, in order; find the respective courses, and differences of latitudes. And hence the present latitudes and meridional differences of latitude.

3. To each course and respective meridional difference of latitude, find a difference of longitude.

4. Between each of these differences of longitude, taken in order, and that given, take the two differences; which call the first and second errors. Take the product of the first departure by the second error, and the product of the second departure by the first error.

5. If the two differences of longitude, found by the 3d article, were both too great, or both too small; let the difference of the said two products be divided by the difference of the errors, and the quotient may be taken for the true departure. But, if the two differences found were one too great, and one too small; let the sum of the said products be divided by the sum of the said errors, and the quotient may be taken for the true departure.

6. With the true departure and distance, find the true course and difference of latitude.

R E M A R K.

This method of ROBERTSON'S is also founded upon middle-latitude sailing, and the remaining part is performed by MERCATOR'S sailing. It may also be observed that, in the case when the ship is sailing towards the Equator, if the augmented departure exceeds the distance, the rule fails.

16.
Applied to
an Example.

EXAMPLE. A ship from the latitude of 37° N. having sailed for some days on a direct course in the N. W. quarter, finds that she has run 1027 miles, and has made 790 miles of difference of longitude: On what course did she steer, and what latitude has she come to?

With radius * and the difference of longitude 790, find a mer. distance, (631,) which being diminished by about its $\frac{1}{4}$ th, or by 100, gives 531.

Then 631 and 531 may be taken for the first and second departures.

The distance 1027, and these departures, give the courses $37^{\circ} 45'$, and 31° , and also the differences of latitude 814 and 883, or $13^{\circ} 34'$ and $14^{\circ} 43'$.

Hence the supposed latitudes are $50^{\circ} 34'$, and $51^{\circ} 43'$; and the meridional differences of latitude will be 1135 and 1245.

With 1135 and 1245, taken with the courses $37^{\circ} 45'$, and 31° , the differences of longitude are 879 and 748.

And $879 - 790 = 89$, the first error, too great:

Also $790 - 748 = 42$, the second error, too small.

The sum of these errors is 131, the divisor.

* This ought to be, With the given latitude.

Now

Now $631 \times 42 = 26502$; and $531 \times 89 = 47259$; the sum of these products is 73761.

Then $73761 \div 131$ gives 563 for the true departure.

With the distance 1027, and departure 563, the course is $N. 33^\circ 15' W.$

The difference of latitude is 860 m. $= 14^\circ 20' N.$

And the present latitude is $51^\circ 20' N.$

SOLUTION OF ROBERTSON'S EXAMPLE.

SOLUTION OF ROBERTSON'S EXAMPLE.

Diff. of long.	13° 16'	-	cosine	9.9884303	-	Diff 17° 7'	cosine	9.9803250	17- And his Example solved by the preceding rule.
Latitude	37 0	-	cotan.	0.1228856	-	-	cosecant	0.2205370	
Arch first	37 44 10 ¹ / ₂	-	cotan.	0.1113159	-	-	fine	9.7867710	
Arch second	13 36 31	-	-	-	-	-	cosine	9.9876330	
Comp. latitude	51 20 41 ¹ / ₂	-	Mer. parts	3601.81	-	-	-	-	
Lat. left	37 0 0	-	Mer. parts	2392.63	-	-	-	-	
Diff. of lat.	{ 14 20 41 ¹ / ₂ or 860' 41 ¹ / ₂	-	M. diff. of lat.	1209.18	-	-	-	3.0824909	
Distance	- 1027	-	-	2.9348476	-	-	-	-	
		-	-	3.0115704	-	-	-	-	
First approx. course	33° 3' 49 ¹ / ₂	-	cos.	9.9232772	-	-	tangent	9.8135747	
First approx. difference of longitude	-	-	-	787.165	-	-	-	2.8960656	
Given difference of longitude	-	-	-	790.	-	-	-	-	
First error	-	-	-	-2.835	-	-	-	-	

Hence the computed latitude is a little greater than the truth; therefore, let a new latitude be assumed a few minutes less, as $51^\circ 18'$.

Assumed lat.	$51^\circ 18'$	-	Mer. parts	-	3597.50	-	-	-	-
Given latitude	37	0	-	Mer. parts	-	2392.63	-	-	-
Diff. of lat.	{ $14^\circ 18'$ 858			-	Mer. diff. of lat.	1204.87	-	-	3.0809404
Distance	-	-	1027	-	-	2.9334873	-	-	-
				-	-	3.0115704	-	-	-
Second approx. course	$33^\circ 20' 16''\frac{1}{2}$	-	-	cos.	9.9219169	-	-	tangent	9.8181116
Second approx. difference of longitude	-	-	-	-	-	792.596	-	-	9.8990520
Given difference of longitude	-	-	-	-	-	790.	-	-	-
Second error	-	-	-	-	-	+2.596	-	-	-
First error	-	-	-	-	-	-2.835	-	-	-
Sum	-	-	-	-	-	5.431	-	-	-

Computed

Computed latitude	-	51° 20' 41".5	-	First approx. course	33° 3' 49".5
Assumed latitude	-	51 18 0	-	Second approx. course	33 20 16.8
Difference	-	2 41.5	-	Difference	16 27.3
Now, As 5.431 : 2.835 :: 2' 41".5 : 1' 24".3					
Computed latitude	-	51 20 41.5	-		
True latitude	-	51 19 17.2	-		
And, As 5.431 : 2.835 :: 16' 27".3 : 8' 36".4					
First approximated course	-	33 3 49.5	-		
True course	-	N 33 12 24.9 W.	-		

IV.

MR. EMERSON'S METHOD OF SOLUTION.

[As given in his *Mathematical Principles of Geography*, &c. page 141.]

P R O B L E M.

One Distance and Difference of Longitude being given, to find the other Latitude.

1. Assume any angle for the course, as near as you can guess, by the circumstances of the question. From this find the difference of latitude, making—

As radius
To the distance,
So the cosine of the course
To the diff. latitude.

2. Both latitudes being had, find the difference of longitude thus—

As radius
To the tangent of the course,
So the mer. diff. of latitude
To the difference of longitude.

3. If the longitude found, differs from the longitude given, as it generally will, correct the course, by making it greater or less, as there is occasion; then find the latitude, and then the longitude again. Repeat the operation till the longitude agrees with that given. But when you come near the matter, you may find the course truly from two trials, by the rule of false.

Or

18.
Emerson's
first rule for
the Solution
of this Problem.

Or thus,

1. Say, as radius : cosine given latitude :: difference of longitude : to ^{19.} Emerson's second rule, the departure.
2. Find the sum and difference of the departure and distance; take the logarithms of the sum and difference, and add them together; half the sum of these logarithms is the log. of the difference of latitude. Again,
3. Say, as meridional diff. of latitude : proper difference of latitude :: difference of longitude : departure more exact.

Repeat the 2d and 3d articles over and over, always with the last departure and difference of latitude, till at last you get the same difference of latitude twice; and this is the true difference of latitude, whence you may get the other latitude exact.

R E M A R K.

EMERSON'S first method is according to MERCATOR'S sailing; and in his second method, both middle-latitude sailing and MERCATOR'S sailing are made use of. The numerous repetitions he desires to be made, will render his methods very tedious. He has not given an example to illustrate his rules; the reader may, however, perform any of the preceeding or following examples by them.

V.

MR. LYONS'S METHOD OF SOLUTION.

[As given in the *Nautical Almanack* for 1772 *.]

^{20.}
Lyons's
method of
Solution,

P R O B L E M.

A Ship sails from a given Latitude, and having sailed a certain number of Miles, has altered her Longitude by a given Quantity; to find the Course steered.

S O L U T I O N.

ADD together the log. cosine of the given latitude, and the log. of the difference of longitude in minutes; and from that sum subtract the log. of the distance, the remainder is the sine of the approximate course.

* As Mr. LYONS says, this Problem was proposed by Dr. HALLEY, it would therefore appear that he had not seen HUTTON'S Geography, nor SMITH'S *Tiphys Batavus*.

In the Traverse Table find the given latitude among the degrees, and the difference of longitude among the distances: half the corresponding departure will be the first correction of the course in minutes; and is to be subtracted from the approximate course, if the given latitude is the lesser of the two; otherwise to be added to it.

In Table A, find the first correction in the first column marked *Arch*; and in the column which is marked at the top with the number of degrees contained in the complement of the approximate course, find the corresponding number of minutes: this number of minutes will be the second correction. In the same Table, find the difference of longitude in the first column; and in the column which is marked at the top with the number of degrees contained in the course, find the corresponding number of minutes; and taking such a part of it as is marked under the given latitude in Table B, such part will be the third correction.

The second and third corrections subtracted from the approximate course, corrected by the first correction, will leave the true course.

TABLE A.										TABLE B.	
Arch.	10°	20°	30°	40°	50°	60°	70°	80°	90°	Lat.	
1°	3'	1'	1'	1'	0'	0'	0'	0'	0'	0°	$\frac{1}{2}$
2	12	6	4	2	2	1	1	0	0	10	$\frac{1}{2}$
3	27	13	8	6	4	3	2	1	0	20	$\frac{1}{2} + \frac{1}{2}$
4	47	23	14	10	7	5	3	1	0	30	$\frac{1}{2} + \frac{1}{2}$
5	74	36	23	16	11	8	5	2	0	40	$\frac{1}{2} + \frac{1}{2}$
6	107	52	33	22	16	11	7	3	0	50	$\frac{1}{2} + \frac{1}{2}$
7	145	70	44	30	21	15	9	4	0	60	$\frac{1}{2} + \frac{1}{2}$
8	190	92	58	40	28	19	12	6	0	70	$\frac{1}{2} + \frac{1}{2}$
										80	$\frac{1}{2}$
										90	$\frac{1}{2}$

E X A M P L E.

21. Applied to an Example. Let a ship sail 272 miles northwards from the latitude of $46^{\circ} 20' N.$ till she has altered her longitude 4° : it is required to find the course.

Cofine latitude	-	-	-	$46^{\circ} 20'$	-	-	9.83914
Log. diff. of longitude	-	-	-	$240'$	-	-	2.38022
							12.21935
Log. distance	-	-	-	$272'$	-	-	2.43457
Approximate course	-	-	-	$37^{\circ} 32'$	-	fine	9.78478

In the Traverse Table, the departure belonging to the distance 240, and course $46^{\circ} 20'$, is 174', the half of which, to wit, 87', or $1^{\circ} 27'$, is the first correction.

correction. In Table A, with $1^{\circ}\frac{1}{4}$, and $52^{\circ}\frac{1}{4}$ the complement of the course, find $1'$, the second correction: with 4° and $37^{\circ}\frac{1}{4}$ we find 11 , $\frac{1}{4}$ th of which (Table B) is $3'$, the third correction.

Approximate course $37^{\circ} 32' - 1^{\circ} 27' - 4' = 36^{\circ} 1'$ the true course.

SOLUTION OF LYONS'S EXAMPLE.

Diff. of loog.	$4^{\circ} 0'$	-	cofine	9.9989408	-	Diff. $4^{\circ} 31'$	cofine	9.9986392	22.
Latitude	$46 20$	-	cotan.	9.9797797	-	-	cosecant	0.1406401	And the
Arch first	$46 24 11.2$	-	cotan.	9.9787205	-	-	fine	9.8598642	same Example
Arch second	$3 35 50$	-	-	-	-	-	cofine	9.9991435	solved by the
Comp. latitude	$50 0 1.2$	-	Mer. parts	3474.78	-	-	-	-	former rules.
Given lat.	$46 20 0$	-	Mer. parts	3144.43	-	-	-	-	-
Diff. of lat.	$\begin{cases} 3 40 1.2 \\ \text{or } 220' 1^{\circ}.2 \end{cases}$	-	M. diff. of lat.	330.35	-	-	-	-	-
Distance	272	-	-	2.3424622	-	-	-	-	-
		-	-	2.4345689	-	-	-	-	-
First approx. course	$36^{\circ} 0' 42''$	-	col.	9.9078933	-	-	tangent	9.8614470	-
First approx. difference of longitude	-	-	-	240.116	-	-	-	2.3804213	-
Given difference of longitude	-	-	-	240.	-	-	-	-	-
First error	-	-	-	-	-	-	-	-	-
	-	-	-	-	-	-	-	+.116	-

The computed latitude is hence too little by a very small quantity: let therefore $50^{\circ} 0'\frac{1}{4}$ be the assumed latitude.

Assumed lat.	$50^{\circ} 0' 30''$	Mer. parts	-	3475.27	-	-	-	-	-
Given latitude	$46 20 0$	Mer. parts	-	3144.43	-	-	-	-	-
Diff. of lat.	$\begin{cases} 3 40 30 \\ \text{or } 220' 30'' \end{cases}$	Mer. diff. of lat.	-	330.84	-	-	-	2.5196180	-
		-	-	2.3434086	-	-	-	-	-
Distance	272	-	-	2.4345689	-	-	-	-	-
Second approx. course	$35^{\circ} 50' 21''.7$	col.	9.9088397	-	-	-	tangent	9.8586981	-
Second approx. difference of longitude	-	-	-	238.955	-	-	-	2.3783161	-
Given difference of longitude	-	-	-	240.	-	-	-	-	-
Second error	-	-	-	-	-	-	-	-1.045	-
First error	-	-	-	-	-	-	-	+.116	-
Sum	-	-	-	-	-	-	-	1.161	-

P p 2

Computed

Computed latitude	-	50° 0' 1".2	-	First approx. course	36° 0' 42".*
Assumed latitude	-	50 0 30.0	-	Second approx. course	35 50 31.7
Difference	-	28.8	-	Difference	10 20.3
Now, As 1.161 : .116 :: 28".8 : 2".8					
Computed latitude	-	-	-	50 0 1.2	
True latitude	-	-	-	50 0 4.	
And, As 1.161 : .116 :: 10' 20".3 : 1' 2"					
First approximated course	-	-	-	36 0 42	
True course	-	-	-	35 59 40	

VI.

M. BEZOUT'S METHOD OF SOLUTION.

[As given in his *Traité de Navigation*, (printed at Paris in the year 1781,) being the sixth volume of his *Cours de Mathématiques*.

23.
Bezout's
Remarks on
this Problem.

At page 107, of the above-mentioned Treatise, Bezout expresses himself, with regard to this Problem, as follows.

Nous pourrions ajouter ici une sixième question, qui ne diffère de la précédente qu'en ce que le rumb de vent y est inconnu, et les lieues de distance, au contraire, sont supposées connues. Elle a également pour objet de faire conclure la latitude, de la longitude. Mais l'usage n'en seroit pas aussi sûr, parce que l'incertitude sur la mesure du fillage est plus grande que sur celle du rumb de vent. D'ailleurs, cette question ne pouvant être résolue que par approximation, nous n'en dirons rien ici. Au reste, ceux qui désireront savoir comment on peut résoudre cette question, le trouveront vers la fin de cet ouvrage.

Again, at page 300, he gives the following formula for this purpose; in which m is the latitude sailed-from, $m + q$ the latitude come-to, a the course, z' the difference of longitude, l the distance expressed in leagues, dq the correction of the latitude come-to, which is supposed to be known nearly; and dz the difference between the given difference of longitude, and that which corresponds to the course and difference of latitude known nearly. The formula is

24.
His formula
for its Solu-
tion.

$$dq = \frac{3l^2 a \sin^2 a \cosine a \cosine (m + q)}{3l \sin^2 a - z' \cosine (m + q)}; \text{ d'où, says M. BEZOUT, connoissant à peu-près le rumb de vent et la latitude, on pourra calculer la correction } dq \text{ qu'on doit faire à cette latitude à-peu-près connue, en mettant pour } a \text{ et } q \text{ leurs}$$

leurs valeurs à-peu-près connues, pour z' la différence de longitude qui répond aux valeurs à-peu-près connues de a et de q , et pour dz la différence entre la différence de longitude donnée, et celle qui répond à la différence de latitude et au rhumb de vent à-peu-près connus.

Par exemple—Supposons qu'étant parti de $42^{\circ} 23'$ de latitude Sud * et 10° de longitude, on ait fait 864 lieues entre le Sud et l'Est, et qu'on soit actuellement dans un lieu dont la longitude est $72^{\circ} 53'$. On estime avoir couru au S. E. $6^{\circ} 50'$ E. et être arrivé par la latitude de 69° ; on demande de confirmer ou de rectifier cette estime.

25.
Applied to
an Example.

Si la latitude et le rhumb estimés étoient exacts, la différence de longitude seroit de $63^{\circ} 3'$, qui excède celle qu'on connoît de $38'$; j'ai donc $dz = -38'$, $z' = 3311$, $a = 51^{\circ} 50'$, et $m + q = 69^{\circ}$, $l = 2592$; substituant ces valeurs, on trouve $dq = 136' = 2^{\circ} 16'$; donc la latitude d'arrivée, corrigée, est de $71^{\circ} 16'$.

Pour connoître si cette correction est suffisante, avec cette nouvelle latitude d'arrivée, je calcule le rhumb de vent et la différence de longitude; je trouve $a = 48^{\circ} 2\frac{1}{2}'$, $z' = 3763'$. Donc la différence de longitude qui résulte de la correction précédente est moindre de $10'$ que la différence de longitude donnée; on a donc $dz = +10'$; substituant ces valeurs comme ci-dessus, dans celle de dq , on trouve $dq = -22'$. Donc la latitude d'arrivée, corrigée de nouveau, est $70^{\circ} 54'$. Se calcule de nouveau le rhumb et la différence de longitude; et je trouve $43^{\circ} 38'$ pour le rhumb, et $3771\frac{1}{2}'$ ou $62^{\circ} 51'\frac{1}{2}'$ pour la différence de longitude. Il y a donc encore une minute et demi de moins sur la longitude. Je fais donc $dz = +1\frac{1}{2}'$; et substituant cette valeur et celles qu'on vient de trouver pour a et pour z' , j'ai enfin $a = 48^{\circ} 50'$ et $m + q = 70^{\circ} 49'$; qui satisfont.

SOLUTION OF BEZOUT'S EXAMPLE.

Diff. of long.	$62^{\circ} 53'$	-	cosine	9.6587780	-	Diff. $43^{\circ} 12'$	-	cosine	9.8627088
Latitude	$42^{\circ} 23'$	-	cotan.	0.0397233	-	-	-	cosecant	0.1712837
Arch first	$63^{\circ} 27' 34''.8$		cotan.	9.6985013	-	-	-	sine	9.9516386
Arch second	$14^{\circ} 39' 27''$		-	-	-	-	-	cosine	9.9856311
Latitude	$78^{\circ} 7' 2''$								
Or	$48^{\circ} 48' 8''$								

26.
The gene-
ral rule ap-
plied to his
Example.

* Bezout makes this "Nord," but, from the given quantities in the Example, it is evident it must be South. If a ship from latitude $42^{\circ} 23' N.$ fails between the South and East, and arrive at a latitude greater than that failed from, that latitude must be South; and to arrive at latitude $69^{\circ} S.$ the distance run must exceed 2227 leagues, although she should sail upon a meridian; but the given distance is only 864 leagues.

Since

^{27.}
By which it
is shewn to
admit of two
different So-
lutions.

Since the sum and difference of arches, first and second, are, each of them, greater than the latitude failed-from, two different Solutions may therefore be obtained to this example. And, since the latitude failed-from, and the other given quantities, are very considerable, the difference between the lengths of the corresponding arch of a great circle and that of a loxodromick curve will also be considerable. Hence, the greatest of the computed latitudes will be considerably greater than the truth, and the least latitude much less. We shall first perform the first of these Solutions, or that in which the greatest latitude is that wanted, and then the other:—and, in order to contract the work, let the latitude come-to be assumed equal to 71° .

The first
Solution by
the general
rule,

Assumed latitude	$71^{\circ} 0'$	-	Mer. parts	-	6145.70	
Given latitude	$42 23$	-	Mer. parts	-	2812.76	
Diff. of lat.	$\left\{ \begin{array}{l} 28 37 \\ \text{or } 1717 \end{array} \right.$	-	Mer. diff. lat.	-	3332.94	3.5228275
Distance	2592	-	-	-	3.2347703	
		-	-	-	3.4136350	
First approx. course	$48^{\circ} 30' 54'' .3$	-	cof.	9.8211353	-	tangent 0.0534120
First approximate difference of longitude		-	-	-	3769.20	- 3.5762495
Given difference of longitude		-	-	-	3773.	
First error	-	-	-	-	-	3.20

As this assumed latitude is too great, let therefore $70^{\circ} 50'$ be assumed.

Second assumed lat.	$70^{\circ} 50'$	-	Mer. parts	-	6115.11	
Given latitude	$42 23$	-	Mer. parts	-	2812.76	
Diff. of latitude	$\left\{ \begin{array}{l} 28 27 \\ \text{or } 1707 \end{array} \right.$	-	Mer. diff. of lat.	3302.35	-	3.5188231
Distance	2592	-	-	-	3.2322375	
		-	-	-	3.4136350	
Second appr. course	$48^{\circ} 48' 32'' .5$	-	cotang	8.8186025	-	tangent 0.0579147
Second approx. diff. of longitude		-	-	-	3773.44	- 3.5767378
Given difference of longitude		-	-	-	3773.	
Second error	-	-	-	-	+ .44	
First error	-	-	-	-	- 3.20	
Sum	-	-	-	-	3.64	
First assumed lat.	$71^{\circ} 0'$	-	-	-	First approx. course	$48^{\circ} 30' 54''$
Second	$70 50$	-	-	-	Second	$48 48 32$
Difference	10	-	-	-	Difference	17 38

As 3.64 + .44 :: 10' + 1' 12"
 Second assumed latitude - - 70 50
 True latitude - - 70 51 12

As

As	3.64	:	.44	::	17' 38"	:	2' 8"
Second approx. course	-				48 48 32		
True course	-				48 46 24		

BEZOUT finds his latitude $2\frac{1}{2}$ less, and his course $3\frac{1}{2}$ greater than the above determination.

Again, for the second solution of this example, let $54^{\circ} 10'$ be taken for the assumed latitude.

Assumed latitude	$54^{\circ} 10'$	-	Mer. parts	3881.69					
Given latitude	$42^{\circ} 23'$	-	Mer. parts	2812.76					
Diff. of latitude	{ $11^{\circ} 47'$ or 707	-	Mer. diff. of lat.	1068.93	-	-	3.0289493		
Distance	-	-	2592	-	-	3.4136350			
First approx. course	$74^{\circ} 10' 16''.6$	-	cofine	9.4357844	-	tangent	0.5474274		
First approx. diff. of longitude	-	-	-	3770.31	-	-	3.5763767		
Given difference of longitude	-	-	-	3773.					
First error	-	-	-	-	-	2.69			

Hence this latitude is too little; let therefore the second

Assumed lat. be	$54^{\circ} 20'$	-	Mer. parts	3898.80					
Given latitude	$42^{\circ} 23'$	-	Mer. parts	2812.76					
Diff. of latitude	{ $11^{\circ} 37'$ or 717	-	Mer. diff. of lat.	1086.04	-	-	3.0358458		
Distance	-	-	2592	-	-	3.4136350			
Second approx. course	$73^{\circ} 56' 29''$	-	cofine	9.4418841	-	tangent	0.5408296		
Second approx. difference of longitude	-	-	-	3772.90	-	-	3.5766754		
Given difference of longitude	-	-	-	3773.					
Second error	-	-	-	-	-	0.1			
First error	-	-	-	-	-	2.69			
Difference	-	-	-	-	-	2.59			

First assumed latitude	-	$54^{\circ} 10'$	-	First approx. course	-	$74^{\circ} 10' 16''.6$
Second assumed lat.	-	$54^{\circ} 20'$	-	Second approx. course	-	$73^{\circ} 56' 29''$
Difference	-	10	-	Difference	-	13 47.6

Theo. As	2.59	:	.1	::	10'	:	23''
Second assumed latitude	-				54 20 0		
True latitude	-				54 20 23 S.		
And, As	2.59	:	.1	::	13' 47''.6	:	32''
Second approx. course	-				73 56 29		
True course	-				S. 73 55 57 E.		

VII. SOLU.

VII.

SOLUTION OF BARON MASERES' FIRST EXAMPLE.

[See page 33d of this Volume.]

I. By ROBERTSON'S Method.

29.
Baron Ma-
seres' first Ex-
ample at-
tempted to
be solved by
Robertson's
rule.

30.
But this
method fails.

IN the Traverse Table to the latitude $51^{\circ} 18'$, taken as a course, and the difference of longitude 786, or its aliquot part, in a distance column, the corresponding difference of latitude $49'$, is the meridional distance; to which, one-sixth part of it being added, because the ship is approaching towards the Equator, gives 573, for a second departure. But this second departure is greater than the given distance 564; that is, one of the legs of a plane triangle is greater than the hypotenuse; which is absurd. Therefore this Example cannot be solved by ROBERTSON'S rules.

II. By the General Rule.

31.
The same
Example
solved by the
general rule.

Diff. of long.	$13^{\circ} 6'$	-	cofine	9.9885482	Diff.	$9^{\circ} 24'$	-	cofine	9.9941289
Latitude	$51^{\circ} 18'$	-	cotan.	9.9737144	-	-	-	cofecant	0.1076658
Arch first	$52^{\circ} 26''$	-	cotan.	9.8922626	-	-	-	fine	9.8967393
Arch second	$4^{\circ} 42' 18''$	-	-	-	-	-	-	cofine	9.9985340
Computed latitude	$47^{\circ} 19' 48''$	-	Mer. parts	3231.83	-	-	-	-	-
Given latitude	$51^{\circ} 18'$	-	Mer. parts	3597.50	-	-	-	-	-
Diff. of latitude	$\left\{ \begin{array}{l} 358' 12'' \\ \text{or } 238' 12'' \end{array} \right.$	-	Mer. diff. of lat.	365.67	-	-	-	-	2.5630893
Distance	564	-	-	2.7512792	-	-	-	-	-
First approx. course	$65^{\circ} 1' 31.2''$	-	cofine	9.6256627	-	tangent	0.3316751	-	-
First approx. diff. of longitude	-	-	-	784.810	-	-	-	2.8947644	-
Given difference of longitude	-	-	-	786.	-	-	-	-	-
First error	-	-	-	-	-	-	-	-1.190	-

The computed latitude is hence too little; let therefore another latitude be assumed two minutes greater, or $47^{\circ} 21' 48''$.

Assumed

Assumed latitude	47° 21' 48"	Mer. parts	-	3234.79
Given latitude	51 18 0	Mer. parts	-	3597.50
Diff. of lat.	{ 3 56 12 or 236 12	Mer. diff. of lat.	362.71	- - 2.5595595
Distance	364		2.3732799 2.7512791	
Second approx. course	65° 14' 29".4	- col.	9.6220008	- tangent 0.3361239
Second approx. difference of longitude	-	-	786.472	- 2.8956834
Given difference of longitude	-	-	796.	
Second error	-	-	+ 0.472	
First error	-	-	- 1.190	
Sum	-	-	1.662	
Computed latitude	- 47° 19' 48"	- First approx. course	65° 1' 3".2	
Assumed latitude	- 47 21 48	- Second approx. course	65 14 29.4	
Difference	- - - 2	- Difference	- - 13 26.2	
Then, As 1.662 : 1.19 :: 2' 0" : 1' 25".9 or 1' 26"				
Computed latitude	- - -	-	47 19 48	
True latitude	- - -	-	47 21 14	
And, As 1.662 : 1.19 :: 13' 26".2 : 9' 37"				
First approximated course	-	-	65 1 3	
True course	- - -	-	S. 65 to 40 W.	

The latitude thus found agrees with that obtained by a different process, within about half a minute; and the course, within three minutes. The results of the former method are given in page 47.

The second example given by Baron MASERES, in page 60, is the same as M. BOUGUER's which has already been solved. We, therefore, now proceed to solve the Baron's third example, which is given in page 95. In that example the latitude sailed-from is 5° N. the distance is 364 miles between the South and West, and the difference of longitude is 324.904 = 5° 24' 54".24.

^{32.} The Baron's Diff. of long.									
third Ex-	Latitude	-	5	24'	34".24	-	cofine	9.9980575	- Diff. 9° 24'
ample per-							-	-	-
formed by	Arch first	-	5	20	27	-	cotan.	1.0580482	- cofecant 1.0597040
new rules									
	Arch second		7	42	34				line 8.9422237
									cofine 9.9960366
	Comp. latitude		2	41	13 7/8		Mer. parts	161.20	
	Given lat.		5				Mer. parts	300.38	
							M. diff. of lat.	461.58	2.6642470
	Diff. of lat.		461'	13"	7/8		2.6639124		
	Distance				564		2.7512791		
	First approx. course		35°	8'	14 1/2"	-	cof.	9.9216333	- tangent 9.8474419
	First approx. difference of longitude						324.854		2.5116889
	Given difference of longitude						324.904		
	First error						- 0.05		
	Diff. of long.		324.904				2.5117551		
	Distance				564		2.7512791		2.7512791
	2d approx. course		35°	10'	28 1/2"	line	9.7604760	cofine	9.9124345 tangent 9.8480415
	Approx. diff of lat.		7	41	0.78	=	461.013	-	2.6637136
	Latitude failed from		5				Mer. parts	300.38	
	Latitude come to		2	41	0.78		Mer. parts	161.07	
							Mer. diff. of latitude	461.45	- 2.6641246
	Second approx. diff. of longitude						325.212		2.5121661
	Given difference of longitude						324.904		
	Second error						+ 0.308		
	First error						- 0.050		
	Sum						- 0.358		
	First computed latitude		-	2°	41' 13 1/2"	-	First approx. course	-	35° 8' 14 1/2"
	Second computed lat.		-	2	41 0.78	-	Second approx. course	-	35 10 28.8
	Difference		-		12.94	-	Difference	-	.2 14.1
	Now, As .358		1	.05	11 1/2".94		1		17.80
	First computed latitude						2		41 13.71
	True latitude						2		41 11.93
	And, As .358		1	.05	11 2' 14 1/2"		1		18".7
	First course						35		8 14.7
	True course						S. 35		8 33.4 W.

C O N .

CONCLUSION.

We have now concluded our Comparison of the different Methods that have been proposed by other writers for solving the preceding Problem, with a new Method which we have given for the same purpose. The first part of the approximation by this new method, generally gives the required latitude very near the truth: and when the place sailed-from is near the Equator, and the other given quantities are small, this method gives sometimes almost an exact solution; as the required terms may in these cases be found true to the difference of a single second: Whereas, in the operations for solving the different Problems in the various sailings in Navigation, it is usual, and always thought sufficiently exact, to bring out the required terms within only a minute of the truth. Again, when the given place is considerably distant from the Equator, and consequently an assumed latitude becomes necessary; we may observe, that, when the ship is sailing towards the Equator, the computed latitude will be less than the true latitude, but that, when the ship is receding from the Equator, it will be greater; and consequently there can be no great difficulty, or danger of committing a great error, in these cases in fixing upon an assumed latitude—that is, a latitude may be readily assumed which will be very near the true latitude. Those who peruse these different methods attentively, will find that every possible case may be solved by the rules we have given; and that these rules will also discover the ambiguity of those examples which admit of two solutions, which is not done in any of the other methods. And one of the other methods, namely, Mr. ROBERTSON'S, (as has been already observed,) fails in the case in which the ship is sailing towards the Equator, when the augmented departure is greater than the given distance.

33-
Conclusion.

To those who incline to examine or perform the preceding operations, it will be necessary to mention that the Tables used in these calculations were TAYLOR'S and SHERWIN'S Logarithmick Tables, and the Table of Meridional Parts for the Sphere, given in the *Connoissance des Temps pour l'année 1793*, by M. DE MENDOZA, a Captain in the Spanish Navy. And to such persons the writer of this paper will venture to predict, that, by the time they shall have gone through these several operations, they will not fail to have discovered, even without intending it, various contractions and limitations of them, by which the labour of performing them might have been lessened; but which the writer did not think it necessary to point out on this occasion, because it was his intention to abide strictly by the rules he had laid down. Also, in drawing up these rules, he used, what appeared to him to be, the simplest methods, or those which would be most generally understood, and he wholly rejected some other methods which might easily have been deduced both from the same and from different principles.

Aberdeen, March,
1796.

Q 9 2

A DIS

A
DISSERTATION
ON THE
RISE AND PROGRESS
OF THE
MODERN ART OF NAVIGATION.

BY
JAMES WILSON, M.D.

[Re-published from Mr. JOHN ROBERTSON's *Elements of Navigation*,
Edition 3d, in 1772.]

A

DISSERTATION

ON THE

RISE AND PROGRESS

OF THE

MODERN ART OF NAVIGATION.

IT has been much disputed, to whom the world was obliged for the mariner's compass. A late *Italian* writer indeed contends, after many¹, that the honour of the invention is due to *Flavio Gioja* of *Amalfi* in *Campania*, who lived about the beginning of the 14th century², though others say it came from the East, and was earlier known in *Europe*³. However that may be, it is certain, this wonderful discovery gave rise to the present art of navigation; which seems to have made some progress during the voyages, that were begun in the year 1420, by *Henry* Duke of *Visto*⁴. This learned Prince, brother to *Edward* King of *Portugal*, was particularly knowing in cosmography, and sent for one master *James* from the island of *Majorca*, to teach navigation, and to make instruments and charts for the sea⁵.

Of the Ma-
riner's Com-
pass.

These voyages being greatly extended, the art was improved under the succeeding monarchs of that nation. For *Roderic* and *Joseph*, physicians to King *John* the Second, together with one *Martin de Bobemia*, a *Portuguese*, born in the island of *Fayal*, scholar to *Regiomontanus*, about the year 1485, calculated, for the use of the sailors, tables of the Sun's declination, and recommended the astrolabe for taking observations at sea⁶.

¹ Suitable to that verse of *Panormitana*,

Prima dedit navis usum magnetis Amalphis.

² See Signor *Gregorio Grimaldi's* Dissertation on this subject, in the *Memoirs* of the *Etruscan Academy* of *Cortona*, Tom. iii. p. 193, printed at *Rome* in 1742.

³ *Histoire des Mathématiques*, par *M. Montucla*, à *Paris*, 1758.

⁴ *Marianæ Hist. Hispan.* lib. xx. cap. 11. and lib. xxvi. cap. 17. *Moguntie*, 1605.

⁵ *Decados d'Asia* par *J. di. Burnos*, lib. xvi. 1552.

⁶ *Maffei Hist. Indic.* lib. i. p. 6. printed at *Florence* in 1598.

The

The famous *Christopher Columbus* is said, before he attempted the discovery of *America*, to have consulted *Martin de Bobemia*, with others, and during the course of his voyage to have instructed the *Spaniards* in navigation¹; for the improvement of which art, the Emperor *Charles* the Fifth afterwards founded a lecture at *Seville*.

Of the variation of the Compass.
A. D. 1492.

The variation of the sea-compass could not be long a secret. *Columbus*, on the 14th of September 1492, observed it, as his son *Ferdinand* asserts², though others seem to attribute that discovery to *Sebastian Cabot*³. And as this variation differs in different places, *Gonzales d'Oviedo* found there was none at the *Azores*⁴; where some geographers have thought fit in their maps to make their first meridian to pass through one of those islands; it not being then known, that the variation altered in time.

The Cross-Staff.
A. D. 1514.

John Werner.
A. D. 1522.

The use of the *Cross-Staff* now began to be introduced amongst the Sailors. This very ancient instrument being described by *John Werner* of *Nuremberg*, in his *Annotations* on the first book of *Ptolemy's Geography*, printed in 1514, he recommends it for observing the distance between the Moon and some Star, in order thence to determine the longitude. *Werner* seems to have been the greatest geometer as well as astronomer of the time. In 1522, he published a tract⁵, containing a specimen of the conick sections, with some solid problems; and also he there determined the precession of the equinox more exactly than it had been done by any before him.

But the Art of Navigation still remained very imperfect, from the constant use of the plane-chart; the gross errors of which must have often misled the mariner, especially in voyages far distant from the Equator. Its precepts were probably at first only set down on the earliest sea-charts, as that custom is continued to this day; and larger directions have been usually premised by the *Dutch*, to collections of their charts called *Wagoner's*, from the name of the publisher; also by many affected titles, such as *Fiery-Columbus*, *Sea-Beacons*, *Mirrors*, *Atlases*, &c.

Peter de Medina.
A. D. 1545.

At length there were published in *Spanish* two treatises, containing a system of the Art, which had a great vogue; the first by *Pedro de Medina* at *Valladolid*,

¹ La Historia general y natural de las Indias par Gonzales de Miedo, en Sevilla, 1535. And Descriptione de las Indias Occidentale, de Antonio de Herrera, en Madrid, 1601.

² Hackluyt, in the dedication of his first volume of Voyages, printed in 1599.

³ In *Columbus's* life written in *Spanish*, which is now become very scarce; but it was printed in *Italian* at *Venice* in 1571.

⁴ See *Levis Sannio's Geography*, at the same place in 1585; Dr. *William Gilbert*, de *Magnete*, London, 1600; and *Purchas's Pilgrim*, in 1625, vol. I.

⁵ *Cabot*, a *Venetian* by birth, first served our King *Henry* the seventh, then the King of Spain, and, lastly, returning to *England*, he was constituted grand pilot by King *Edward* the Sixth, with an annual salary of above 200 pounds. Of this famous navigator and his expeditions, many writers have made mention, both foreigners, and *English*, as *Petrus Masius*, *Ramusio*, *Herrera*, *Hakluyt*, *Lord Bacon*, and particularly *Hakluyt* and *Purchas*, in their Collections of Voyages.

⁶ *Opera Mathematica*, at *Nuremberg*, in quarto.

in 1545, called *Arte de Nauegar*; the other at *Seville*, in 1556, by *Martin Cortes*, Martin Cortes. with this title, *Breve Compendio de la Sphera, y de la Arte de Navegar con nuevas Instrumentos y Reglas*. The author of this last tract says, he composed it at *Cadiz* in 1545.

These seem to have been the oldest writers who had fully handled this subject; for *Medina*, in his dedication to *Philip*, Prince of *Spain*, laments, that multitudes of ships daily perished at sea, because there were neither teachers of the art, nor books whereby it might be learnt; and *Cortes*, in his dedication, boasts to the Emperor, that he was the first who had reduced Navigation into a *compendium*, enlarging much on what he had performed¹.

Medina gave ridiculous directions, how to guess at the place of the horizon, when it could not be seen; and he likewise defended the errors of the plane-chart, and advanced against the variation of the magnetic needle the same sort of absurd arguments as *Aristotle* and his followers had brought to prove the impossibility of the Earth's motion. But *Cortes* briefly and clearly made out the errors of the plane chart, and seemed to reflect on what had been said against the variation of the compass, when he advised the mariner rather to be guided by experience, than to mind subtle reasonings. Besides, he endeavoured to account for this variation, by imagining the needle to be influenced by a magnetic pole (which he called *the point attractive*) different from that of the world; and this notion has since been farther prosecuted by others.

However, *Medina's* book, being perhaps the first of its kind, was soon translated into *Italian*, *French*, and *Flemish*², and served for a long time as a guide to the navigators of foreign countries.

But *Cortes* was our favourite author; a translation of whose work by Mr. *Richard Eden* was, on the recommendation of that great navigator Mr. *Stephen Burrough*, and the encouragement of the Society for making discoveries at sea, published at *London* in 1561: which underwent various impressions³, whilst the English translation of *Medina's* work, though made within twenty years after the other, seems to have been neglected, notwithstanding the encomiums bestowed on it by Mr. *John Frampton*, the translator.

A system of Navigation at that time consisted of some such particulars as these:—An account of the *Ptolemaic* hypothesis, and the circles of the sphere; of the roundness of the Earth, its longitudes, latitudes, and climates, and the eclipses of the luminaries; a kalendar; directions how to find the prime, equat, &c. and, by the last, the Moon's age, and thence the tides; a description of the sea-compass,

¹ The learned Don *Nicolo Antonio*, in his *Biblioteca Hispanica*, printed at *Rome* in 1672, Tom. i. p. 323, puts down a book, intitled, *Tratado de la Sphera y del marear con el regimento de las skuras*, written by *Francisco Falero*, a *Portuguese*, and printed at *Seville* in 1535; but perhaps there is a mistake in the date. He also mentions an edition of *Cortes* in 1551.

² The *Italian* and *French* translations were printed in 1554, the first at *Venice*, the other at *Lyon*; the *Flemish* edition, I have seen, was at *Antwerp* in 1580; perhaps it had been printed before.

³ In the latter editions some mistakes in the translation are corrected.

not forgetting the load-stone, with something about the variation, called its *merit-casting* and *north-westing*, for the discovering of which by night as well as by day, *Cortes* said, an instrument might easily be contrived; tables of the Sun's declination for four years¹, in order to find the latitude, from his meridian altitude; to do the same thing by the stars called *the guard-stars* in the north, and by those called *the crofters* in the south; of the course of the Sun and Moon; the length of the days; of time and its divisions; to find the hour of the day, and, by the nocturnal, that of the night; and, lastly, a description of the sea-chart; on which to discover where the ship is, they made use of a small table, that shewed, upon an alteration of one degree in the latitude, how many leagues were run on each rumb, together with the departure from the meridian. Besides some instruments were described, especially by *Cortes*; as, for instance, an instrument for finding the place and declination of the Sun, with the days and place of the Moon; and certain dials, the altrolabe and cross-staff, with a complex machine to discover the hour and latitude at once.

And after this manner the Art continued to be treated, though from time to time improvements were made in it by the following authors.

Gemma
Frifius.
A. D. 1530.

As *Werner* had proposed to find the longitude by observations on the Moon; so *Gemma Frifius*, in a tract intitled *De Principiis Astronomiæ et Cosmographiæ*, printed at *Antwerp* in 1530, advised, for the same purpose, the keeping the time by the means of small clocks, or watches, then, as he says, lately invented. He also contrived a new sort of cross staff, which he describes in his treatise *De Radio Astronomico et Geometrico*, printed at the same place in the year 1545; and, in his additions to *Peter Apian's* Cosmography, he gives the figure of an instrument which he calls a *Nautical Quadrant*, and represents it as very useful in Navigation, promising to write largely on the subject. Accordingly, in an edition he made anno 1553, of his above-mentioned book *de Principiis Astronomiæ*, &c., he delivers several *nautical axioms*, as he calls them; which, with some alterations, were repeated by his son *Cornelius Gemma*, in a posthumous piece of his father on the *Universal Astrolabe*, published in 1556. *Gemma Frifius* died in 1555, aged 45 years.

Dr. Cun-
ningham.
A. D. 1559.

With us *Dr. William Cunningham*, in his *Cosmographical Glass*, printed in 1559, amongst other things, briefly treats of Navigation, and especially shews the use of the *Nautical Quadrant*, and bestows much praise on that instrument.

Peter Nunez,
or Nonius.
A. D. 1567.

But a greater genius than any of these undertook this subject. For the famous mathematician *Pedro Nunez*, or *Nonius*, so early as 1537 published a book written in the *Portuguese* language, to explain a difficulty in Navigation proposed to him by the commander *Don Martin Alphonso de Sosa*; which was thirty years after printed at *Basil*, in *Latin*, with the addition of a second book, and the whole intitled *de Arte et Ratione Navigandi*; in which he both truly and learnedly exposes the errors of the plane-chart; and also gives the solutions of several curious

¹ *Cortes* sets down the places of the Sun for a twelvemonth, with an equation-table to correct those places, serving for many years to come; and also another table to find the Sun's declination from his longitude being given.

Astronomical Problems, amongst which is that of determining the latitude from two observations of the Sun's altitude and the intermediate azimuth being given. He also delivers many useful advices about the Art of Navigation, particularly how to perform its operations on the globe. He observed, that, though the oblique rhumbs are spiral lines, yet the direct course of a ship will always be the arch of some great circle, whereby the angle with the meridians will be continually changing; and consequently that all that the steersman can here do in order to preserve the original rhumb, is to correct these deviations as soon as they become sensible. But thus the ship will in reality describe a course somewhat different from the rhumb-line intended; and therefore the steersman's calculations for assigning the latitude, where any rhumb-line crosses the several meridians, will be in some measure erroneous. He also again sets down his method of dividing a quadrant by concentric circles¹, which he had described in his ingenious treatise *de Crepusculis*, printed in 1542, and which he supposed to have been practised by *Ptolemy*. There were also other tracts of his contained in this edition of this work: but another and still more compleat edition of his *Latin* works was made by himself at *Coimbra*, in 1573. His treatise of *Algebra*, written in *Spanish*, was printed at *Antwerp* six years before.

In 1577 Mr. *William Bourne* published his Treatise², intitled, *A Regiment for the Sea*, which he designed as a supplement to *Cortes*, whom he frequently quotes. Besides many things common with others, *Bourne* gives a table of the places and declinations of thirty-two principal stars, in order to find the latitude and hour; as also a larger tide-table than that published by Mr. *Leonard Digges*, in 1556³. He shews, by considering the irregularities in the Moon's motion, the errors of the sailors in finding her age by the epact; and also in their determining the hour from observing upon what point of the compass the Sun and Moon appeared. He advises, in sailing towards high latitudes, to keep the reckoning by the globe, as in those latitudes the plane-chart errs most. He despairs of our ever being able to find the longitude by any instrument, unless the variation of the compass should be caused by some such attractive point as *Cortes* had imagined, though of this he doubts. And, as he had shewn how to find the variation of the compass at all times, he advises mariners to keep an account of the observations, as useful to discover thereby the place of a ship; which advice the famous *Simon Stevin* prosecuted at large in a treatise published at *Leyden* in 1599, intitled *Portuum Investigandorum Ratio, metaphysico Hugone Grotio*; the substance of which was the same year printed at *London*, in *English*, by Mr. *Edward Wright*, intitled *The Haven-finding Art*.

But the most remarkable thing in this ancient tract is, the describing the way our sailors estimated the rate a ship made in her course, by an instrument

William Bourne.
A. D. 1577.

Of the Log.
A. D. 1607.

¹ The admirable division now so much in use is a very great improvement of this; so that when the famous Dr. *Edmund Halley*, the Royal Astronomer, revived that by adapting it to his *Maral Arch*, some body here named it *A Nonius*, in which he has been followed by many.

² He had ten years before published what he calls *Rules of Navigation*, as appears from his *Almanac*, printed in 1571.

³ In his Treatise, intitled, *A Proposition everlasting*, fol. 23.

called the *log*. This was so named from the piece of wood, or log, that floats in the water, while the time is reckoned during which the line to which it is fastened is veering out. The author of this device is not known; and I find no farther mention made of it till the year 1607, when it is said to have been made use of in an *East-India* voyage, of which an account is published by *Purchas*; but from that time its name occurs in other voyages that are published in *Purchas's* collections. And henceforward it became famous, being taken notice of, both by our own authors, and by foreigners; as by *Gunter* in 1623, *Snellius* in 1624, *Melius* in 1631, *Oughtred* in 1633, *Herrigone* in 1634, *Saltonstall* in 1636, *Norwood* in 1637, *Fournier* in 1643; and indeed by almost all the succeeding writers on Navigation, of every country. And it continues to be still in use as at first, though attempts have been often made to improve it, and other contrivances proposed to supply its place. Many of these have succeeded in quiet water, but have proved useless in a troubled Sea.

A following edition of this book was revised by the author, where, in the preface, he sets forth the gross ignorance of the old ship-masters, repeating some of the insipid jests they made use of to justify their want of knowledge in their art. Amongst the additions, he enlarges on the account of the log-line. And at the end subjoins an *Hydrographical Discourse touching the five several Passages into Catbay*.

Bourne published other tracts, as one called *Inventions or Devices*, where he describes a method by *wheel-work* of measuring the velocity of a ship at sea, which artifice he attributes to one Mr. *Humphrey Cole*.

Michael
Coignet.
A. D. 1581.

At *Antwerp*, in 1581, *Michael Coignet*, a native of the place, published a small treatise, intitled, *Instruction nouvelle des Points plus excellents et nécessaires touchant l'Art de Naviguer*¹. This served as a supplement to *Medina*, whose mistakes *Coignet* well exposed. He there shewed, that as the rhumbs are spirals, making endless revolutions about the poles, numerous errors must arise from their being represented by straight lines on the sea-charts; and expressed his hopes of discovering a rule to remedy those errors; saying, that most of the speculations delivered by the great mathematician, *Peter Nonius*, for that purpose, were scarce practicable; and therefore, in a manner, useless to sailors. In treating of the Sun's declination, he took notice of the gradual decrease in the obliquity of the ecliptic, a point long disputed, but now settled from the theory of attraction. He also described the cross-staff with three transverse pieces, as it is at present made, which he acknowledged to be then in common use amongst the mariners; but he preferred that of *Genma Frijsus*. He likewise gave some instruments of his own invention, which are now quite laid aside, except perhaps his *noëturnal*. As the old sea-table, mentioned above, erred more and more in advancing towards the poles; he set down another to be used by such as sailed beyond the 60th degree of latitude. At the end of the book is

¹ It had been published in *Flemish*; but the *French* edition is the fullest. *Coignet* died in 1625, leaving many mathematical manuscripts. See *Valerii Andrea Bibliotheca Belgica*, printed at *Larvoux* in 1643.

delivered

delivered a method of sailing on a parallel of latitude by means of a ring-dial, and a 24 hour glass; on which the author very much values himself.

The same year Mr. *Rbert Norman*¹ published a discovery, he had long before made, of the dipping of the magnetic needle, in a small pamphlet, called *The Newe Attraction*, where he shews how to determine its quantity; and in speaking of the loadstone, he disputes against *Cortes's* notion, that the variation of the compass was caused by a point fixed in the heavens, contending that it should be sought for in the earth, and proposes how to discover its place. He also treats of the various sorts of compasses, setting forth at large the dangers that must arise from the then prevailing practice of not fixing, on account of the variation, the wire directly under the *flower-de-luce*; as compasses made in different countries have it placed differently. *Bourne* indeed had warned against this abuse, and there are many things common to both authors.

Of the dipping of the Magnetick Needle.
Robert Norman. A. D. 1581.

To *Norman's* piece is always subjoined, Mr. *William Burrough's Discourse of the Variation of the Compass or Magnetick Needle*. The author had been a famous navigator, having used the sea from fifteen years of age, and for his merit was promoted to be Controller of the Navy by *Queen Elizabeth*². He shews how to determine the variation several ways, setting down many observations of it made by an azimuth-compass of *Norman's* invention, but improved by himself. He demonstrates the falshood of the rules commonly used, to find the latitude by the guard-stars. He particularizes many errors in the then sea-charts, occasioned by the neglect of the variation; adding, *But of these coastes (towards the north,) and of the inward partes of the countries of Russia, Muscovia, &c., I have made a perfect plat and description, by myne owne experience in sundrie voiaiges and travailes, bothe by sea and lande to and fro in those partes, whiche I gave to her Majestie in anno 1578.* And lastly, he justly finds fault with *Coignet's* instrument, called a nautical hemisphere; but speaks too severely against the writers of Navigation, concluding thus,

William Burroughs.

But (as I have already sufficientlie declared,) the compas sheweth not alwaies the pole of the worlde, but varieth from the same diversly, and in sayling describeth circles accordingly. Whiche thing, if Petrus Nonius, and the rest that have written of Navigation, had jointlie considered in the tractation of their rules and instruments, then might they have been availeable to the use of Navigation; but they perceiuing the difficultie of the thyng, and that, if they had dealt therewith, it would have utterly overwhelmed their former plausible conceits, with Pedro de Medina (who, as it appeareth, having some small suspicion of the matter, reasoneth very clerkly, that it is not necessary that such an absurdity as the Variation should be admitted in such an excellent art as Navigation is) they have all thought best to passe it over with silence.

The Spaniards too continued to publish treatises of the Art: and particularly an excellent *Compendium* by *Roderico Zamorano* was printed at *Seville* in the year 1585; which is written clearly and with brevity, not being incumbered with

Roderico Zamorano. A. D. 1585.

¹ He is commended for an excellent Artist by our authors, as *Bourne*, *Burroughs*, *Sir Humphrey Gilbert*, *Hues*, *Potter*, *Blundeville*, *Wright*, and *Dr. Gilbert*.

² *Hackluy's Voyages*, Vol. i. p. 417. Printed in 1599.

such

such idle speculations as abound in *Medina* and *Cortes*. The author was Royal Lecturer at *Seville*, and contributed much to the reforming the sea-charts; as we are told by his successor, *Andres Garcia de Céspedes*, who also published a *Treatise of Navigation* at *Madrid*, in 1606.

Large Globes
made by Ed-
ward Mulli-
neux. A. D.
1592.

As globes may be very serviceable for the mariner, Mr. *Edward Mullineux* set forth, in 1592, at the charges of Mr. *William Sanderfon*, a merchant*, a pair of globes much larger than those exhibited in the year 1541 by the famous geographer, *Gerard Mercator*. On the terrestrial one were described many new-discovered countries; and the several voyages made round the world by Sir *Francis Drake* in 1577, and by Mr. *Thomas Cavendish* in 1586, with the progress Sir *Martin Frobisher* had made towards the north in 1576, to a place called his *Straits*, were traced out on it.

Robert Hues
on the use of
the Globes.
A. D. 1594.

These globes were accompanied with a tract describing their uses, written in *English*; but in 1594 Mr. *Robert Hues* published a more elaborate work on the same subject in *Latin*; wherein, amongst others, he solves by the globe the problem of determining the latitude from two observations of the height of the Sun, when the time between the two observations is given[†]; and in the last part of his book he performs the usual questions in Navigation, premising a very sensible discourse on the Rhumb-lines, where he refutes the error of *Gemma Frisius*, who had asserted, that they met in the poles. At the conclusion he highly praises a treatise of Mr. *Thomas Hariot*, hoping it would be soon published, in which that author had treated of this subject upon geometrical principles, with great sagacity and judgement. But all the manuscripts of that great mathematician were lost, except his *Artis Analytica Praxis*, which was published in the year 1631, some years after his death; in which tract that idea of the generation of algebraic equations one from another by multiplication, which has since been very generally followed, was first advanced.

Hues[‡] was a person of letters, and besides had been far at sea. Amongst other curious particulars, he gives a good account of the attempts that had been made at various times to measure the Earth. In the *Epistle* to Sir *Walter Raleigh* he takes an occasion to enumerate the many discoveries of our mariners in very different parts of the world. His book was received with great applause, and has been indeed a pattern for such as afterwards handled the same subject. It has been often printed abroad, particularly in 1617, with the notes of *Jobu Isaac Pentanous*, who omitted the *Epistle* and the mentioning of *Hariot*. However, from this mutilated edition it was translated into *English* by one Mr. *Jobu Chilmead*, M. A. of *Christ Church College, Oxford*, and published at *London* in 1639.

Captain
John Davis.
A. D. 1594.

Amongst our sailors, none were more famous than Captain *John Davis*[§], who gave

* This Mr. *Sanderfon* was much respected for his knowledge as well as for his liberality to ingenious men.

† This problem has been discussed by Dr. *Henry Pemberton*, in the *Philosophical Transactions*, Vol. II. part 2d. 1760, p. 910, where he has also given some improvements in Trigonometry.

‡ There is an account of *Hues* and *Hariot* in *Anthony Wood's Athen. Oxon.* Vol. I. printed in 1721, as being both members of that University.

§ Several of his voyages are in *Hackluyt's* and *Purchas's* collections. He and Captain *Abraham Ainkut* are greatly praised by Sir *Robert Dudley*, in his *Arcana del Mare*, as keeping a perfect reckoning

gave name to the Straits he discovered; and great matters were expected from his long experience and skill. In 1594 he published a small treatise, intitled, *The Seaman's Secrets*. This is writ with brevity, though somewhat pedantically, and was esteemed in its time, an eighth edition being printed in 1657; so it seems to have supplanted *Cortes*. *Davis* treats of plane sailing, calling it *horizontal*, and sets down the form of keeping a reckoning at sea. He likewise shews how to sail by the globe, and boasts of what he intended to do; much commending great-circle-sailing, without describing it, as also what he calls *paradoxal*; that is, by a projection on the plane of the Equator with spiral rhumbs, saying, he will publish a chart for that purpose. But above all, he extols the use of calculations in the cases of Navigation, and promises to handle that subject.

At the end of the book is given the figure of a staff of his contrivance, to make a back observation. Of this the author is so vain as to say, *Then which instrument (in my opinion) the seaman shall not finde any so good, and in all clymates of so great certaintie, the invention and demonstration whereof, I may boldly challenge to appertain unto my selfe (as a portion of the talent which God hath bestowed upon me) I hope without abuse or offence to any.*

This instrument seems to have for some time been in use; for *Adrian Metius*, in his treatise, intitled *Astronomicæ Institutio*, printed in 1605, gives a figure of it from an original in the possession of M. *Frederic Hautman*, governor of *Amboyna*. But it soon yielded to one of a more commodious form, which is now commonly called *Davis's Quadrant*¹; as if it was also of his invention, and that perhaps only because a back observation is made by both instruments; so the quadrant itself was at first styled a *Staff* and *Back-Staff*.

The famous traveller Signor *Pietro della Valle* in going, in the year 1623, from *Ormus* to *Surat*, as a passenger on board an *English* vessel, observed that great use was made of this quadrant by the seamen: and from this circumstance, (as the instrument was quite new to him,) he takes occasion, in his account of his voyages, to shew its use very distinctly, and says, the seamen told him, that it had been lately invented and was called *David's Staff*² from its author. Also Captain *Charles Saltonstall*, in his *Navigation*, describes it under the name of a *Back-Staff*; and in Captain *Thomas James's* famous voyage for discovering a north-west passage, begun in 1631, amongst the many instruments which he carried along with him, are mentioned two of Mr. *Davis's Back-staves*, which were doubtles these quadrants.

ing by the way of longitude and latitude; in proof of which Sir *Robert* has there given us two of their *Journals*. This Sir *Robert Dudley* was a natural son of the great Earl of *Leicester*, Queen *Elizabeth's* favourite, and had commanded, in 1594, a fleet against the *Spaniards*; but retiring to *Florence*, he assumed the titles of Duke of *Northumberland* and Earl of *Warwick*. His *Arcano* was printed at that place in two volumes, in 1646 and 1647.

¹ It is called by the French *Quartier Anglois*.

² *David Staff*, che in lingua *Inglese* vale à dir legno di David: *Viaggi*, Part 3. Letter 1. à *Roma*. This author not only praises the Captain *Nicholas Woodcock*, and other officers, but also the common sailors, for their care and skill; and says, the *Portuguese* lose great number of ships for not being so exact in their observations as the *English*.

Con-

Contemporary with *Davis* was Mr. *Richard Palter*, who, it is said, had been a principal master on board the *Royal Navy*. He wrote a very small book, intitled, *The Pathway to Perfect Sailing*, where, from an observation he made in 1586, he would infer¹, that different load-stones communicated different degrees of variation to the magnetic needle, and therefore despises the publishing observations of that kind, as needless. His book was not printed till 1644; and it did not deserve to be published at all, as it abounds with mistakes, and is written fantastically, obscurely, and arrogantly.

But all this while the plane chart, notwithstanding its errors were frequently complained-of, continued to be followed; as its use is easy, and serves tolerably well in short voyages, especially near the Equator.

Gerard
Mercator.
A. D. 1569.

However, a way to remedy these errors had, for some time, been inquired after. And *Gerard Mercator* seems to be the first, who conceived the means of effecting this, in a manner convenient for seamen, by continuing to represent both the meridians and parallels of latitude by parallel straight lines, as in the plane-chart, but gradually augmenting the distances between the degrees of latitude in advancing from the Equator towards either Pole, that the rhumbs also might be extended into straight lines, so that a straight line drawn between any two places, laid down in this chart by their longitudes and latitudes, should make an angle with the meridians, expressing the rhumb leading from one to the other. But, though *Mercator*, in 1569, set forth an universal map thus constructed², it does not appear upon what principles he proceeded; probably, it was by observing, in a globe furnished with rhumbs, what meridians the rhumbs passed at each degree of latitude. That he knew not the genuine principles, I shall make evident; our countryman, Mr. *Edward Wright*, was certainly the first who discovered them.

Wright insinuates, but without sufficient grounds, that this enlargement of the intervals between the parallels had been suggested before by *Cortes*³, and even by *Ptolemy* himself.

As to *Cortes*, he speaks of the *number* of the degrees of latitude, and not of the *extent* of them: for his expression amounts to no more than this, "that the degrees of latitude are to be numbered from the Equator, and consequently that both northwards and southwards from that line the numbers affixed to them must continually increase; and from any place having latitude (suppose Cape *St. Vincent* in *Spain*, which is his instance) the degrees of latitude will be denoted by numbers increasing towards the Pole, and decreasing towards the Equator." He had before expressly directed, that they should be all equal by measurement on a scale of leagues adapted to the map⁴.

¹ Perhaps he should have thence concluded that the variation altered, which was afterwards found to be the case.

² See his life, written by his intimate friend, *Gualterus Gymnius*, which was prefixed to an enlarged edition of his *Atlas*, published at *Duisburg*, in 1593, by *Ramoldus* his son, a year after his father's death. *Gerard Mercator* was born in 1512.

³ See the 2d chapter of *Wright's* book.

⁴ Part 3d, cap. 2d. fol. 58.

The

The passage in *Ptolemy*¹, referred to by *Wright*², does indeed relate to the proportion between the distances of the parallels and meridians, but contains no shadow of *Mercator's* scheme: for, instead of proposing any gradual enlargement of the distances of the parallels in a general chart, that passage relates only to particular maps, and is more distinctly explained in the first chapter of his last book; where he advises explicitly not to confine a system of such maps to one and the same scale, but to plan them out by a different measure, as occasion shall require, with this only caution, that the degrees of longitude should in each bear, in some measure, that proportion to the degrees of latitude, which the magnitude of the respective parallels bear to a great circle of the sphere; and subjoins, that in particular maps, if this proportion be observed in regard to the middle parallel, the inconvenience will be not great, though the meridians should be straight parallels to each other; wherein his design is plainly no other, than that the maps should in some sort represent the figures of the countries they are drawn for. *Mercator*, (who drew maps for *Ptolemy's* tables³,) understood him in no other sense, thinking it an improvement not to regulate the meridians by one parallel, but by two; one distant from the northern, and the other from the southern extremity of the map by a fourth part of the whole depth; whereby in his maps, though the meridians are straight lines, they are generally drawn inclining to each other towards the Pole.

But *Mercator's* universal map, mentioned above, though the author designed it for the benefit of sailors, was so far from being readily adopted, that some of the most skillful amongst them objected to its usefulness. Thus Mr. *Burrough* says of it—*By augmenting his degrees of latitude towards the poles, the same is more fitte for such to behold, as studie in cosmographie, by readyng outbours upon the lande, then to bee used in Navigation at the sea.*

And Mr. *Thomas Blundeville*, in his *Briefe Description of Universal Mappes and Cardes*, first printed in 1589, gives an account of this map, observing that *Bartholomæus Ptolemy*, of *Bruges*, had published, in 1579, one altogether like it. And, though *Blundeville* is so particular as to set down numbers expressing the distances between every two parallels of latitude in those maps, yet he seems to slight them, by saying, that no better rules than those given by *Ptolemy* can be devised. But this is so far from being true, that what is delivered by that ancient geographer about the construction of a general map, is a very indifferent performance, altogether unworthy of the author of the *Almagest*, and not in the least corresponding with the sagacity shewn in two treatises on the *Plainsphere* and *Analemma*, which the *Arabians* have handed down to us as compositions of *Ptolemy*⁴.

Thomas
Blundeville,
A. D. 1589.

¹ *Geograph. lib. ii. cap. 1.*

² In an advertisement set down on his universal map, at the end of his second edition of his book; and in this mistake he has been followed by others.

³ In an edition he made of *Ptolemy's Geography*, in 1584.

⁴ These were published by *Fed. Commandinus*, one at *Venice*, in 1518, the other at *Rome*, in 1562.

Maginus, A. D. 1596. *Maginus* also, at the end of his *Geographia Universa*, (the former part of which is a translation of *Ptolemy's*) first printed at *Venice*, in 1596, mentions this map of *Mercator*, and even gives a sketch of it; but seems to have no distinct conception of the author's design.

That *Mercator's* map was not rightly described, is manifest from the numbers given by *Blundeville*; and that he was ignorant of a genuine method of dividing the meridian, appears from a passage in an account that was published of his life, where the writer says, that *Mercator* often assured him, that this extending a sphere into a plane answered to the quadrature of the circle, so that nothing seemed to be wanting to make it a just solution of that celebrated Problem, but the demonstration.

However, though they did not perfectly understand the principles of this map of *Mercator*, yet our authors about this time began to entertain favourable thoughts of it; which might, perhaps, be owing to a report that then prevailed, that the learned and accurate Mr. *Wright* was about to treat of that subject. For Dr. *Thomas Hood*, to the first edition he made, in 1592, of *Bourne's Regiment*, added a *Dialogue* of his own, called *The Mariner's Guide*, written only to shew the use of the plane-chart, where he acknowledges and sets forth its errors, and highly praises *Mercator's*, saying, he had composed a treatise concerning it; but the indistinct account he gives thereof, shews it would not be this author's lot to render it fit for the use of Navigation. And Mr. *Blundeville*, in the following editions of his above-mentioned tract, omitted the commendation he had given of *Ptolemy's* method of delineating an universal map.

Mercator's scheme was not indeed contrived for representing the parts of a country in a just proportion to each other; but is appropriated to the use of mariners, who sail upon rhumbs by the guidance of the compass: and our countryman, Mr. *Edward Wright*, brought it to perfection¹, by discovering a true way of dividing the meridian. Of this discovery Mr. *Wright* lent an account from *Caius College*, in *Cambridge*, (of which College he was then a Fellow,) to his friend the above-mentioned Mr. *Blundeville*, containing a short table for that purpose, with a specimen of a chart so divided, together with the manner of dividing it: All which *Blundeville* published, in 1594, amongst his *Exercijes*, in that part of them which treats of Navigation²; where he has well delivered what had been before written on that Art; inasmuch that his book was long in great repute, a seventh edition of it having been printed in 1636. To the second edition, anno 1606, and the following ones, was added his former discourse of universal maps.

In 1597, the Reverend Mr. *William Barlowe*, in his *Navigators Supply*, gave a demonstration of this division, as communicated by a friend; saying, *This manner of cards has been publicly extant in print these thirtie yeares at least*³; but a cloude (as it were) and thicke miste of ignorance doth keepe it hitherto concealed: And so much the more, because some who were reckoned for men of good knowledge,

¹ Some of our modern writers have said, *Mercator* took the hint from *Wright*: but that is a mistake; for *Mercator's* map was published thirty years before *Wright's* book, who frequently refers to it. See *Edward Sherburne's* translation of the first book of *Manilius*, in 1675, p. 86.

² Chap. 29.

³ He should have said 28 only.

have by glancing speeches (but never by any one reason of moment) gone about what they could, to disgrace it.

This book of *Barlowe's* contains descriptions of several instruments for the use of Navigation, the principal of which is an *azimuth compass*, with two upright sights¹; and, as the author was very curious in making experiments on the loadstone, he discourses well and largely on the sea-compass; and still farther handles that subject in a tract he published some years after, intitled, *Magnetical Advertisements*.

At length, in 1599, Mr. *Wright* himself printed his famous treatise, intitled, *The Correction of certain Errors in Navigation*, which had been written many years before; where he shews the reason of this division², the manner of constructing his table, and its uses in navigation, with other improvements: A book, as Dr. *Halley* says, well deserving the perusal of all such as design to use the sea³. Edward Wright's principal work, in A. D. 1599.

In the preface, *Wright* complains of the obstinacy of our mariners, for not liking an improvement in their Art, saying, that they were like those whose ignorance Master *Bourne* had exposed, repeating *Bourne's* very words⁴.

Though this great improvement in Navigation by *Wright* has been embraced and followed by all proper judges; yet some undiscerning persons have of late, even amongst us, found fault with it: as one *Henry Wilson*, by a proposal for a *curvilinear sea-chart*, in 1720; and one Mr. *West*, of *Exeter*, in a posthumous piece, printed in 1762. But their cavils have been sufficiently obviated; those of the first by Mr. *Hafelden*, in his *Mercator's Chart*, and in his *Reply*, both printed in 1722; and those of the second, by Mr. *William Mountaine*, in the *Philosophical Transactions*, Vol. LIII. p. 69. Anno 1763.

In 1610 Mr. *Wright* published a second edition of this valuable work, and dedicated it to *Henry*, Prince of Wales, his royal pupil⁵, with the addition of some new improvements: particularly, he proposed an excellent way of determining the magnitude of the Earth; and at the same time recommended very judiciously the making our common measures in some settled proportion to that of a degree of a great circle on its surface, that they might not depend any longer on the uncertain length of a barley-corn.

¹ Many of these instruments are in the *Arcano del Mar*; together with the demonstration above-mentioned.

² Maps with their meridians thus divided, had been published at *Amsterdam* by *Johannes Hondius*, who, when in *London*, working as an engraver, learnt the manner of doing it from Mr. *Wright's Manuscript*; the fourth chapter of which he had transcribed into one of his maps. *Hondius* afterwards in his letters, both to Mr. *Briggs*, and also to Mr. *Wright*, begged pardon for not having acknowledged the obligation. See *Wright's* preface; where he complains of *Hondius's* proceeding, and farther relates, how his book, (a copy of which had been presented to the Earl of *Cumberland*;) had like to have come out under the name of a famous navigator, whom, from some circumstances there mentioned, I imagine to have been *Abraham Kendall*.

³ *Philosophical Transactions*, for 1696. N^o 219.

⁴ See above, page 308, lines 16 and 17.

⁵ In 1657 a 3d edition of this book was published by Mr. *Joseph Mossey*; in which the dedication to Prince *Henry* is unwisely left out; and at the end is added, by the editor, the above-mentioned *Heaven-finding Art*, as also *Wright's* universal map, improved by the discoveries made since his time.

Some of his other improvements were as follows: 1st, The Table of Latitudes for dividing the meridian, computed to minutes; whereas before it was but to every tenth minute, and the short table sent by him to *Blundeville* to degrees only: 2dly, An Instrument, he calls *the Sea-rings*; by which the variation of the compass, the altitude of the Sun, and the time of the day, may be determined readily at once in any place, provided the latitude be known: 3dly, The correcting the errors arising from the excentricity of the eye in observing by the cross-staff: 4thly, A total amendment in the Tables of the declinations and places of the Sun and Stars from his own observations, made with a six-foot quadrant, in the years 1594, 95, 96 and 97: and, 5thly, A sea-quadrant, to take altitudes by a forward or backward observation, and likewise with a contrivance for the ready finding the latitude by the height of the pole-star, when not upon the meridian. And that his book might be the better understood by beginners, he has, in this edition of it, subjoined a translation of the above-mentioned *Zamorano's Compendium*; in which he has corrected some mistakes in the original, and added a large table of the variation of the compass observed in very different parts of the world, to shew that it is not occasioned by any magnetical pole.

This excellent person was allowed fifty pounds a year (no inconsiderable sum at that time of day) by the *East-India Company*, for reading a lecture on Navigation. He also formed a project for conveying water to *London*; but was prevented from executing it by designing men: which is frequently the case. Whilst he led a studious and retired life in the University of *Cambridge*, his reputation was so far known, that Queen *Elizabeth* granted, in 1589, a *dispensation* for his absence from the University, to the end that he might accompany the Earl of *Cumberland* in the expedition to the *Azores*; as I have been informed by Sir *James Burrough*, Master of *Cains College*, (of which Mr. *Wright* had formerly been a fellow,) a gentleman of fine taste in architecture, as appears from some of the new buildings in *Cambridge*, which render the older buildings in the same place a disgrace to that famous seat of learning, which has produced many great men, as, (to mention here only mathematicians) *Wright*, *Briggs*, *Oughtred*, *Dr. Pell*, *Foster*, *Horrox*, *Bainbridge*, *Bishop Ward*, *Dr. Wallis*, *Dr. Barrow*, *Rooke*, *Sir Isaac Newton*, *Cotes*, and *Dr. Brook Taylor*.

Wright's improvements on *Mercator's* chart became soon known abroad.

Simon Stevin.
vin. A. D.
1608.

In 1608 were published the *Hypomnemata Mathematica* of the above-mentioned *Simon Stevin*, composed for the use of Prince *Maurice*. In the part concerning Navigation, the author, having treated of sailing on a great circle, and shewn how to draw mechanically the rhumbs on a globe, sets down *Wright's* two tables of latitudes and of rhumbs, in order to describe those lines more accurately; and in an appendix he commends *Hues*, shews a mistake committed by *Nonius* in relation to the rhumbs, and pretends to have discovered an error in *Wright's* latter table; but *Wright* himself, in the second edition of his book, has fully answered all *Stevin's* objections, demonstrating that they arose from his gross way of calculating.

And,

And, in 1624, the learned *Willebrordus Snellius*, Professor of the Mathematics at *Leyden*, published his *Typis Batavus*¹, a Treatise of Navigation on *Wright's* plan, written somewhat obscurely. In the introduction are praised *Nonius*, *Mercator*, *Stevin*, *Hues*, and *Wright*. But since what had been performed by our artists on this subject is not there particularly declared, as are the improvements made by the others; it has happened that some have attributed *Wright's* principal discovery to this author. Thus *Albert Girard*, who, in 1634, published a *French* translation of *Stevin's* Works, with notes, in one of them observes, that *Snellius* had calculated, what he calls, *Tabula Canonica Parallelorum*, to minutes as far as 70 degrees; whereas *Wright* had set forth, in 1610, such a table, so calculated to 89 degrees 59 minutes; notwithstanding which, *M. de Lagry*, in the Memoirs of the Royal Academy of Sciences at *Paris* for 1703, treating of the *Corrected Chart*, says, *C'est Willebrord Snellius qui en est l'inventeur*. But the *French* writers now acknowledge our countryman, *Wright*, to have been its author².

Willebrordus Snellius.
A. D. 1624.

Snellius was followed in *Holland* by *Adrian Metius*, in a treatise, intitled, *Primum Mobile*, printed at *Amsterdam*, in 1631; and in *France*, by the learned *Peter Herigone*, in his *Curfus Mathematicus*, where, in the dedication of the fourth tome to the Marſhal *B. ſompiere*, the author ſays, *Artem navigandi in cenſu Mathematicis non reſpuere plerique noſtrum; neque ſane in hunc ordinem aſcribi meruit, quandiū eorū tantum nautarum praxi celebrata eſt; nunc verò, cū inventis tabulis loxodromicis (quas nos primum Gallis exhibemus) formam certam firmāque leges acceperit, ſine injuriā omitti non poteſt*. But to return to our countryman—

Adrian Metius. A. D. 1631.
Peter Herigone.

Mr. Wright, in the 12th chapter, having ſhewn how to find the place of a ſhip on his chart, obſerved, the ſame might be performed more accurately by calculation; but conſidering, as he ſays, that the latitudes, and eſpecially the courſes at ſea, could not be determined ſo precisely, he forbore ſetting down particular examples; as the mariner may be allowed to ſave himſelf this trouble, and only mark out upon his chart, when truly conſtructed, the ſhip's way after the manner then uſually practiſed.

However, in 1614³, *Mr. Raphe Handſon*, among his nautical queſtions, ſubjoined to a tranſlation of *Pitſcu's Trigonometry*, ſolved very diſtinctly every caſe of Navigation, by applying arithmetical calculations to *Wright's* table of latitudes, or of *meridional parts*, as it has ſince been called.

Mr. Ralph Handſon.
A. D. 1614.

And beſides, though the method *Wright* diſcovered for determining the change of longitude by a ſhip ſailing on a rhumb, is the adequate means of performing it; *Handſon* propoſed two ways of approximation for that purpoſe,

Of finding the change of Longitude.

¹ In 1617 had been published his *Eratosthenes Batavus*, where is given an account of his meaſuring the Earth.

² ——— *c'eſt qu'on appelle les cartes réduites, invention admirable, de la quelle on eſt redevable à Edouard Wright, quoiqu'on l'ait ſouvent attribué à Mercator.* Hiſt. de l'Acad. Royale des Sciences, An. 1753. p. 275.

³ It was reprinted in 1630.

without

without the assistance of *Wright's* division of the meridian line. The first was computed by the arithmetical mean between the co-sines of both latitudes; the other by the same mean between their secants, as an alternative, when *Wright's* book was not at hand, though this latter is wider from the truth than the first; and farther he shewed by the forsaide calculations, how much each of these *compendiums* deviates from the truth, and also how erroneously the computations on the principles of the plain-chart differ from them all.

There is another method of approximation, by what is called, the Middle Latitude³; which, though it errs more than that by the arithmetical mean between the co-sines, yet, being less operose, is the only one now used by our sailors; notwithstanding the arithmetical mean between the logarithmick co-sines, equivalent to the geometrical mean between the co-sines themselves, had been since proposed by Mr. *J. bu Basset*⁴, which in high latitudes is somewhat preferable.

Mr. John
Basset.
A. D. 1630.

The computation by the middle-latitude, will always fall short of the true change of longitude; that, by the geometrical mean, will always exceed it; but that, by the arithmetical mean, will fall short in latitudes above 45 degrees, and exceed in lesser latitudes. However, none of these methods, when the change in latitude is sufficiently small, will deviate greatly from the real change in longitude.

About this time Logarithms⁵ began to be introduced into the practice of the mathematicks; and as they are of excellent use in the art of Navigation, we shall here say something about their original.

Of Logarithms.
Lord Napier.
A. D. 1614.

These were invented by *John Napier*, Baron of *Marchisoun* in *Scotland*, as appears from his treatise, intitled, *Mirifici Logarithmorum Canonis Descriptio*, first printed in 1614⁶. Soon after, the author communicated to Mr. *Henry Briggs*, Professor of Geometry at *Gresham College* in *London*⁷, another form of logarithms; with which Mr. *Briggs* was so well pleased, that he immediately set about computing a very large table of them, which he published in 1624, with his *Arithmetica Logarithmica*⁸. But in the mean time, as a specimen, he printed,

in

³ *Gunter's* works, first printed in 1623.

⁴ About 1630, in a dialogue which was published after the author's death, in an appendix to the *Pathway to Perfect Sailing*. *Basset* had been a teacher of Navigation at *Chatham*, and well made out what he undertook, to wit, "that a ship would return to the place it departed from, by sailing on the same rhumb," contrary to what *Fuller* and others had maintained. At the end of this discourse, he applies his compendium to the three principal problems in sailing.

⁵ The foundation of logarithms is a property of two serieses of numbers, one in arithmetical, the other in geometrical proportion; which property is declared by *Archimedes* in his *Arenarius*.

⁶ In 1619 was made, after the author's death, a second edition, with his farther improvements in Spherical Trigonometry.

⁷ He was in 1619 appointed by Sir *Henry Saville*, his professor of geometry at *Oxford*.

⁸ *Adrian Vlacq* made an edition of this book at *Torow*, in 1618, where the table of logarithms was continued by him to one hundred thousand numbers, though the logarithms themselves are but to ten places, whereas in *Briggs's* book they were to fourteen. Some copies of *Vlacq's* tables were

in 1617, a few copies for his own use and that of his friends, of a very small one, not exceeding a thousand natural numbers.

From this table Mr. *Edmund Gunter*, Mr. *Briggs's* colleague in Astronomy, computed one of artificial sines and tangents to every minute of the quadrant; which he published in 1620, being the first of its kind¹. And when he made an edition of his works three years after, both these tables were subjoined to his book.

Henry
Briggs, A. D.
1619.
Edmund
Gunter.
A. D. 1620.

There he applied to Navigation, according to *Wright's* table of meridional parts, as well as to other branches of the mathematicks, his admirable Ruler², on which were inscribed the logarithmick lines for numbers, and for sines and tangents of arches. He also greatly improved the Sector³ for the same purposes. And he shewed how to take a back-observation by the cross-staff, whereby the error, arising from the eccentricity of the eye, is avoided; describing likewise an instrument of his invention, named by him *A Cross-Bow*, for taking altitudes of the Sun or Stars, with some contrivances for the more ready collecting the latitude from the observation⁴.

The discoveries relating to the logarithms were carried to France by Mr. *Edmund Wingate*, who, going to Paris in 1624, published in that city two small tracts in *French*⁵, and dedicated them both to *Gaston*, Duke of Orleans, the King's only brother. In the first he teaches the use of *Gunter's* Ruler; and in the other, he teaches the use of the tables of logarithms and artificial sines and tangents, as modelled according to *Napier's* last form, which he attributed to *Briggs*. But this was a mistake; as appears from the dedication of *Napier's Rabb logia*, printed in 1616, and from what Mr. *Briggs* himself said in the preface of his *Arithmetica Logarithmica*.

Edmund
Wingate.
A. D. 1624.

The Reverend Mr. *William Oughtred* projected this Ruler into a circular arch, shewing fully its uses in a treatise first printed in 1633, intitled, *The Circles of Proportion*; where, in an appendix, are well handled several important points in Navigation. It has been made in the form of a Sliding Ruler. See *Setb Partridge's* use of the double scale, in 1662.

William
Oughtred.
A. D. 1633.

were purchased by our bookellers, and published at London, with an *English* explanation premised, dated 1631.

¹ *Vlacq* also published, at the same place, in 1633, his *Trigonometria Artificialis*, with tables of logarithmic sines and tangents to every tenth second of the quadrant. *Vlacq's* tables have a great reputation for their exactness, as have amongst us *Sherwin's* tables, the first edition in 1706, and *Gardiner's* edition of the same tables in 1743. M. de *Foucault*, in the History of the Academy of Sciences for 1717, commends an edition of *Vlacq's* smaller tables, made at Lyons, in 1670, as does M. de la Lande, in his Astronomy, printed at Paris, in 1764, tables published there in 1760.

² This Ruler is so constantly in the practice of our artists, that it has got the name of *The Gunter*.

³ The uses of a Sector had been shewn by Dr. *Robert Hood*, in a tract he published in 1598.

⁴ This ingenious person died in 1626, aged 45 years. His works have been several times reprinted with successive additions; the second edition was made in 1636 from his own manuscript; then from those of Mr. *Samuel Foster*, Professor of Astronomy at *Gresham* College, again by Mr. *Henry Bond*, and Mr. *William Leybourn*. The fullest and last, being the fifth, was in 1673.

⁵ These were afterwards printed at London in *English* with improvements.

As by the logarithmick tables all trigonometrical calculations are greatly facilitated; so the first author, who, I find, has applied them to the cases of sailing, was Mr. *Thomas Addison*, in his treatise, intitled, *Arithmetical Navigation*, printed in 1625. He also gives two traverse tables with their uses, the one to quarter points of the compass, the other to degrees.

Thomas
Addison.
A. D. 1625.

Mr. *Henry Gellibrand*, Mr. *Gunter's* successor at *Gresham College*, published his discovery of the changes in the variation of the compass, in a small quarto pamphlet, intitled, *A Discourse Mathematical on the Variation of the Magnetical Needle*, printed in 1635. This extraordinary phenomenon he found out by comparing the observations made at different times near the same place by Mr. *Burroughs*, Mr. *Gunter*, and himself, all persons of great skill and experience in these matters. And this discovery was soon known abroad; for father *Aibanofius Kircher*, in his treatise intitled *Magnes*, first printed at *Rome* in 1641, says, our countryman, Mr. *John Greaves*, had informed him of it, and then gives a letter of the famous *Marinus Mercennus* containing a very distinct account thereof. *Gellibrand* had been famous for the part he bore in the *Trigonometria Britannica* of his deceased friend Mr. *Briggs*, which was printed in 1633, at *Tergou*, under the care of *Adrian Vlacq*. *Gellibrand* also, in 1635, published in English an *Institution Trigonometricell*.

Richard
Norwood.
A. D. 1631.

In 1631 Mr. *Richard Norwood* had published an excellent treatise of *Trigonometry*, adapted to the invention of logarithms, particularly in applying *Napier's* general canons*. The author having, as he says, acquired his knowledge in the mathemaricks at sea, especially shewed the use of trigonometry in the three principal kinds of Navigation. And towards the farther improvement of that Art, he undertook a laborious work for examining the divisions of the log-line.

As altitudes of the Sun are taken on ship-board, by observing his elevation above the visible horizon; to collect from thence the Sun's true altitude with correctness, *Wright* observes it to be necessary, that the dip of the horizon below the observer's eye should be brought into the account; which cannot be calculated without knowing the magnitude of the Earth. Hence he was led to propose different methods for finding this; but complains, that the most effectual one was out of his power to execute; and therefore he contented himself with a rude attempt, which was in some measure sufficient for his purpose: and the dimensions of the Earth, deduced by him, corresponded so well with the usual divisions of the log-line, that, as he did not write an express treatise on Navigation,

* In the History of the Royal Academy of Sciences, at *Paris*, for 1712, p. 19, it is said by M. *de Fontenelle*, that the learned *Peter Gassendi* was the principal discoverer of this property; but *Gassendi* himself acknowledged that he had before received information of *Gellibrand's* discoveries. *Gassend. Oper.* vol. ii. p. 152. *Lugd.* 1658.

* A very advantageous report of it was made by M. *Mariotte* at a meeting in 1668, of the Academy. *Du Hamel, Hist. Acad. Scient.* p. 51. 1701.

* From a sailor he became a teacher, flying himself before his books, A Reader of the Mathematicks in *London*.

but

but only meant to correct such errors as prevailed in general practice, the log-line did not fall under his notice. But Mr. Norwood, in order to regulate this instrument upon genuine principles, put in execution the method which Mr. Wright had recommended, as the most perfect, for finding the dimension of the Earth, with the true length of the degrees of a great circle upon it; and, in 1635, actually measured the distance between London and York; from which, and from the altitudes of the Sun in the Summer-Solstice, observed on the meridian at both places, he found a degree on the great circle of the Earth to contain 367,196 English feet, (which are equal to 57300 French fathoms or toises,) which is very near the true length of it; as appears from many measures of a degree that have been made since that time.

The measurement of a degree of Latitude. A. D. 1635.

Of this affair Mr. Norwood gives a full and clear account in his treatise, called *The Seaman's Practice*, which was first published in 1637. There, with unaffected modesty, he apologizes for the hardness of a private person's undertaking so difficult a task; and very cautiously points out the true reason why so great a mathematician as Snellius had failed in his attempt. He also shews various uses of his discovery, particularly for correcting the gross errors hitherto committed in the divisions of the log-line. But such necessary amendments have been little attended to by the sailors; whose obstinacy in adhering to inveterate mistakes has been always complained of by the best writers on Navigation.

Norwood's *Seaman's Practice*. A. D. 1637.

Farther, Mr. Norwood likewise there describes his own excellent method of setting down and perfecting a Sea-Reckoning, using a traverse table for that purpose; which method he had followed and taught for many years: and besides, he shews how to rectify the course, by taking into consideration the variation of the compass; as also how to discover currents, and to make proper allowance on their account.

This treatise and his treatise on *Trigonometry* were continually reprinted, as the principal books for learning the Art of Navigation scientifically. What he had delivered in these two books, and especially in the latter of them, concerning this subject, was contracted as a manual for sailors, in a very small piece, called his *Epitome*; which is a very useful performance, and has gone through numberless editions.

No alterations were ever made in *The Seaman's Practice*, till in the 12th edition, (printed in 1676, after the author's decease,) there began to be inserted, at page 59, the following paragraph in a smaller character—[*About the year 1672 Monsieur Picart has published an account in French, concerning the measure of the Earth, a breviate whereof may be seen in the Philosophical Transactions, No. 112, wherein he concludes one degree to contain 365,184 English feet, nearly agreeing to Mr. Norwood's experiment.*] And this advertisement is continued in the subsequent editions, as I find it in one printed so lately as 1732.

Norwood's measure therefore, though it was not known to the great Sir Isaac Newton in his youth, was not buried in oblivion, on account of the confusions occasioned by our civil wars, as M. de l'Isle has been pleased to say¹; on the contrary, it has been constantly commended by our writers on Navigation:

¹ *Elements de la Philosophie de Newton*, chap. xviii. printed at Paris in 1759.

as, for example, by Mr. *Henry Bond*, soon after its publication, in a note at page 107 of the *Seaman's Kalender*, (which ancient book he reprinted and improved, whose use, through numberless editions, is continued amongst our sailors to this day;) and by Mr. *Henry Phillips*, in his *Geometrical Seaman*, in 1652; and in his *Advancement of Navigation* in 1657; and by Mr. *John Collins*, in his *Navigation by the Plane Scale*, in 1659; and by the Reverend Dr. *John Newton*, in his *Mathematical Elements*, in 1660; and by Mr. *John Seller*, in his *Practical Navigation*, in 1669; and by Mr. *John Brown*, in his *Triangular Quadrant*, in 1671.

And in the *Philosophical Transactions* for 1676, No. 126, there is given a very particular account of it. Nor had it escaped the royal notice; for when King *James*, in 1690, honoured the Observatory at *Paris* with a visit, he informed the gentlemen, then present, of this measure of the Earth; and upon their acquainting his Majesty how that had been determined by M. *Picard*, the King wished the two measures might be compared together.

But "that it was not commonly known in *France*" is no wonder, seeing that English books were not then so much inquired after as at present, by that polite and learned people.

Thus, for example, in the *Journal des Sçavans* for December 1666, it was observed of Dr. *Hooke's Micrographia*, *qu'il est écrit en une langue que peu de personnes entendent*; but, of late years, in the same *Journal* for February 1730, it is said of the English tongue, that it was *une langue que tous les vrais sçavans devoient servir*. And now, as *Norwood* is taken notice of in the latter editions of Sir *Isaac Newton's Principia*, his name and merit are become almost universally known. Insomuch that a particular account of his measure is given by M. *de Maupertuis*, in the preface to his Treatise of the figure of the Earth, printed at *Paris* in 1738; wherein he describes his method of determining the length of a degree on the Earth in *Lapland*; and *Norwood* is likewise mentioned by two learned Spanish sea-officers, Don *Jorge Juan* and Don *Antonio d'Ulloa*, in the account of their voyage printed at *Madrid* in 1748; which voyage had been undertaken for the same purpose of discovering the true length of a degree of latitude, they having been appointed to accompany the French mathematicians who were sent out by the King of *France* to measure such a degree near the Equator.

Henry Bond. About the year 1645 Mr. *Bond* published, in *Norwood's Epitome*, a very
A. D. 1645. great improvement in *Wright's* method, by a property in his meridian line, whereby its divisions are more scientifically assigned, than the author himself was able to effect; which was from this theorem, That these divisions are analogous to the excesses of the logarithmick tangents of half the respective latitudes, augmented by 45 degrees, above the logarithm of the radius.

This he afterwards explained somewhat more fully in the third edition of *Gunter's* works, printed in 1653; where, after observing that the logarithmick tangents from 45° upwards, increase in the same manner (as he expresses it,) that the secants added together do, if every half degree be accounted as one whole degree of *Mercator's* meridional line; his rule for computing the meri-

* *Du Hamel*, Hist. Academ. Regal. Scient. p. 285.

dional parts appertaining to any two latitudes (supposed on the same side of the Equator) is laid down to this effect; To take the logarithmick tangent, rejecting the radius, of half each latitude augmented by 45 degrees, and dividing the difference of those numbers by the logarithmick tangent of $45^{\circ} 30'$, the radius being likewise rejected, and the quotient will be the meridional parts required, expressed in degrees. And this rule is the immediate consequence from the general theorem, That the degrees of latitude bear to 1 degree (or 60 minutes, which in *Wright's* table stands as the meridional parts for 1 degree,) the same proportion as the logarithmick tangent of half any latitude augmented by 45 degrees, and the radius neglected, to the like tangent of half a degree augmented by 45 degrees, with the radius likewise rejected.

But here was farther wanting the demonstration of this general theorem, James Gregory, of Aberdeen, in his *Exercitationes Geometricæ*, printed at London in 1668; and since more concisely demonstrated, together with a scientific determination of the divisor, by Dr. Halley, in the *Philosophical Transactions* for 1695, No. 219, from the consideration of the spirals into which the rhumbs are transformed in the stereographic projection of the sphere upon the plane of the equinoctial; which the excellent Mr. Roger Cotes has rendered still more simple, in his *Logometria*, first published in the *Philosophical Transactions* for 1714, No. 388.

It is moreover added in *Gunter's* book, that, if $\frac{1}{2}u$ of this divisor (which does not sensibly differ from the logarithmick tangent of $45^{\circ} 1' 30''$ curtailed of the radius) be used, the quotient will exhibit the meridional parts expressed in leagues: and this is the divisor set down in *Norwood's Epitome*.

After the same manner the meridional parts will be found in minutes, if the like logarithmick tangent of $45^{\circ} 0' 30''$ diminished by the radius be taken, that is, the number used by others^a being 12633, when the logarithmick tables consist of eight places besides the index.

This Mr. Bond, who introduced so useful a discovery into the Art, was a teacher of the mathematics in London, and employed to take care of and improve the impressions of the current Treatises of Navigation. In an edition of the *Seaman's Kalender*, p. 103, he declared, he had discovered the longitude, by having found out the true theory of the magnetic variation; and to gain credit to his assertion, he foretold, that at London, in 1657, there would be no variation of the compass, and from that time it would gradually increase the other way, which happened accordingly. Again, in the *Philosophical Transactions* for 1668, No. 40, he published a table of the variations for 49 years to come.

Of the Variation of the Magnetic Needle.

This joyful news to all sailors acquired Mr. Bond a great reputation; inasmuch that the Treatise he had composed, called *The Longitude found*, was, in 1676, published by the special command of King Charles the Second, and

^a See Mr. Perkins's *Treatise of Navigation*, in Vol. I. of Sir Jonas Moore's *New System of the Mathematics*, p. 208, printed at London in 1681. Perkins's book was published by itself the year following, under the title of the *Seaman's Tutor*.

ushered into the world with the approbation of several of the most eminent mathematicians of that time¹.

But it was soon opposed, there being published at London, a book, in 1678, called *The Longitude not found*, written by one Mr. *Beckborow*. And, indeed, as *Bond*'s hypothesis did not in any wise answer its author's sanguine expectations, the famous Dr. *Halley* again undertook this affair; and from a multitude of observations he was induced to conclude, that the magnetic needle was influenced by four poles. His speculations on this subject are delivered in the *Philosophical Transactions* for 1683, No. 148, and for 1692, No. 195. But this wonderful phenomenon seems to have hitherto eluded all our researches.

However, that excellent person, in 1700, published a general map, on which were delineated curve lines expressing the paths, where the magnetic needle had the same variation. This was received with universal applause², as it may lead to some discovery in so abstruse an affair, and at present be useful on many occasions in determining the longitude. The positions of these curves will indeed continually suffer alterations; but then they should be corrected from time to time; as they have been for the year 1744, and the year 1756, by two ingenious persons, Mr. *William Mountaine* and Mr. *James Dodson*, Fellows of the Royal Society. The latter died not long after he had been chosen, for his merit, mathematical master at *Christ's Hospital* in London.

Of the Mon-
soons.

Dr. *Halley* also gave, in the *Philosophical Transactions* for 1690, No. 183, a Dissertation on the Monsoons, containing many observations very useful for all such as sail to the places that are subject to those winds.

The true principles of Navigation having been settled by *Wright*, *Norwood*, and *Bond*, many authors amongst us trod in their steps, making some little improvements. It would be impossible to enumerate each particular. Of the writers already mentioned, *Phillips* and *Collins*, in the title-pages of their books, declare what they aimed at; *Phillips* also in his tract, called the *Advancement of Navigation*, recommends a pendulum instead of a half-minute glass, to estimate the time the log-line is running out. He also proposes to do the same thing by wheel-work. Besides, in the *Philosophical Transactions* for 1668, No. 34, he delivers a better method to determine the tides than what was commonly practised; for which purpose Mr. *John Flamsteed*, the Royal Astronomer, gave still more perfect directions in the same *Transactions* for 1683, No. 143; and he likewise first ordered a glass lens to be fixed on the shade-vane, in what is called

¹ In the *Philosophical Transactions* for the same year, No. 130, it is said, the Lord *Brouncker*'s name was inserted by mistake.

² It is particularly commended in the History and Memoirs of the Royal Academy of Sciences at Paris, for the years 1701, 1705, 1706, 1708, and 1710. See also Mr. *Robinson*'s Reflections in his Introduction to Lord *Anson*'s Voyage round the World, made in 1743, &c. as also in the ninth chapter of the first book, and eighth of the third.

Davis's quadrant', which contrivance Dr. Robert Hook, Professor of Geometry at Gresham College, had before thought of².

Seller's *Practical Navigation*, though without demonstrations, has the rules of failing in the different kinds, as performed by calculation, by the plane-scale, ^{Seller's} *Practical Navigation*, by the gunter, and by the sinical quadrant, with various other matters relative to the Art; as also the use of the azimuth-compass (as now modelled,) the ring-dial, the sea-ring, the cross-staff, Davis's quadrant, plough, nocturnal, inclinatory needle and globe, together with all the necessary tables; the whole being delivered in a manner to well adapted to the general humour of mariners, that it has undergone numberless editions: the last, I have seen, was in 1739; but some late writers seem to have abated the run of this book. ^{A. D. 1669.}

As in failing especial regard ought to be had to the lee-way a ship makes, so many authors have touched upon this point; but the allowances usually made on that account are very particularly set down by Mr. John Buckler, and published in a small tract first printed in 1702, intitled, *A New Compendium of the whole Art of Navigation*, written by Mr. William Jones. ^{Of the Lee-way of a Ship.}

We ought not here to pass over in silence the very useful invention of Dr. Gawin Knight, which is the making artificial magnets, that are of greater efficacy than the natural ones. Though the Doctor has not thought fit to reveal his secret; yet others have found it out, who have made it public, particularly the Reverend Mr. John Mitchell, of Queen's College, Cambridge, and Mr. John Canton; the first in a treatise of *Artificial Magnets*, printed in 1750; the other in the Philosophical Transactions, vol. XLVII. *Ann.* 1751. ^{Of artificial Magnets.}

The Earth being now universally agreed to be not a perfect globe, but a spheroid, whose diameter at the poles is shorter than any other; the Rev. Dr. Patrick Murdoch published a tract in 1741, where he accommodated Wright's failing to such a figure; and Mr. Colin Maclaurin, Professor of Mathematics in the University of Edinburgh, the same year, in the Philosophical Transactions, No. 461, gave a rule to determine the meridional-parts of a spheroid, which speculation he farther prosecutes in his book of *Fluxions*, printed at Edinburgh, in 1742. ^{Of the Spheroidal Figure of the Earth.}

Though Sir Isaac Newton in his *Principia*, (which was first printed in 1686,) had demonstrated from the theory of gravity, that this must be the real form of the Earth, as it revolved about an axis; yet, in the year 1718, M. Cassini again³ undertook from observations to shew the contrary, and that the Earth was a spheroid, having its longest diameter passing through its poles⁴; and, in 1720, M.

² See the above-mentioned *Perkin's Navigation*, page 350.

³ See the Bishop of Rochester's (Dr. Thomas Sprat's,) excellent History of the Royal Society in 1667, pag. 246; and Hook's Posthumous Works, published by Richard Waller, Esq. in 1705, p. 557.

⁴ In the Memoirs of the Royal Academy of Sciences at Paris, his father in 1701, and he in 1713, attempted to prove that the Earth was an oblong spheroid.

⁵ M. John Bernoulli in his *Essai d'une Nouvelle Physique Cléste*, printed at Paris in 1735, triumphs over Sir Isaac Newton's vainly imagining these precarious observations could invalidate what Sir Isaac had demonstrated.

Dr. Defagulier.
A. D. 1725.

de Mairan advanced arguments, supposed to be strengthened by geometrical demonstrations, to farther confirm *M. Cassini's* assertion. But in the Philosophical Transactions for 1725, No. 386, 387, and 388, *Dr. Defagulier* published a dissertation, wherein he made appear the weakness of the reasoning, and the insufficiency of the observations, as they were managed, to settle so nice an affair. He there also proposed a proper method for adjusting this point, when he says, *If any consequence of this kind could be drawn from actual measuring, a degree of latitude should be measured at the Equator, and a degree of longitude likewise measured there; and a degree very northerly, as for example, a whole degree might be actually measured upon the Baltick sea, when frozen, in the latitude of sixty degrees. There, according to M. Cassini's last supposition, a degree would be 56653 toises; whereas at the Equator it would be of 58019 toises, the difference being 1366 toises, about the two and fortieth part of a degree; which must be sensible: and likewise the degree of longitude would according to him be of 56817 toises, less by 1202, or the forty-eighth part, than a degree of latitude at the same place.*

On this admonition, in 1735, were sent from *France* two sets of mathematicians, members of the Royal Academy of Sciences; one towards the Pole, the other to the Equator; in order to measure at each place the length of a degree on the meridian. The report they brought home quite overset what had been urged in favour of the oblong figure; a degree towards the north, in the latitude of $66^{\circ} 20'$ being found to contain about 57438 toises, and near the Equator but 56750.

This unwelcome news caused again a degree to be measured in *France*, which at length came out to be consonant with those which had been brought from very distant parts of the world. Thus these mathematicians confirmed by painful observations, what *Sir Isaac Newton* had, as *M. de Maupertuis* used to say, determined in his elbow-chair; *Sir Isaac* making the length of a degree under the Pole to be 57382, and at the Equator 56637 toises. And perhaps no observations can be exact enough to determine this matter more precisely.

But I will now mention some of the foreign writers on Navigation.

Italian and
French writers
on Navigation.

At *Rome*, in 1607, came forth a treatise, intitled, *Nautica Mediterranea*, written in *Italian* by *Bartolomeo Crescenti*, the Pope's engineer. The author misses no opportunity of exposing the errors of *Medina*; but scarce gives any thing of his own, except a machine to serve to measure the way a ship has made.

As the Jesuits have treated of most branches of learning, so this Art has not been beneath their consideration; the three following authors having been of their society.

Father
George Fournier.
A. D. 1633.

At *Paris*, in 1633, Father *George Fournier* published an *Hydrography*, principally relating to Navigation. The author would fain persuade us, that a native of *Dieppe* was the first person that had corrected the plane-chart; and that the *Hollanders* had learnt of the *French* the art of making charts so corrected; whereas charts so corrected had been engraved long before at *Amsterdam* by *Jodocus Hondius* and others.

John Baptist Riccioli, in his *Geographia & Hydrographia Reformata*, printed at *Belegna* in 1661, inserts a Treatise of Navigation, collecting his materials from almost every writer, as he does in his *Almagest* and *Chronology*, which is indeed the chief merit of his works. Riccioli.
A. D. 1661.

Father *Claudius Francis Milliet de Chales*, a learned Jesuit, wrote on this subject after a more masterly manner, both in his *Curjus Mathematicus*, first printed at *Lyons* in 1674, and in a French Treatise, published in 1677, intitled, *L'Art de Naviguer, démontré par Principes*. Father de Chales. A. D. 1677.

These three authors, besides treating of the different kinds of sailing, abound in methods for taking of altitudes, finding the variation, and estimating the way a ship makes, &c. They also describe a machine resembling that of *Crescenti*. *Riccioli* gives a very faulty measure of the Earth, made by himself; and *Dechaies* advises the use of a pendulum in reckoning by the log-line, as also of wheel-work for the same purpose, as *Phillips* and *Cole* had done before.

But there were writers in *France* between *Fournier* and *Dechaies*. As in 1666, and the following years, had been printed at *Dieppe* several tracts, handling different parts of Navigation, composed by *M. G. Denys*, which have been often reprinted. Monsieur G. Denys.
A. D. 1666.

And, in 1671, the *Sieur Blondel Saint Aubin* published a book, called, *L'Art de Naviguer par le Quartier de Réduction*, describing an instrument much in use amongst the French sailors, by which may be performed, as by the sinical quadrant, the operations of Navigation, though not much more speedily than by the *Traverse Table*, and not near so accurately. He also published, in 1673, his *Trésor de la Navigation*, where the Art is well treated of, particularly by calculations. M. Blondel Saint Aubin.
A. D. 1671.

M. Savérien in his *Marine Dictionary*, printed at *Paris* in 1758, says, that *M. Daffier* seems to have been the first of the French writers that shewed the use of *Gunter's scales* [*échelles Angloises*] in his *Pilote expert*, printed in 1683. M. Daffier.
A. D. 1683.

At *Paris*, ten years after, was published the first part of a magnificent work, intitled, *Le Neptune François*, by order of the French king, consisting of sea-charts, according to *Wright's* scheme, made from the latest observations, and reviewed by Messrs. *Pene*, *Cassini*, and others. As this contained the charts of *Europe* only, there were added others of different parts of the world, printed the same year at *Amsterdam*. The whole was preceded by a discourse of *M. Sauvour*, who had formed some of the charts, where he shews how to perform the problems of Astronomy and Navigation by scales; which discourse had been published by itself at *Paris*, in 1692. Le Neptune François.
A. D. 1693.

M. John Bouguer composed by authority his *Traité Complet de la Navigation*, first printed in 1698, which was well received, as containing most of the practices then known; and Father *Pézenas*, Jesuit and Royal Professor of Hydrography at *Marseilles*, published there, in 1733, a tract, called, *E'léments de Pilotage*; and at *Avignon*, in 1741, a larger work, intitled, *Pratique du Pilotage*. Mr. John Bouguer.
A. D. 1698.
Father Pézenas. A. D. 1733.

* It is only a kind of skeleton of *Wright's* universal map.

This author shews how to find the meridional parts by the *Artificial Tangents*, an old discovery amongst us, declared so long ago as 1645, in *Norwood's Epitome*; he also has been industrious in translating several of our mathematical books into *French*.

M. Peter Bouguer.

A. D. 1753.

But, in 1753, M. *Peter Bouguer*, son of the former, published a very elaborate treatise on this subject, intitled, *Nouveau Traité de Navigation*; which is written sensibly, the author being an excellent mathematician, and famous for other productions. He there gives a variation-compass⁴ of his own invention, and attempts to reform the log, as he had done in the Memoirs of the Academy of Sciences for 1747. He is also very particular in determining the lunations more accurately than by the common methods, and in describing the corrections of the dead reckonings.

M. de la Caille. A. D.

1760.

M. le Moonicier. A. D. 1766.

The excellent astronomer, M. *de la Caille*, in 1760, made an edition of M. *Bouguer's* book, which he somewhat abridged and improved.

In 1766, at *Paris*, came out a treatise, with this title, *Abîgé du Pilotage; divisé en deux parties. Où on traite principalement des Amplitudes, des Loxodromies, dans l'hypothèse de la Sphère et de Sphéroïde, des marées, des variations de l'aîman*.

The former part of this book was first published in 1693. Here the whole is improved by M. *le Moonicier*.

Spanish authors.

Antonio de Nairera. A. D. 1628.

Manuel Pimentel. A. D. 1712.

Don Jorge Juan. A. D. 1757.

Though the *Spaniards* were the earliest writers on Navigation, yet they were very backward to adopt its improvements. Indeed *Antonio de Nairera* published at *Lisbon*, in 1628, a treatise, intitled, *Navegacion speculativa y practica*; where, though the author rectifies the tables of the Sun and Fixed Stars from *Jycho Brabé's* observations, he proceeds no farther in the theory of Navigation than what had been advanced by *Nonius*, as followed by *Cespedes*. But of late, in 1712, was printed at the same place, *Arte de Navegar*; por *Manuel Pimentel*; where is shewn the use of *Wright's* chart, which, in imitation of the *French*, the author calls *Charta Reduxida*. He likewise describes *Davis's* quadrant, and mentions *Norwood* and *Picard's* measures of the Earth. In 1757 was printed at *Cadiz*, a Treatise, intitled, *Compendio de Navegacion para el uso de los Cavalleros Guardias Marinas*, written by the ingenious gentleman mentioned above, *Don Jorge Juan*. This is a good performance, delivering very distinctly the several parts of the Art, as now improved. Some things are here omitted, that usually occur in books on this subject; but for the knowledge of such particulars, references are made to tracts composed expressly for the use of the society of gentlemen destined for the sea-service.

Bouguer and *Jorge Juan*, describe and commend the method of dividing instruments for taking of angles, published by *Peter Vernier*, in a treatise, intitled,

⁴ Many of these sorts of compasses have been proposed at different times: and, particularly, by M. *Buache*, in the Memoirs of the French Academy of Sciences for 1752, page 327; by Captain *Christopher Middleton*, in the Philosophical Transactions, No. 450, Ann. 1738; and by Dr. *Knight*, as improved by the ingenious Mr. *John Smooten*, *ibid.* No. 495, Ann. 1750.

La Construction, &c. du quadrant nouveau, printed at *Brussels*, in 1631. This division, (an improvement of *Curtius's*, as that of *Feretus* is of the division by diagonals *) readily flows from the first *Lemma* of *Clavius's* Treatise on the *Astronomie* †, as has been observed by *Pézenas*, in a book he published at *Avignon* in 1762, intitled, *Astronomie des Marins*.

As to their treating of *Wright's* chart, I mentioned above *Snellius* and *Metius*. To an edition, in 1665, of *Vlacq's* small tables of logarithms, &c. is added, by *Abraham de Grue* ‡, a table of meridional parts, of which he explains the use, with other parts of Navigation, in his *Course of Mathematic*, written in *Dutch*, and printed at *Amsterdam*, in 1676, as had been done by *John Viret*, in his *Flambeau reluisant, ou Trésor de la Navigation*, at the same place, in 1677.

The *Dutch* are great navigators, and have been famous for their *Atlases*, before which they usually publish Treatises of Navigation, as has been already observed, Dutch au-
thors. The oldest I have seen of these, was published at *Leyden* in 1584, intitled, *Spiegel der Zee-Vaert*, (or *Mirror of Navigation*,) by *Lucas Jansz Wagenaer*. In their later *Atlases* there is described an instrument to be used after the manner of *Davis's* quadrant, but in which, instead of circular arches, are substituted straight lines.

Notwithstanding all the improvements hitherto mentioned, the *sea- reckonings*, though kept by persons that were deemed very skilful mariners, have been often found widely different from the truth. But this often happens through negligence, as I have heard *Dr. Halley*, who had used the sea, say.

These errors would be avoided, if from time to time the latitude and longi- Of finding
the Latitude. tude could be determined. The first is generally obtained by the meridian altitude and declination of the Sun being given. The declination is got by the help of tables of the Sun, with an easy trigonometrical process.

The first could not be very exact, before the famous *Kepler* had determined the true form of the Earth's orbit †. Hence were fabricated his *Tabula Rudolphina*. After these tables of *Kepler*, those of *Mr. Thomas Street* were in great request *. But they now yield to those of *Dr. Halley* ‡; which, however, will want to be corrected hereafter: For, as *Sir Isaac Newton* has shewn that all bodies mutually attract one another, the Earth will be disturbed in its motion by the actions of some of the other planets.

To find the longitude is a much more difficult affair. For this end, at present, the societies of learned men in *Europe* offer, from time to time, rewards Of finding
the Longi-
tude. to such as shall best treat of particular subjects in *Mathematicks* or *Physicks*. Some of these have been relating to Navigation, when *Poleni*, *Bernoulli*, *Bouguet*,

* See *Mr. Robins's Mathematical Tracts*, where these divisions are largely treated of.

† First printed at *Rome*, in 1693.

‡ In his treatise de *Motu Martis*, in 1609.

§ In his *Astronomia Carolina*, in 1661.

|| In 1719, many years after their construction.

and others have obtained the prizes. And, it is hoped, this encouragement may contribute to the advancement of the Art.

Eclipses of the Moon were used of old; and *Kepler* recommended those of the Sun as preferable^a.

The satellites of *Jupiter* were no sooner discovered by the great *Galileo*^b, than the frequency of their eclipses recommended them for this purpose; and, amongst those who attempted this subject, none were more successful than *Sig- nor Dominic Cassini*.

This great astronomer, in 1688, published at *Bologna* tables for calculating the appearances of their eclipses, with directions for finding thence the longitudes of places; and being invited to *France* by *Lewis* the Fourteenth, he there published correcter tables in 1693. But the mutual attractions of the satellites on one another rendering their motions excessively irregular, the tables soon run out; inasmuch, that they require to be renewed from time to time, which is performed by ingenious persons, as *Dr. James Pound*, *Dr. James Bradley*^c, *M. Cassini* the son, and *M. Peter Wargentin*^d; so that now many of the common *Almanaks* set down when these eclipses happen throughout the year.

Mr. Nevil Maskelyne, the Royal Astronomer, has published annually, ever since 1767; by order of the Commissioners of the Longitude, a *Nautical Almanack*, and *Astronomical Ephemeris*, containing not only the eclipses of the satellites, but also many other tables, to enable the mariner to determine the longitude. The same thing has been attempted likewise in the *Connoissance des Temps*, especially for these latter years, by *M. de la Lande*.

The large reward granted by the *British Parliament* for a practical way of discovering the Longitude at sea, has put many upon the search; inasmuch that several idle and absurd schemes have been offered by ignorant and wrong-headed men. But the perfecting the methods proposed long ago by *John Werner* and *Gemma Frisius*, seems at present to engage the attention of the publick.

The theory of the Moon, though much amended by the noble *Tycho Brahe* and *Mr. Jeremy Horrox*^e, was found to be insufficient to answer this end. But

^a *Tabule Rudolphæ*, printed at *Ulm* in 1627, cap. xvi & xxii.

^b In his *Synteresis Nunciis*, first printed at *Venice* in 1610.

^c He succeeded *Dr. Halley* as Astronomer Royal at *Greenwich*, where he made a complete set of *Astronomical Observations*, which, as they are very accurate, it is hoped will not be lost. He became famous on publishing an account of an apparent motion in the fixed stars, which was immediately exhibited by the great mathematician *Dr. Brook Taylor*, according to the exact theory of the Earth's motion. See *Mr. Robin's Mathematical Tracts*, vol. II. page 276.

^d *Wargentin's* tables are much esteemed; they were first published at *Stockholm*, in the *Acta Societatis Regiæ Scientiarum Upsalensis*, for the year 1741; but there has since that time been a second edition of them, more correct than the former, made from a new copy of the author's, at *Paris*, in 1759, by *M. de la Lande*.

^e This great genius died in 1644, scarce 23 years old. See his *Opera Posthuma*, published by the famous *Dr. John Wallis* at *London*, in 1673. *Horrox* first observed the Transit of *Venus* over the Sun, in 1639. He wrote an account of this Phenomenon, which was published by the great astronomer *Hewelius*, at *Danzic*, in 1661.

the

the causes of her various irregularities having been discovered by Sir *Isaac Newton*, and her theory thence improved beyond expectation, gave great hopes of success; which are now increased by the farther improvements made on his principles by *M. Euler* and others; as they have enabled *Mr. Tobias Mayer* of *Gottingen*¹, to compose at length tables agreeing to the Moon's motion in every part of her orbit, with very surprising exactness. And this ingenious person has left behind him tables still more exact², for which the Parliament have rewarded his widow, as also *Mr. Euler*. These tables were published in 1770, by *Mr. Maskelyne*. Mayer's Lunar Tables. A. D. 1770.

As to the method of *Gemma Frisius*, *M. Huygens* was persuaded it might be accomplished by his inventions of pendulum-clocks and watches; a description of the first of which he published in a small tract, printed at the *Hague*, in 1658, and of the second, as improved, in the *Journal des Sçavans* for the month of *February*, 1675. And great expectations of success had been raised from some trials made in a voyage with these watches of the first construction, by *Major Holmes*; an account whereof is given in the *Philosophical Transactions*, *Ann.* 1669. But the various accidents those movements are liable to, soon caused that way to be laid aside.

Notwithstanding which, the ingenious *Mr. John Harrison* has, for many years past, employed himself in contriving a machine, that shall be free from all imaginable inconveniences; and his endeavours have been so well approved-of by gentlemen of the greatest knowledge in these subjects, that the commissioners for the longitude have thought fit to allow him some gratifications for his pains. He has been since farther considered, upon disclosing the internal structure of his instrument³, and he still perseveres in pushing for the whole reward.

The difficulty of making observations at sea with sufficient exactness, was feared to be unformountable; but attempts have been made to overcome it. In the History of the Royal Society, at page 246, is mentioned an invention in these words: *A new instrument for taking angles by reflection, by which means the eye at the same time sees the two objects both as touching the same point, though distant almost to a semicircle; which is of great use for making exact observations at sea.* A figure of this instrument, drawn by *Dr. Hook*, the inventor, is given in the Doctor's posthumous works, with a description, at page 503. But here as one reflection only is made, it would not answer the purpose. However, this was at last effected by *Sir Isaac Newton*, who communicated to *Dr. Halley* a paper of his own writing, containing a description of an instrument with two reflections, which soon after the Doctor's death was found among his papers by *Mr. Jones*, who communicated it to the Royal Society; and it was published in the *Philosophical Transactions*, No. 465, *Ann.* 1742. Of making Observations at Sea.

¹ *Com. Societ. Reg. Gottingen.* tom. II. p. 283.

² See his *Eulogium* in the *Nova Acta Eruditorum* for *March* 1762.

³ May not a machine be produced that shall answer to the words, but not to the intention of the act of parliament? That act was procured by *Whiston* and *Dutton*, two very weak men; and their project was treated with the utmost scorn.

How it happened that Dr. *Halley* never mentioned this in his life-time, is very extraordinary; seeing that *John Hadley*, Esq^r had described, in No. 430, *Ann.* 1731, an instrument grounded on the same principles, which is so well esteemed, that our shops abound with them, accommodated with *Vernier's* division, as they are made by our most skilful workmen; and they are now in general use amongst the skilful seamen of most of the maritime nations.

Of finding
the Horizon
at Sea.

Though *Medina's* method for finding the place of the horizon was absurd; yet, for this end, several plausible ones have been proposed by ingenious persons, as Messrs. *Elton*, *Hadley*, *Godfrey*, and *Leigh*; and that chiefly by applying a level to *Davis's* quadrant. Their devices are described in the *Philosophical Transactions* for 1732, 33, 34, and 37. And, lastly, an *Horizontal Top*, invented by the late Mr. *Serfon*, (who was unfortunately lost at sea on board the *Victory* man of war,) has been approved-of, and published by Mr. *Smeaton* in the *Philosophical Transactions*, vol. XLVII. for 1752. part ii. page 352.

Some methods used for obtaining the place of the horizon, and of observing with Mr. *Hadley's* *Reflecting Sector*, are described by Mr. *Robertson*, in his *Elements of Navigation*; which treatise has deservedly met with the approbation of the publick.

Thus have I endeavoured to trace out the principal steps by which the Art of Navigation has advanced to its present height; nor without hopes that the attempt may not prove altogether unacceptable to those, whose business or curiosity lead them to be acquainted with this very useful branch of the Mathematicks: on the successful practising of which depends, in an especial manner, the flourishing state of our country.

This Dissertation, written at first by desire, is now reprinted with alterations. Though I may be thought to have dwelt too long on some particulars not directly relating to the subject; yet, I hope, that what is so delivered, will not be altogether unentertaining to the candid reader. As to any apology for having handled a matter quite foreign to my way of life, I shall only plead, that when very young, I lived in a sea-port town, which made me eagerly desirous of becoming acquainted with an Art that could enable the Mariner to cross the wide and pathless ocean with security, and arrive at his desired harbour.

² Mr. *Hadley* being well acquainted with Sir *Isaac Newton*, might have heard him say, *Hooke's* proposal could be perfected by means of a double reflection. However, Mr. *Hadley*, being a very ingenious person, might have hit on the same thought, as well as Mr. *Godfrey* of Pennsylvania; to whom the invention of this admirable instrument has been ascribed by some gentlemen of that colony: This is not the only case, wherein different persons have produced similar inventions.

London.

JAMES WILSON.

A PRO-

A
P R O B L E M

CONCERNING

THE CONSTRUCTION OF A TRIANGLE,

BY MEANS OF A CIRCLE ONLY,

Though the Properties of its Sides lead to an Equation of the Tenth Order;

PROPOSED AND SOLVED

BY *JAMES GLENIE, ESQ.*

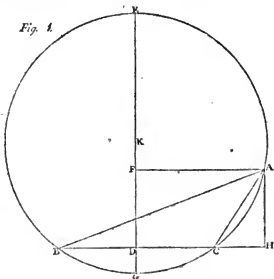
LATE A LIEUTENANT IN THE CORPS OF ENGINEERS.

PROBLEM

Concerning the CONSTRUCTION of a TRIANGLE, by means of a Circle only, though the Properties of its Sides lead to an Equation of the Tenth Order; Proposed and Solved by JAMES GLENIE, Esq. late a Lieutenant in the Corps of Engineers.

P R O B L E M.

Article 1. **O**N the given line BC, as a base, to describe, by drawing only right lines and circles, a triangle BAC, of which the perpendicular altitude shall be equal to half the base BC, and the sides of which, to wit, AB and AC, shall be of such magnitudes that the sum of their cubes shall be equal to three times the cube of the base BC.



CON-

CONSTRUCTION, *by* JAMES GLENIE, *E/q.*

Bisect the base BC in the point D ; and from D draw the right line DE at right angles to BC , and equal to three times DC ; and prolong the said line DE on the other side of BC to the point G , so that DC shall be to DG in the same proportion as DE is to DC , or that DC shall be equal to three times DG . Then bisect the line EG in the point K ; and, with K as a center, and KE , or KG , as a radius, describe a circle. Then, because DC is a mean proportional between ED and DG , the two segments of the diameter EG , it is evident that the circumference of this circle will pass through the point C , and consequently through the opposite point B , which is at the same distance as C from the diameter EG . In the next place, from the point D in the diameter EG , take the line DF equal to DC ; and from the point F draw the line FA parallel to the base BC and meeting the circumference of the circle $ECGB$ in the point A . And, lastly, from the point A draw the right lines AB and AC to the extremities of the given base BC .

Then will the triangle ABC be the triangle required, or the triangle of which the perpendicular altitude FD is equal to half the base BC , and of which the two sides AB and AC are of such magnitudes that the sum of their cubes shall be equal to three times the cube of the given base BC . $Q. E. F.$

DEMONSTRATION.

Art. 2. For, in the 1st place, the perpendicular altitude of this triangle, to wit, DF , is, by the construction, equal to DC , or half the base BC ; which is the first condition required in the Problem.

And, 2dly, the sum of the cubes of the two sides AB and AC will be equal to three times the cube of the base BC ; as may be shewn in the manner following.

Let DC , or half the base BC , be denoted by the letter a . And we shall have $BC = 2a$, and consequently $BC^3 (= 2a^3) = 8a^3$, and $3BC^3 (= 3 \times 8a^3) = 24a^3$. We must therefore shew that $AB^3 + AC^3$ will be equal to $24a^3$.

Art. 3. Now let the right line AH be drawn at right angles to the base BC , produced, and meeting it in the point H .

Then, because ACH and ABH are right-angled triangles, having a right angle

angle at H, we shall, by El. 1, 47, have $AB^2 = BH^2 + AH^2 = BH^2 + DF^2 = BH^2 + DC^2 = BH^2 + aa = \overline{BD + DH}^2 + aa = BD^2 + 2BD \times DH + DH^2 + aa = aa + 2a \times DH + DH^2 + aa = 2aa + 2a \times DH + DH^2 = 2aa + 2a \times DH + FA^2 = 2aa + 2a \times FA + FA^2 =$ (by the nature of the circle,) $2aa + 2a \times FA + EF \times FG = 2aa + 2a \times FA + EF \times FD + DG =$ (by the construction,) $2aa + 2a \times FA + 2FD \times FD + \frac{FD^2}{3} = 2aa + 2a \times FA + 2a \times a + \frac{a^2}{3} = 2aa + 2a \times FA + 2a \times \frac{4a}{3} = 2aa + 2a \times FA + \frac{8aa}{3} = \frac{6aa}{3} + 2a \times FA + \frac{8aa}{3} = \frac{6aa}{3} + 2a \times \sqrt{\frac{8aa}{3}} + \frac{8aa}{3} = \frac{14aa}{3} + 2a \times a \times \frac{\sqrt{8}}{\sqrt{3}} = \frac{14aa}{3} + 2aa \times \frac{\sqrt{3} \times \sqrt{8}}{\sqrt{3} \times \sqrt{3}} = \frac{14aa}{3} + 2aa \times \frac{\sqrt{24}}{3} = \frac{14aa}{3} + \frac{2aa}{3} \times \sqrt{24} = \frac{14aa}{3} + \frac{aa \times \sqrt{4} \times \sqrt{24}}{3} = \frac{14aa}{3} + \frac{aa \times \sqrt{96}}{3} = aa \times \frac{14 + \sqrt{96}}{3}$. Therefore AB will be $= a \times \frac{\sqrt{14 + \sqrt{96}}}{\sqrt{3}}$, and consequently AB^3 will be $= a^3 \times \frac{14 + \sqrt{96}}{3} \times \frac{\sqrt{14 + \sqrt{96}}}{\sqrt{3}}$.

And we shall have $AC^2 =$ (by El. 1, 47,) $AH^2 + CH^2 = DC^2 + CH^2 = aa + CH^2 = aa + \overline{DH - DC}^2 = aa + \overline{FA - DC}^2 = aa + FA^2 - 2FA \times DC + DC^2 = aa + FA^2 - 2DC \times FA + DC^2 = aa + FA^2 - 2a \times FA + aa = 2aa + FA^2 - 2a \times FA = 2aa + \frac{8aa}{3} - 2a \times FA = 2aa + \frac{8aa}{3} - 2a \times \sqrt{\frac{8aa}{3}} = 2aa + \frac{8aa}{3} - 2a \times a \times \frac{\sqrt{8}}{\sqrt{3}} = 2aa + \frac{8aa}{3} - 2aa \times \frac{\sqrt{3} \times \sqrt{8}}{\sqrt{3} \times \sqrt{3}} = 2aa + \frac{8aa}{3} - 2aa \times \frac{\sqrt{24}}{3} = 2aa + \frac{8aa}{3} - aa \times \frac{\sqrt{96}}{3} = \frac{6aa}{3} + \frac{8aa}{3} - aa \times \frac{\sqrt{96}}{3} = \frac{14aa}{3} - aa \times \frac{\sqrt{96}}{3} = aa \times \frac{14 - \sqrt{96}}{3}$. Therefore AC will be $= a \times \frac{\sqrt{14 - \sqrt{96}}}{\sqrt{3}}$, and consequently AC^3 will be $= a^3 \times \frac{14 - \sqrt{96}}{3} \times \frac{\sqrt{14 - \sqrt{96}}}{\sqrt{3}}$.

Therefore $AB^3 + AC^3$ will be $= a^3 \times \frac{14 + \sqrt{96}}{3} \times \frac{\sqrt{14 + \sqrt{96}}}{\sqrt{3}} + a^3 \times \frac{14 - \sqrt{96}}{3} \times \frac{\sqrt{14 - \sqrt{96}}}{\sqrt{3}}$.

Art. 4. Now $\frac{14 + \sqrt{96}}{3}$ is $= \frac{12 + 2 \times \sqrt{24} + 2}{3} = \frac{12 + 2 \times \sqrt{8 \times \sqrt{3}} + 2}{3} =$
 $\frac{12 + 2 \times \sqrt{4 \times \sqrt{2} \times \sqrt{3}} + 2}{3} = \frac{12 + 2 \times 2 \times \sqrt{2} \times \sqrt{3} + 2}{3} = \frac{2\sqrt{3} + \sqrt{2}^2}{3}.$

Therefore $\frac{\sqrt{14 + \sqrt{96}}}{\sqrt{3}}$ will be $= \frac{2\sqrt{3} + \sqrt{2}^2}{\sqrt{3}} = 2 + \frac{\sqrt{2}}{\sqrt{3}}$. And consequently
 AB, or $a \times \frac{\sqrt{14 + \sqrt{96}}}{\sqrt{3}}$, will be $= a \times \left[2 + \frac{\sqrt{2}}{\sqrt{3}} \right]$, and AB² will be $=$
 $a^2 \times \left[2 + \frac{\sqrt{2}}{\sqrt{3}} \right]^2 = a^2 \times 8 + \frac{12\sqrt{2}}{\sqrt{3}} + 6 \times \frac{2}{3} + \frac{2\sqrt{2}}{3\sqrt{3}}.$

And in like manner $\frac{14 - \sqrt{96}}{3}$ is $= \frac{12 - 2\sqrt{24} + 2}{3} = \frac{12 - 2 \times \sqrt{8 \times \sqrt{3}} + 2}{3} =$
 $= \frac{12 - 2 \times \sqrt{4 \times \sqrt{2} \times \sqrt{3}} + 2}{3} = \frac{12 - 2 \times 2 \times \sqrt{2} \times \sqrt{3} + 2}{3} = \frac{2\sqrt{3} - \sqrt{2}^2}{3}.$

Therefore $\frac{\sqrt{14 - \sqrt{96}}}{\sqrt{3}}$ will be $= \frac{2\sqrt{3} - \sqrt{2}^2}{\sqrt{3}} = 2 - \frac{\sqrt{2}}{\sqrt{3}}$. And consequently
 AC, or $a \times \frac{\sqrt{14 - \sqrt{96}}}{\sqrt{3}}$ will be $= a \times \left[2 - \frac{\sqrt{2}}{\sqrt{3}} \right]$, and AC² will be $=$
 $a^2 \times \left[2 - \frac{\sqrt{2}}{\sqrt{3}} \right]^2 = a^2 \times 8 - \frac{12\sqrt{2}}{\sqrt{3}} + \frac{6 \times 2}{3} - \frac{2\sqrt{2}}{3\sqrt{3}}.$

Therefore AB² + AC² will be $= a^2 \times \left[8 + \frac{12\sqrt{2}}{\sqrt{3}} + \frac{6 \times 2}{3} + \frac{2\sqrt{2}}{3\sqrt{3}} + \right.$
 $a^2 \times 8 - \frac{12\sqrt{2}}{\sqrt{3}} + \frac{6 \times 2}{3} - \frac{2\sqrt{2}}{3\sqrt{3}} \left. \right] = a^2 \times 8 + 8 + \frac{6 \times 2}{3} + \frac{6 \times 2}{3} =$
 $a^2 \times 16 + 4 + 4 = 24a^2$, or $3 \times 8a^2$, or $3 \times BC^2$. Q. E. D.

A S C H O L I U M.

Art. 5. In the course of the foregoing demonstration it was shewn that AB²,
 or $aa \times \frac{14 + \sqrt{96}}{3}$ was equal to $aa \times \left[2 + \frac{\sqrt{2}}{\sqrt{3}} \right]^2$, and consequently that AB,
 or $a \times \frac{\sqrt{14 + \sqrt{96}}}{\sqrt{3}}$, or the square-root of $aa \times \frac{14 + \sqrt{96}}{3}$, was equal to
 $a \times \left[2 + \frac{\sqrt{2}}{\sqrt{3}} \right]$. But it may be asked, how, when we had found AB² to be
 equal to $aa \times \frac{14 + \sqrt{96}}{3}$, we came to discover that AB, or the square-root of

$aa \times$

as $\times \frac{14 + \sqrt{96}}{3}$, would be equal to $a \times \sqrt{2 + \frac{\sqrt{2}}{\sqrt{3}}}$, or that the square-root of the fraction $\frac{14 + \sqrt{96}}{3}$ would be the binomial quantity $2 + \frac{\sqrt{2}}{\sqrt{3}}$. Now this might have been discovered in the following manner.

An Investigation of the Binomial Square-Root of the Fraction $\frac{14 + \sqrt{96}}{3}$.

Art. 6. In the first place we may observe that the fraction $\frac{14 + \sqrt{96}}{3}$ is ($= \frac{2 \times 7 + \sqrt{4 \times 24}}{3} = \frac{2 \times 7 + 2 \times \sqrt{24}}{3} = \frac{2}{3} \times \sqrt{7 + \sqrt{24}}$). Therefore the square-root of the fraction $\frac{14 + \sqrt{96}}{3}$ will be $= \frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{7 + \sqrt{24}}$. We will therefore now seek for the square root of the binomial quantity $7 + \sqrt{24}$; which may be found in the following manner.

Art. 7. In the first place we may observe that, if the binomial quantity $7 + \sqrt{24}$ has another binomial quantity for its square-root, (which is not the case with every such binomial quantity as $7 + \sqrt{24}$, of which one member is a furd quantity, or the square-root of a non-square number,) one member at least of the said binomial square-root must be a furd quantity, or the square-root of a non-square number. And we cannot be certain before-hand that both the members of such binomial square-root will not be such furd quantities. But we are sure that both the members of such binomial square-root cannot be rational numbers: for, if they were so, their sum, or the whole of the said binomial square-root, would also be a rational number; and so would its square; and consequently it could not be equal to the binomial quantity $7 + \sqrt{24}$, which is an irrational quantity. It is therefore certain that either one, or both, the members of the said binomial quantity, which is equal to the square-root of the binomial quantity $7 + \sqrt{24}$, must be furd quantities. Let the greater of the two members of this supposed binomial square-root be called x , and the lesser be called y , so that $x + y$ shall be equal to the square-root of $7 + \sqrt{24}$.

Then will $xx + 2xy + yy$ (which is equal to the square of $x + y$), be equal to $7 + \sqrt{24}$.

Now, whether x and y are, both of them, furd quantities, that is, square-roots of non-square rational numbers, or only one of them is a furd of that kind, and the other is a rational number; their squares xx and yy must, in both cases, be rational numbers. Therefore the sum of these squares, or the quantity $xx + yy$, must also be a rational number. Let us therefore suppose that

X x 2

this sum is equal to 7, or the rational member of the binomial quantity $7 + \sqrt{24}$, to which the whole trinomial quantity $xx + 2xy + yy$ is equal, and consequently that $2xy$, or the remaining part of that trinomial quantity, is equal to $\sqrt{24}$, or to the surd member of the said binomial quantity $7 + \sqrt{24}$. This supposition is not inconsistent with any thing before supposed; and it seems likely to enable us to find the values of x and y , of which we are in search.

Art. 8. Now, if $xx + yy$ is equal to 7, we shall have $xx = 7 - yy$. And, because x is supposed to be greater than y , it follows that xx must be greater than yy , and consequently greater than half of $xx + yy$, or than half of 7; and yy will be less than half of $xx + yy$, or than half of 7.

Further, since $2xy$ is $= \sqrt{24}$, the square of $2xy$ will be equal to 24; that is, $4xxyy$ will be $= 24$; and consequently $xxyy$ will be $(= \frac{24}{4}) = 6$, and xx will be $= \frac{6}{yy}$.

But it has been shewn that xx is $= 7 - yy$.

Therefore $\frac{6}{yy}$ will be $= 7 - yy$, and consequently 6 will be $= yy^2 - y^4$. Therefore (subtracting both sides from $\frac{49}{4}$, or the square of $\frac{7}{2}$) we shall have $\frac{49}{4} - 6 = \frac{49}{4} - (yy^2 - y^4) = \frac{49}{4} - yy^2 + y^4$, or $\frac{49}{4} - \frac{24}{4} = \frac{49}{4} - yy^2 + y^4$, or $\frac{25}{4} = \frac{49}{4} - yy^2 + y^4$. Therefore, (extracting the square-roots of both sides,) we shall have $\frac{5}{2} = \frac{7}{2} - yy$, and consequently $\frac{5}{2} + yy = \frac{7}{2}$, and $yy = \frac{7}{2} - \frac{5}{2} = \frac{2}{2} = 1$. Therefore y , or the square-root of yy , will be equal to the square-root of 1, that is, to 1 likewise; and xx (which is equal to $7 - yy$,) will be $= 7 - 1 = 6$, and consequently x will be $= \sqrt{6}$.

Therefore $x + y$ will be $= \sqrt{6} + 1$, or the square-root of the binomial quantity $7 + \sqrt{24}$, will be $= \sqrt{6} + 1$. Q. E. I.

Art. 9. Now "that $\sqrt{6} + 1$ is truly the square-root of the binomial quantity $7 + \sqrt{24}$ " will easily appear by squaring it. For the square of $\sqrt{6} + 1$ is $6 + 2\sqrt{6} + 1 (= 7 + 2\sqrt{6} = 7 + \sqrt{4} \times \sqrt{6} = 7 + \sqrt{4 \times 6}) = 7 + \sqrt{24}$. Q. E. D.

Art. 10. Since the square-root of the binomial quantity $7 + \sqrt{24}$ is $= \sqrt{6} + 1$, the square-root of $\frac{14 + \sqrt{96}}{3}$ (which has been shewn, in art. 6, to be $=$

$$= \frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{7 + \sqrt{24}} \text{ will be } = \frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{6 + 1} \left(= \frac{\sqrt{2} \times \sqrt{6}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} \times 1 \right) \\ = \frac{\sqrt{2} \times \sqrt{2} \times \sqrt{3}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} = 2 + \frac{\sqrt{2}}{3}. \quad \text{Q. E. I.}$$

An Investigation of the Binomial Square-Root of the Fraction $\frac{14 - \sqrt{96}}{3}$.

Art. 11. In like manner we may investigate the square-root of the fraction $\frac{14 - \sqrt{96}}{3}$, which we shall find to be $2 - \frac{\sqrt{2}}{\sqrt{3}}$. This may be done as follows.

The fraction $\frac{14 - \sqrt{96}}{3}$ is $= \frac{2 \times 7 - 2 \times \sqrt{24}}{3} = \frac{2}{3} \times \sqrt{7 - \sqrt{24}}$; and consequently the square root of $\frac{14 - \sqrt{96}}{3}$ will be $= \frac{\sqrt{2}}{\sqrt{3}} \times$ the square root of the binomial quantity $7 - \sqrt{24}$. We must therefore endeavour to find the square-root of $7 - \sqrt{24}$.

Art. 12. Now let $x - y$ be supposed to be equal to this square-root. And we shall have $xx - 2xy + yy (= x - y)^2 = 7 - \sqrt{24}$.

Further, let $xx + yy$ be supposed to be $= 7$, and consequently $2xy$ to be $= \sqrt{24}$.

Then will xx be $= 7 - yy$, and $4xyy$ will be $= 24$, and consequently xyy will be $(= \frac{24}{4}) = 6$, and xy will be $= \frac{6}{y}$.

Therefore $7 - yy$ will be $= \frac{6}{yy}$, and $yyy - y^4$ will be $= 6$, and $\frac{49}{4} - (yyy - y^4)$ will be $= \frac{49}{4} - 6 = \frac{49}{4} - \frac{24}{4} = \frac{25}{4}$, or $\frac{49}{4} - yyy + y^4$ will be $= \frac{25}{4}$. Therefore $\frac{7}{2} - yy$ will be $= \frac{5}{2}$, and $\frac{7}{2}$ will be $= \frac{5}{2} + yy$, and yy will be $= \frac{7}{2} - \frac{5}{2} = \frac{2}{2} = 1$. Therefore y , or the square-root of yy , will be equal to the square-root of 1, that is, to 1. And xx (which is equal to $7 - yy$) will be $= 7 - 1 = 6$. Therefore x will be $= \sqrt{6}$, and $x - y$ will be $= \sqrt{6} - 1$; that is, the square-root of the binomial quantity $7 - \sqrt{24}$ will be $= \sqrt{6} - 1$. Q. E. I.

Art. 13.

Art. 13. Now "that $\sqrt{6} - 1$ is truly the square-root of the binomial quantity $7 - \sqrt{24}$," will easily appear by squaring it. For the square of $\sqrt{6} - 1$ is $6 - 2\sqrt{6} + 1 (= 7 - 2\sqrt{6} = 7 - \sqrt{4} \times \sqrt{6} = 7 - \sqrt{4 \times 6}) = 7 - \sqrt{24}$. Q. E. D.

Art. 14. Since the square-root of the binomial quantity $7 - \sqrt{24}$ is $= \sqrt{6} - 1$, the square-root of the fraction $\frac{14 - \sqrt{96}}{3}$ (which has been shewn in art. 11 to be $= \frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{7 - \sqrt{24}}$), will be $= \frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{6} - 1 (= \frac{\sqrt{2} \times \sqrt{6}}{\sqrt{3}} - \frac{\sqrt{2} \times 1}{\sqrt{3}} = \frac{\sqrt{2} \times \sqrt{2} \times \sqrt{3}}{\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}}) = 2 - \frac{\sqrt{2}}{\sqrt{3}}$. Q. E. I.

Observations on the foregoing Methods of Investigating the Square-Roots of Irrational Binomial and Residual Quantities, such as $7 + \sqrt{24}$ and $7 - \sqrt{24}$.

Art. 15. When the quantity of which the square-root is sought is a residual quantity consisting of two members, the one rational, and the other irrational, and the irrational member is less than the rational one, and is subtracted from it, (as is the case with the residual quantity $7 - \sqrt{24}$, or, in general, with the residual quantity $A - \sqrt{B}$), the two suppositions made above, to wit, 1st, that $x - y$, or the difference of two other quantities denoted by x and y , is equal to the square-root of such given residual quantity $A - \sqrt{B}$, and, 2dly, that $xx + yy$, or the sum of the squares of the said two other quantities, is equal to A , or the rational member of the said given residual quantity, will always be possible, or compatible with each other.

For, if we suppose the greater quantity x to be at first equal to $\sqrt{A - \sqrt{B}}$, or the square-root of the given residual quantity $A - \sqrt{B}$, and the lesser quantity y to be at first equal to 0; and we further suppose the quantity x to increase gradually from its first magnitude, to wit, $\sqrt{A - \sqrt{B}}$, *ad infinitum*, and the quantity y to increase gradually at the same time from 0 *ad infinitum*, and the increments received by y to be always equal to the contemporary increments received by x ; it is evident that $x - y$, which is at first equal to $\sqrt{A - \sqrt{B}} - 0$, or $\sqrt{A - \sqrt{B}}$, will continue always of the same magnitude, notwithstanding the increase of both x and y , and therefore will be always, as at first, equal to $\sqrt{A - \sqrt{B}}$. And in the course of this increase of x and y from $\sqrt{A - \sqrt{B}}$ and from 0 *ad infinitum*, it is evident that xx will increase from $A -$

$A - \sqrt{B}$ *ad infinitum*, and that xy will increase from 0 *ad infinitum*. Therefore $xx + yy$ will increase gradually at the same time from $A - \sqrt{B} + 0$, or from $A - \sqrt{B}$, *ad infinitum*, and consequently will, at some point of time during its increase, become equal to any given quantity whatsoever that is greater than $A - \sqrt{B}$. But A is greater than $A - \sqrt{B}$. Therefore the quantity $xx + yy$ will, at some point of time during its increase from $A - \sqrt{B}$ *ad infinitum*, become equal to A . And consequently the second supposition, to wit, that $xx + yy$ is equal to A , or the rational member of the given residual quantity $A - \sqrt{B}$, is compatible with the first supposition, to wit, that $x - y$ is equal to $\sqrt{A - \sqrt{B}}$, or the square-root of the said given residual quantity $A - \sqrt{B}$; or, in other words, it is possible for two quantities, x and y , whereof x is the greater, to exist, that shall be of such magnitudes that the excess of x above y shall be equal to $\sqrt{A - \sqrt{B}}$, or to the square-root of the given residual quantity $A - \sqrt{B}$, and that the sum of their squares, or the quantity $xx + yy$, shall be equal to A , or the rational member of the said given residual quantity.

Q. E. D.

Art. 16. And the like two suppositions will also be compatible with each other in the case of a given binomial quantity, as $7 + \sqrt{24}$, or, in general, $A + \sqrt{B}$, when the rational member A is greater than the irrational member $\sqrt{B}$; but not otherwise.

For, whatever be the magnitudes of x and y , the sum of their squares will always be greater than twice their product, or $xx + yy$ will always be greater than $2xy$. Therefore, to the end that $xx + yy$ may be equal to A , and $2xy$ to $\sqrt{B}$, it is necessary that A should be greater than $\sqrt{B}$. And, if we attempt to find the square-root of a binomial quantity $A + \sqrt{B}$, when A is less than $\sqrt{B}$, by making these two suppositions, to wit, 1st, that $x + y$ is equal to the square-root of $A + \sqrt{B}$, and, 2dly, that $xx + yy$ is equal to A , we shall be sure to come to a conclusion that will point out the incompatibility of these two suppositions with each other, or the impossibility of the second supposition, to wit, that $xx + yy$ is equal to A .

Thus, for example, if we endeavour to find out the square-root of the binomial quantity $3 + \sqrt{24}$; and, with this view, we denote the said square-root by the binomial quantity $x + y$, and then make the second supposition above-mentioned, to wit, that $xx + yy$ is equal to 3, and consequently that $2xy$ is equal to $\sqrt{24}$, we shall have $xx = 3 - yy$, and $4xxyy (= (2xy)^2) = 24$, and $xxyy (= \frac{24}{4}) = 6$, and $xx = \frac{6}{y}$. Therefore $3 - yy$ will be $= \frac{6}{y}$, and $3y - y^2$ will be $= 6$. But $3y - y^2$ is $= (3 - y) \times y$, which (by Euclid's Elements, Book 2d, Prop. 5.) can never be greater than the square of half the quantity 3, of which $3 - y$ and y are the component parts, that is, than the square of $\frac{3}{2}$, or than $\frac{9}{4}$, or than $2 + \frac{1}{4}$; which is much less than 6. There-

fore

fore the equation $3yy - y^4 = 6$ is an impossible equation. And consequently the supposition from which it is derived, to wit, "that $xx + yy$ is equal to 3," is a false supposition.

Art. 17. But, so long as A is greater than $\sqrt{B}$, the two suppositions that $x + y$ is equal to the square-root of the binomial quantity $A + \sqrt{B}$, and that $xx + yy$ is equal to A , will be possible suppositions, or will be compatible with each other, though the excess of A above $\sqrt{B}$ should be ever so small; of which the binomial quantity $5 + \sqrt{24}$ may afford us an example.

- For, if we suppose $x + y$ to be equal to the square-root of the binomial quantity $5 + \sqrt{24}$, and $xx + yy$ to be equal to 5, and consequently $2xy$ to be equal to $\sqrt{24}$, we shall have $xx = 5 - yy$, and $4xxyy = 24$, and consequently $xxyy (= \frac{24}{4}) = 6$, and $xx = \frac{6}{yy}$. Therefore $5 - yy$ will be $= \frac{6}{yy}$, and consequently $5yy - y^4$ will be $= 6$, and $\frac{25}{4} - \sqrt{5yy - y^4}$ will be $= \frac{25}{4} - 6$, or $\frac{25}{4} - 5yy + y^4$ will be $= \frac{25}{4} - \frac{24}{4} = \frac{1}{4}$. Therefore (extracting the square-roots on both sides,) we shall have $\frac{5}{2} - yy = \frac{1}{2}$, and $\frac{5}{2} = \frac{1}{2} + yy$, and $yy = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$. Therefore y will be $= \sqrt{2}$, and xx (which is equal to $5 - yy$) will be $= 5 - 2 = 3$, and consequently x will be $= \sqrt{3}$. Therefore $x + y$, or the square-root of the given binomial quantity $5 + \sqrt{24}$ will be $= \sqrt{3} + \sqrt{2}$. Q. E. I.

Art. 18. Now "that $\sqrt{3} + \sqrt{2}$ is truly the square-root of the binomial quantity $5 + \sqrt{24}$," will easily appear by squaring it. For the square of $\sqrt{3} + \sqrt{2}$ is $= 3 + 2\sqrt{3} \times \sqrt{2} + 2 (= 3 + 2\sqrt{6} + 2 = 5 + 2\sqrt{6} = 5 + \sqrt{4 \times 6} = 5 + \sqrt{24}$. Q. E. D.

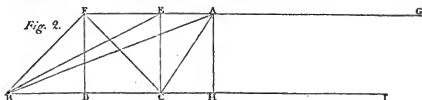
End of the Observations begun in Art. 15.

Art. 19. Hitherto we have only set forth and demonstrated Mr. Glenie's elegant construction of the foregoing Problem, and, in the Scholium subjoined to the demonstration of that construction, have given an explanation of the methods by which, in the course of that demonstration, we discovered the square-roots of the two fractions $\frac{14 + \sqrt{96}}{3}$ and $\frac{14 - \sqrt{96}}{3}$ to be the quantities $2 + \frac{\sqrt{2}}{\sqrt{3}}$ and $2 - \frac{\sqrt{2}}{\sqrt{3}}$, which were necessary steps of the said demonstration: and we afterwards added to the said Scholium some useful observations relating to

to the extractions of the square-roots of irrational binomial and residual quantities, such as $A + \sqrt{B}$ and $A - \sqrt{B}$. But we will now undertake the more arduous task of directly solving Mr. Glenie's Problem without the help of his construction. And with this view it will be convenient to state the Problem over again in the following terms.

A Problem concerning the Constrution of a certain Triangle, the Base and perpendicular Altitude of which are given, and likewise the Proportion of the Sum of the Cubes of its Sides to the Cube of the given Base.

Art. 20. To find the two sides AB and AC of a triangle ABC, fig. 2, of which the base BC is given, and of which the perpendicular altitude is equal to half the base BC, and the two sides AB and AC are of such magnitudes that the sum of their cubes is equal to three times the cube of the base BC.



Before we enter upon the solution of this Problem, or the investigation of the lengths of the sides AB , AC , of the said triangle ABC , it will be proper to shew that a triangle of this description, or of which the sides are so related to the base, is possible. This may be shewn as follows.

A previous Demonstration that it is possible for a Triangle with the Properties described in the Conditions of this Problem to exist.

Let the given base BC be bisected in the point D . And at D draw the line DF at right angles to the base BC and equal to half BC . Then will DF be the perpendicular altitude of the triangle sought.

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From F draw the lines FB and FC to the extremities of the base BC, and likewise draw the line FG parallel to the base BC. Then, since DF is the perpendicular altitude of the triangle sought, it is evident that the vertex A of the said triangle must lie somewhere in the right line FG.

Now let us first inquire whether F is the vertex of the said triangle, that is, whether the point A co-incides with the point F, or whether the sum of the cubes of FB and FC (the two sides of the triangle FBC,) is equal to three times the cube of its base BC.

Now the cube of the base BC is equal to 8 times the cube of DC, or half the base. Therefore, if F is the vertex of the triangle sought, the sum of the cubes of FB and FC ought to be equal to three times $8DC^3$, or to $24DC^3$. But the sum of those two cubes is equal only to $4 \times \sqrt{2} \times DC^3$, or to $4 \times 1.414,213$, &c $\times DC^3$, or to $5.656,852$, &c $\times DC^3$; as may be shewn in the following manner.

By El. 1, 47, FC^2 is $= FD^2 + DC^2 (= DC^2 + DC^2) = 2DC^2$, and FB^2 is $= FD^2 + BD^2 (= BD^2 + BD^2 = 2BD^2) = 2DC^2$. Therefore FC is $= DC \times \sqrt{2}$, and FC^3 is $= DC^3 \times 2\sqrt{2}$; and FB is also $= DC \times \sqrt{2}$, and FB^3 is $= DC^3 \times 2\sqrt{2}$. Therefore $FB^3 + FC^3$ is $(= DC^3 \times 2\sqrt{2} + DC^3 \times 2\sqrt{2}) = DC^3 \times 4\sqrt{2} = DC^3 \times 4 \times 1.414,213$, &c $= 5.656,852$, &c $\times DC^3$. Q. E. D.

Therefore $FB^3 + FC^3$ is less than $24DC^3$, and consequently the triangle FBC is not the triangle required.

Now let us suppose the point A to move from the point F along the line FG (which is parallel to the given base BC,) towards the distant point G. Then it is evident that the line AB will continually increase, from being equal at first to the line FB, *ad infinitum*; and consequently the cube of the line AB will also increase *ad infinitum*. Therefore the sum of the two cubes AB and AC (which, when the point A co-incided with the point F, was equal to $5.656,852$, &c $\times DC^3$), will also increase from the said first magnitude, $5.656,852$, &c $\times DC^3$, *ad infinitum*, and consequently will, at some point of time during its said increase, become equal to any quantity that is greater than $5.656,852$, &c $\times DC^3$; and therefore it will, at one point of time, be equal to $24DC^3$. It is therefore possible for a triangle to exist which shall have the property required by the conditions of the Problem, or of which the two sides AB and AC shall be of such magnitudes that the sum of their cubes shall be equal to $24DC^3$, (or $3 \times 8DC^3$, or $3BC^3$), or three times the cube of the given base BC.

Q. E. D.

ART. 21. But the discovery of the point A in the indefinite line FG, from which if we draw the lines AB and AC to the extremities of the given base BC, the sum of their cubes shall be equal to three times the cube of the given base BC,

BC, or to 24 times the cube of DC, or of half the base BC, is a matter of great difficulty. The following method of discovering it by means of an Algebraick investigation, (which seems to be the most obvious that can be taken,) will bring us to an equation of the tenth power. But, before we enter upon this investigation, it will be proper to make another attempt towards fixing the limits of the position of the point A, or the vertex of the proposed triangle ABC.

From the point C draw the line CE at right angles to the base BC, and meeting the line FG (which is parallel to BC,) in the point E. And from the point E draw the right line EB to the other extremity B of the given base BC. And let us inquire whether the triangle EBC, arising from this construction, will answer the conditions of the Problem, or whether the sum of the cubes of EB and EC will be triple, or more, or less, than triple, of the cube of the base BC, or will be equal to, or greater than, or less than, $(3 \times 8DC^3,$ or) 24-times the cube of DC, or of half the said base.

Now EC is $(= DF) = DC$, and consequently $EC^3 = DC^3$. And, by El. 1, 47, $EB^3 = EC^3 + BC^3 (= DC^3 + BC^3 = DC^3 + 2DC^3 = DC^3 + 4DC^3) = 5DC^3$; and consequently EB is $= DC \times \sqrt{5}$, and therefore EB^3 will be $= DC^3 \times 5\sqrt{5} (= DC^3 \times 5 \times 2.236,067, \&c) = 11.180,335, \&c \times DC^3$. Therefore $EB^3 + EC^3$ will be $= 11.180,335, \&c \times DC^3 + DC^3 = 12.180,335, \&c \times DC^3$; which is less than $24DC^3$, or $3BC^3$. And therefore the triangle EBC will not answer the conditions of the Problem, and the point A, or the vertex of the triangle ABC, which has the property required, will not co-incide with the point E, but will lie somewhat beyond the point E towards the point G, so as to make the line AF be greater than the line EF, and the line AB be greater than EB, and the sum of the cubes of AB and AC be greater than $EB^3 + EC^3$, or $12.180,335, \&c \times DC^3$.

Art. 22. Now let the base BC be produced indefinitely on the side of C to the point I. And let the point A, taken by conjecture in the line FG somewhat beyond the point E towards the point G, be supposed to be the vertex of the triangle sought; and from A draw the right lines AB and AC to the extremities of the given base BC; so that AB and AC shall be supposed to be the sides of the said triangle. And from the same point A draw the right line AH at right angles to the line BCI, and meeting it in the point H. Then, it is evident, if we can ascertain the length of the external line CH, lying between the point H and the given point C, or the nearer extremity of the given base BC, we shall be able to determine the position of the point A, and consequently the lengths of the lines AB and AC of the triangle required by the Problem.

An Algebraick Investigation of the Length of the Line CH.

Art. 23. Now to investigate the length of this line CH, let us denote it by the letter x ; and let the line DC, or half the given base BC, be denoted by the letter a .

Then will BC be $= 2a$, and BH, or BC + CH, will be $= 2a + x$, and consequently BH^2 will be $= 2a + x^2 = 4aa + 4ax + xx$. And AH (which is $= FD = DC$) will be $= a$, and consequently AH^2 will be $= aa$.

Therefore $AH^2 + BH^2$ will be $= aa + 4aa + 4ax + xx = 5aa + 4ax + xx$, and $AH^2 + CH^2$ will be $= aa + xx$.

But, by El. 1, 47, AB^2 is $= AH^2 + BH^2$, and AC^2 is $= AH^2 + CH^2$.

Therefore AB^2 will be $= 5aa + 4ax + xx$, and AC^2 will be $= aa + xx$. Therefore AB will be $= \sqrt{5aa + 4ax + xx}$, and AC will be $= \sqrt{aa + xx}$. Therefore AB^3 will be $= \sqrt{5aa + 4ax + xx}^3$, and AC^3 will be $= \sqrt{aa + xx}^3$.

But, by the conditions of the Problem, $AB^3 + AC^3$ is $= 3BC^3 (= 3 \times 12DC^3 = 3 \times 8DC^3 = 24DC^3) = 24a^3$.

Therefore $\sqrt{5aa + 4ax + xx}^3 + \sqrt{aa + xx}^3$ will be $= 24a^3$.

Therefore, squaring both sides of this equation, we shall have $\sqrt{5aa + 4ax + xx}^6 + 2 \times \sqrt{5aa + 4ax + xx}^3 \times \sqrt{aa + xx}^3 + \sqrt{aa + xx}^6 = 576a^6$.

Art. 24. Now the cube of $5aa + 4ax + xx$ is the septinomial quantity $125a^6 + 300a^5x + 315a^4x^2 + 184a^3x^3 + 63a^2x^4 + 12ax^5 + x^6$, as will appear by the following multiplications.

5aa

$$\begin{array}{r}
5aa + 4ax + xx \\
5aa + 4ax + xx \\
\hline
25a^4 + 20a^3x + 5a^2x^2 \\
+ 20a^3x + 16a^2x^2 + 4ax^3 \\
+ 5a^2x^2 + 4ax^3 + x^4 \\
\hline
25a^4 + 40a^3x + 26a^2x^2 + 8ax^3 + x^4 \\
5aa + 4ax + xx \\
\hline
125a^6 + 200a^5x + 130a^4x^2 + 40a^3x^3 + 5aax^4 \\
+ 100a^4x + 160a^3x^2 + 104a^2x^3 + 32aax^4 + 4ax^5 \\
+ 25a^4x^2 + 40a^3x^3 + 26a^2x^4 + 8ax^5 + x^6 \\
\hline
125a^6 + 300a^5x + 315a^4x^2 + 184a^3x^3 + 63a^2x^4 + 12ax^5 + x^6.
\end{array}$$

And the cube of $aa + xx$ is $a^6 + 3a^4x^2 + 3a^2x^4 + x^6$.

Therefore $(5aa + 4ax + xx)^2 + 2 \times (5aa + 4ax + xx)^{\frac{1}{2}} \times (aa + xx)^{\frac{1}{2}} + (aa + xx)^2$ will be $= 2 \times (5aa + 4ax + xx)^{\frac{1}{2}} \times (aa + xx)^{\frac{1}{2}} + 125a^6 + 300a^5x + 315a^4x^2 + 184a^3x^3 + 63a^2x^4 + 12ax^5 + x^6 + a^6 + 3a^4x^2 + 3a^2x^4 + x^6$

$$= 2 \times (5aa + 4ax + xx)^{\frac{1}{2}} \times (aa + xx)^{\frac{1}{2}} + 126a^6 + 300a^5x + 318a^4x^2 + 184a^3x^3 + 66a^2x^4 + 12ax^5 + 2x^6.$$

But it has been shewn that $(5aa + 4ax + xx)^2 + 2 \times (5aa + 4ax + xx)^{\frac{1}{2}} \times (aa + xx)^{\frac{1}{2}} + (aa + xx)^2$ is $= 576a^6$.

Therefore $2 \times (5aa + 4ax + xx)^{\frac{1}{2}} \times (aa + xx)^{\frac{1}{2}} + 126a^6 + 300a^5x + 318a^4x^2 + 184a^3x^3 + 66a^2x^4 + 12ax^5 + 2x^6$ will be $= 576a^6$; and (dividing all the terms by 2) $(5aa + 4ax + xx)^{\frac{1}{2}} \times (aa + xx)^{\frac{1}{2}} + 63a^6 + 150a^5x + 159a^4x^2 + 92a^3x^3 + 33a^2x^4 + 6ax^5 + x^6$ will be $= 288a^6$, and consequently (subtracting $63a^6 + 150a^5x + 159a^4x^2 + 92a^3x^3 + 33a^2x^4 + 6ax^5 + x^6$ from both sides,) we shall have $(5aa + 4ax + xx)^{\frac{1}{2}} \times (aa + xx)^{\frac{1}{2}} = 225a^6 - 150a^5x - 159a^4x^2 - 92a^3x^3 - 33a^2x^4 - 6ax^5 - x^6$.

Art. 25. Now let both sides of this equation be squared. The square of $(5aa + 4ax + xx)^{\frac{1}{2}} \times (aa + xx)^{\frac{1}{2}}$, (which forms the left-hand side of this equation,) is $= 5aa + 4ax + xx)^{\frac{1}{2}} \times (aa + xx)^{\frac{1}{2}} =$ the septinomial quantity $125a^6 + 300a^5x + 315a^4x^2 + 184a^3x^3 + 63a^2x^4 + 12ax^5 + x^6 \times$ the quadrinomial quantity $a^6 + 3a^4x^2 + 3a^2x^4 + x^6$, which product is equal to the following compound quantity, consisting of sixteen terms, to wit, $125a^{12} + 300a^{11}x + 690a^{10}x^2 + 1084a^9x^3 + 1383a^8x^4 + 1464a^7x^5 + 1260a^6x^6 + 888a^5x^7 + 507a^4x^8 + 220a^3x^9 + 66a^2x^{10} + 12ax^{11} + x^{12}$; as will appear by the following multiplication.

125a⁶

$$125x^0 + 300x^1 + 315x^2 + 184x^3 + 63x^4 + 12x^5 + x^6$$

$$125a^{18} + 300a^{15}x + 315a^{12}x^2 + 184a^9x^3 + 63a^6x^4 + 12a^3x^5 + 2a^0x^6$$

$$+ 375a^{10}x^8 + 900a^9x^7 + 945a^8x^6 + 512a^7x^5 + 180a^6x^4 + 36a^5x^3 + 3a^4x^2 +$$

$$+ 900a^7x^6 + 945a^6x^7 + 189a^5x^8 + 36a^4x^9 + 3a^3x^{10} +$$

$$125x^{12} + 304x^{11} + 690x^{10}x^2 + 1084x^9x^3 + 1383x^8x^4 + 1464x^7x^5 + 888x^6x^6 + 507x^5x^7 + 220x^4x^8 + 66x^3x^{10} + 12x^2x^{11} + x^{12}$$

And the square of the septinomial quantity $225a^6 - 150a^5x^3 - 159a^4x^3 - 92a^3x^3 - 33a^2x^4 - 6ax^5 - x^6$, (which forms the right-hand side of the last equation), is the compound quantity $50,625a^{12} - 67,500a^{11}x - 99,050a^{10}x^2 + 6,300a^9x^3 + 36,456a^8x^4 + 20,308a^7x^5 + 8280a^6x^6 + 2,511a^5x^7 + 580a^4x^8 + 102a^3x^9 + 12ax^{11} + x^{13}$; as will appear by the following multiplication.

x^6	$-$	$150x^5$	$-$	$150x^4$	$-$	$9x^3x^4$	$-$	$33x^2x^4$	$-$	$6ax^5$	$-$	x^6
x^6	$-$	$150x^5$	$-$	$150x^4$	$-$	$9x^3x^4$	$-$	$33x^2x^4$	$-$	$6ax^5$	$-$	x^6

$$1.9625x^{11} - 33.75x^{10} + 35.55x^{10}x - 20.5000x^9 - 7.425x^8x^4 - 1.35x^7x^3 - .235x^6x^6$$

$$-33,75\omega^{11}x + 23,50\omega^{10}x^2 + 3,85\omega^9x^3 + 13,80\omega^8x^4 + 4,95\omega^7x^5 + 90\omega^6x^6 + 15\omega^5x^7$$

$$+ 23,850x^3 + 25,281x^4 + 14,628x^5 + 5,247x^6 + 954x^7 + 159x^8$$

$$+ 13,800x^4 + 14,618x^5 + 8,464x^6 + 3036x^7 + 52x^8 + 92x^9$$

$$+ 4,950x^5 + 5,747x^6 + 3,036x^7 + 1089x^8 + 198x^9 + 33x^{10}$$

$$-1,350x^2 + 900x^3 + 954x^4 + 552x^5 + 198x^6 + 30x^7 + 6x^8 + x^9$$

$$10.62x^{18} - 67.7504x^{17} - 40.0091x^{16} + 6.10009x^{15} + 18.0318x^{14} + 36.4568x^{13} + 20.3088x^{12} + 8.2803x^{11} + 2.5182x^8 + 5882x^9 + 1018^9x^{10} + 128x^{11} + x^{12}$$

Therefore

Therefore we shall now have $125a^{12} + 300a^{11}x + 690a^{10}x^2 + 1084a^9x^3 + 1383a^8x^4 + 1464a^7x^5 + 1260a^6x^6 + 888a^5x^7 + 507a^4x^8 + 220a^3x^9 + 66a^2x^{10} + 12ax^{11} + x^{12} = 50,625a^{12} - 67,500a^{11}x - 49,050a^{10}x^2 + 6,300a^9x^3 + 38,031a^8x^4 + 36,456a^7x^5 + 20,308a^6x^6 + 8,280a^5x^7 + 2,511a^4x^8 + 580a^3x^9 + 102a^2x^{10} + 12ax^{11} + x^{12}$.

In this equation we may observe that the two terms $12ax^{11}$ and x^{12} are found on both sides of the mark of equality, and with the same sign & prefixed to them. They will therefore counterbalance each other, and consequently may be left out of the equation without affecting the equality of its two opposite sides. And then the said equation will rise only to the 10th power of x , and will be as follows, to wit, $125a^{12} + 300a^{11}x + 690a^{10}x^2 + 1,084a^9x^3 + 1,383a^8x^4 + 1,464a^7x^5 + 1,260a^6x^6 + 888a^5x^7 + 507a^4x^8 + 220a^3x^9 + 66a^2x^{10} = 50,625a^{12} - 67,500a^{11}x - 49,050a^{10}x^2 + 6,300a^9x^3 + 38,031a^8x^4 + 36,456a^7x^5 + 20,308a^6x^6 + 8,280a^5x^7 + 2,511a^4x^8 + 580a^3x^9 + 102a^2x^{10}$.

Art. 26. Now let the two terms $67,500a^{11}x$ and $49,050a^{10}x^2$ be added to both sides of this equation. And we shall then have $125a^{12} + 67,800a^{11}x + 49,740a^{10}x^2 + 1084a^9x^3 + 1383a^8x^4 + 1464a^7x^5 + 1260a^6x^6 + 888a^5x^7 + 507a^4x^8 + 220a^3x^9 + 66a^2x^{10} = 50,625a^{12} + 6,300a^9x^3 + 38,031a^8x^4 + 36,456a^7x^5 + 20,308a^6x^6 + 8,280a^5x^7 + 2,511a^4x^8 + 580a^3x^9 + 102a^2x^{10}$.

In the next place let the terms $125a^{12}$, $1084a^9x^3$, $1383a^8x^4$, $1464a^7x^5$, $1260a^6x^6$, $888a^5x^7$, $507a^4x^8$, $220a^3x^9$, and $66a^2x^{10}$, be subtracted from both sides of the equation. And we shall have $67,800a^{11}x + 49,740a^{10}x^2 = 50,500a^{12} + 5,216a^9x^3 + 36,648a^8x^4 + 34,992a^7x^5 + 19,048a^6x^6 + 7,392a^5x^7 + 2004a^4x^8 + 360a^3x^9 + 36a^2x^{10}$; and consequently (dividing all the terms by aa ,) $67,800a^9x + 49,740a^8x^2 = 50,500a^{10} + 5,216a^7x^3 + 36,648a^6x^4 + 34,992a^5x^5 + 19,048a^4x^6 + 7,392a^3x^7 + 2004a^2x^8 + 360ax^9 + 36x^{10}$; and (by subtracting the terms $5,216a^7x^3$ and $36,648a^6x^4$, and all the following terms on the right-hand side of the equation, from both sides of it,) we shall have $67,800a^9x + 49,740a^8x^2 - 5,216a^7x^3 - 36,648a^6x^4 - 34,992a^5x^5 - 19,048a^4x^6 - 7,392a^3x^7 - 2004a^2x^8 - 360ax^9 - 36x^{10} = 50,500a^{10}$; and (by dividing all the terms by 4,) we shall have $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$. By resolving this equation we shall discover the value of x ,

or CH, from which we may derive those of $aa + xx$, or AC², and of $5aa + 4ax + xx$, or AB², and consequently those of AC and AB, or the sides of the triangle ABC which we are required to describe. Q. E. I.

Answer

Another Method of obtaining the foregoing Equation for determining the Value of x, or CH.

Art. 27. As the foregoing algebraïck computations are very long and intricate, and there may therefore be some ground for apprehending that some arithmetical errors may have been committed in the performing them, it will, I think, be prudent and expedient to have recourse also to another method of obtaining the final equation above-mentioned, to wit, the equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 + 90ax^9 - 9x^{10} = 12,625a^{11}$, by which the value of x , or CH, is to be determined; to the end that no doubt may remain concerning the accuracy of the said equation, and its fitness for determining the value of x , or CH. Now this equation may likewise be obtained in the following manner.

Art. 28. Let AB be put $= y$, and AC $= z$.

Then, by the conditions of the Problem, we shall have $y^3 + z^3 = 24a^3$. Therefore (squaring both sides,) we shall have $y^6 + 2y^3z^3 + z^6 = 576a^6$, and consequently $2y^3z^3 = 576a^6 - y^6 - z^6$, and (by squaring both sides,) $4y^6z^6 = 331,776a^{12} - 1152a^6y^6 - 1152a^6z^6 + 2y^6z^6 + y^{12} + z^{12}$, and (by subtracting $2y^6z^6$ from both sides,) $2y^6z^6 = 331,776a^{12} - 1152a^6y^6 - 1152a^6z^6 + y^{12} + z^{12}$, and (by adding $1152a^6y^6 + 1152a^6z^6$ to both sides,) $2y^6z^6 + 1152a^6y^6 + 1152a^6z^6 = 331,776a^{12} + y^{12} + z^{12}$, and (by subtracting $y^{12} + z^{12}$ from both sides,) $2y^6z^6 + 1152a^6y^6 + 1152a^6z^6 - y^{12} - z^{12} = 331,776a^{12}$.

Art. 29. Now let $aa + xx$ be substituted in this equation instead of zz , to which we have seen above that it is equal. And we shall have $2y^6 \times aa + xx + 1152a^6y^6 + 1152a^6 \times aa + xx - y^{12} - aa + xx = 331,776a^{12}$; that is, $2y^6 \times a^6 + 3a^4x^2 + 3a^2x^4 + x^6 + 1152a^6y^6 + 1152a^6 \times (a^6 + 3a^4x^2 + 3a^2x^4 + x^6) - y^{12} - a^{12} - 6a^{10}x^2 - 15a^8x^4 - 20a^6x^6 - 15a^4x^8 - 6a^2x^{10} - x^{12}$ will be $= 331,776a^{12}$, or $2y^6 \times a^6 + 3a^4x^2 + 3a^2x^4 + x^6 + 1152a^6y^6 + 1152a^{12} + 3456a^{10}x^2 + 3456a^8x^4 + 1152a^6x^6 - y^{12} - a^{12} - 6a^{10}x^2 - 15a^8x^4 - 20a^6x^6 - 15a^4x^8 - 6a^2x^{10} - x^{12}$ will be $= 331,776a^{12}$; or, if we place the terms that involve the powers of y before the rest, we shall have

$$\left. \begin{aligned} & 2y^6 \times a^6 + 3a^4x^2 + 3a^2x^4 + x^6 \\ & + 1152a^6y^6 - y^{12} \\ & + 1152a^{12} + 3456a^{10}x^2 + 3456a^8x^4 \\ & - a^{12} - 6a^{10}x^2 - 15a^8x^4 \\ & + 1152a^6x^6 - 15a^4x^8 - 6a^2x^{10} - x^{12} \\ & - 20a^6x^6 \end{aligned} \right\} = 331,776a^{12},$$

or

$$\left. \begin{aligned} & \text{or } 2y^6 \times a^6 + 3a^4x^2 + 3a^2x^4 + x^6 \\ & + 1152a^3y^6 - y^{12} \\ & + 1151a^{12} + 3450a^{10}x^2 + 3441a^8x^4 \\ & + 1132a^6x^6 - 15a^4x^8 - 6a^2x^{10} - x^{12} \end{aligned} \right\} = 331,776a^{12}.$$

ART. 30. Now let $5aa + 4ax + xx$, (which has been shewn to be equal to $A B^2$, or yy ,) be substituted instead of yy in this equation. And it will then be as follows, to wit,

$$\left. \begin{aligned} & 2 \times \overbrace{5aa + 4ax + xx}^2 \times a^6 + 3a^4x^2 + 3a^2x^4 + x^6 \\ & + 1152 \times a^6 \times \overbrace{5aa + 4ax + xx}^2 - \overbrace{5aa + 4ax + xx}^6 \\ & + 1151a^{12} + 3450a^{10}x^2 + 3441a^8x^4 + 1132a^6x^6 \\ & - 15a^4x^8 - 6a^2x^{10} - x^{12} \end{aligned} \right\} = 331,776a^{12},$$

$$\begin{aligned} & \text{or } 2 \times \overbrace{125a^6 + 300a^4x + 315a^2x^2 + 184a^2x^2 + 63a^2x^4 + 12ax^5 + x^6}^2 \\ & \quad \times \overbrace{a^6 + 3a^4x^2 + 3a^2x^4 + x^6}^2 \\ & + 1152a^6 \times \overbrace{125a^6 + 300a^4x + 315a^2x^2 + 184a^2x^2 + 63a^2x^4 + 12ax^5 + x^6}^2 \\ & - \overbrace{5aa + 4ax + xx}^6 + 1,151a^{12} + 3,450a^{10}x^2 + 3,441a^8x^4 + 1,132a^6x^6 - 15a^4x^8 \\ & - 6a^2x^{10} - x^{12} = 331,776a^{12}, \text{ or (if we multiply the septinomial quantity } 125a^6 \\ & + 300a^4x + 315a^2x^2 + 184a^2x^2 + 63a^2x^4 + 12ax^5 + x^6 \text{ into the quadrimomial} \\ & \text{quantity } a^6 + 3a^4x^2 + 3a^2x^4 + x^6) 2 \times \text{the compound quantity } 125a^{12} + 300a^{10}x^2 \\ & + 690a^{10}x^2 + 1,084a^8x^2 + 1,383a^6x^4 + 1,464a^6x^2 + 1,260a^6x^6 + 888a^4x^2 + 507a^4x^4 \\ & + 220a^2x^6 + 66a^2x^{10} + 12ax^{12} + x^{12}, \text{ consisting of thirteen terms, } + 1,152a^6 \\ & \times \text{the septinomial quantity } 125a^6 + 300a^4x + 315a^2x^2 + 184a^2x^2 + 63a^2x^4 + 12ax^5 \\ & + 12ax^5 + x^6 - \overbrace{5aa + 4ax + xx}^6 + 1,151a^{12} + 3,450a^{10}x^2 + 3,441a^8x^4 + 1,132a^6x^6 \\ & - 15a^4x^8 - 6a^2x^{10} - x^{12} = 331,776a^{12}, \text{ or (if we multiply the first compound} \\ & \text{quantity, consisting of thirteen terms, into 2,)} 250a^{12} + 600a^{10}x^2 + 1,380a^{10}x^2 \\ & + 2,168a^8x^2 + 2,766a^6x^4 + 2,928a^6x^2 + 2,520a^6x^6 + 1,776a^4x^2 + 1,014a^4x^4 \\ & + 440a^2x^6 + 132a^2x^{10} + 24ax^{12} + 2x^{12} + 1152a^6 \times \text{the septinomial quan-} \\ & \text{tity } 125a^6 + 300a^4x + 315a^2x^2 + 184a^2x^2 + 63a^2x^4 + 12ax^5 + x^6 \\ & - \overbrace{5aa + 4ax + xx}^6 + 1151a^{12} + 3,450a^{10}x^2 + 3,441a^8x^4 + 1,132a^6x^6 - 15a^4x^8 \\ & - 6a^2x^{10} - x^{12} = 331,776a^{12}, \text{ or (if we multiply the septinomial quan-} \\ & \text{tity } 125a^6 + 300a^4x + 315a^2x^2 + 184a^2x^2 + 63a^2x^4 + 12ax^5 + x^6 \text{ into} \\ & 1152a^6,) 250a^{12} + 600a^{10}x^2 + 1,380a^{10}x^2 + 2,168a^8x^2 + 2,766a^6x^4 + 2,928a^6x^2 \\ & + 2,520a^6x^6 + 1,776a^4x^2 + 1,014a^4x^4 + 440a^2x^6 + 132a^2x^{10} + 24ax^{12} \\ & + 2x^{12} + 144,000a^{12} + 345,600a^{10}x^2 + 362,880a^{10}x^2 + 211,968a^8x^2 + 72,576a^6x^4 \\ & + 13,824a^6x^2 + 1,152a^6x^6 - \overbrace{5aa + 4ax + xx}^6 + 1,151a^{12} + 3,450a^{10}x^2 + 3,441a^8x^4 \\ & + 1,132a^6x^6 - 15a^4x^8 - 6a^2x^{10} - x^{12} = 331,776a^{12}, \text{ or (if we place} \\ & \text{the terms that involve the same powers of } x \text{ near each other,)} \end{aligned}$$

VOL. IV.

Z z

250a¹²

$$\begin{array}{r}
 250a^{13} + 600a^{11}x + 1,380a^9x^2 \\
 + 144,000a^{12} + 345,600a^{10}x + 362,880a^8x^2 \\
 + 1,151a^{12} + 3,450a^{10}x^2 \\
 + 2,168a^8x^3 + 2,766a^6x^4 + 2,928a^4x^5 \\
 + 211,968a^8x^3 + 72,576a^6x^4 + 13,824a^4x^5 \\
 \quad + 3,441a^4x^5 \\
 + 2,520a^6x^6 + 1,776a^5x^7 + 1,014a^4x^8 \\
 + 1,152a^6x^6 - 15a^4x^8 + 440a^7x^9 \\
 + 1,132a^6x^6 \\
 \quad + 132a^2x^{10} + 24ax^{11} + 2x^{12} \\
 \quad - \frac{6x^{10}}{5aa + 4ax + xx} - \frac{x^{12}}{x^{12}}
 \end{array} \left. \vphantom{\begin{array}{l} 250a^{13} \\ 144,000a^{12} \\ 1,151a^{12} \\ 2,168a^8x^3 \\ 211,968a^8x^3 \\ 2,520a^6x^6 \\ 1,152a^6x^6 \\ 1,132a^6x^6 \end{array}} \right\} = 331,776a^{13},$$

or (by making the proper additions and subtractions,) $145,401a^{12} + 346,200a^{11}x + 367,710a^{10}x^2 + 214,136a^9x^3 + 78,783a^8x^4 + 16,752a^7x^5 + 4,804a^6x^6 + 1,776a^5x^7 + 999a^4x^8 + 440a^3x^9 + 126a^2x^{10} + 24ax^{11} + x^{12} - (5aa + 4ax + xx)^6 = 331,776a^{13}$, or (subtracting $145,401a^{12}$ from both sides of the equation,) $346,200a^{11}x + 367,710a^{10}x^2 + 214,136a^9x^3 + 78,783a^8x^4 + 16,752a^7x^5 + 4,804a^6x^6 + 1,776a^5x^7 + 999a^4x^8 + 440a^3x^9 + 126a^2x^{10} + 24ax^{11} + x^{12} - (5aa + 4ax + xx)^6 = 186,375a^{13}$.

Art. 31. We must next raise the trinomial quantity $5aa + 4ax + xx$ to the sixth power.

Now the cube of this quantity has already been shewn to be $125a^6 + 300a^5x + 315a^4x^2 + 184a^3x^3 + 63a^2x^4 + 12ax^5 + x^6$. Therefore its fourth power will be $625a^6 + 2000a^5x + 2900a^4x^2 + 2480a^3x^3 + 1366a^2x^4 + 496a^1x^5 + 116a^0x^6 + 16ax^7 + x^8$; and its fifth power will be $3125a^6 + 12,500a^5x + 23,125a^4x^2 + 26,000a^3x^3 + 19,650a^2x^4 + 10,424a^1x^5 + 3,930a^0x^6 + 1040a^5x^7 + 185a^4x^8 + 20ax^9 + x^{10}$; and its sixth power will be $15,625a^6 + 75,000a^5x + 168,750a^4x^2 + 235,000a^3x^3 + 225,375a^2x^4 + 156,720a^1x^5 + 80,996a^0x^6 + 31,344a^5x^7 + 9,015a^4x^8 + 1880a^3x^9 + 270a^2x^{10} + 24ax^{11} + x^{12}$.

Art. 32. Therefore the last equation will now be as follows, to wit,

$$\begin{array}{r}
 346,200a^{11}x + 367,710a^{10}x^2 + 214,136a^9x^3 \\
 - 75,000a^{11}x - 168,750a^{10}x^2 - 235,000a^9x^3 \\
 + 78,783a^8x^4 + 16,752a^7x^5 + 4,804a^6x^6 \\
 - 225,375a^8x^3 - 156,720a^7x^4 - 80,996a^6x^5 \\
 + 1,776a^5x^7 + 999a^4x^8 + 440a^3x^9 + 126a^2x^{10} \\
 - 31,344a^5x^7 - 9,015a^4x^8 - 1,880a^3x^9 - 270a^2x^{10} \\
 + 24ax^{11} + x^{12} \\
 - 24ax^{11} - x^{12} = 15,625a^6 = 186,375a^{13}, \text{ or (by making the proper}
 \end{array}$$

per subtractions,) $271,200a^{11}x + 198,960a^{10}x^2 - 20,864a^9x^3 - 146,592a^8x^4 - 139,968a^7x^5 - 76,192a^6x^6 - 29,568a^5x^7 - 8,016a^4x^8 - 144a^3x^9 - 144a^2x^{10} - 15,625a^{11} = 186,375a^3$; and (adding $15,625a^3$ to both sides,) we shall have $271,200a^{11}x + 198,960a^{10}x^2 - 20,864a^9x^3 - 146,592a^8x^4 - 139,968a^7x^5 - 76,192a^6x^6 - 29,568a^5x^7 - 8,016a^4x^8 - 144a^3x^9 - 144a^2x^{10} = 202,000a^{11}$; and (by dividing all the terms by aa ;) we shall have $271,200a^9x + 198,960a^8x^2 - 20,864a^7x^3 - 146,592a^6x^4 - 139,968a^5x^5 - 76,192a^4x^6 - 29,568a^3x^7 - 8,016a^2x^8 - 144ax^9 - 144x^{10} = 202,000a^{10}$; and (by dividing all the terms by 2,) $135,600a^9x + 99,480a^8x^2 - 10,432a^7x^3 - 73,296a^6x^4 - 69,984a^5x^5 - 38,096a^4x^6 - 14,784a^3x^7 - 4,008a^2x^8 - 72ax^9 - 72x^{10} = 101,000a^{10}$; and (by dividing all the terms by 8,) $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$; which is the same equation we obtained above, in art. 26, by the former method of proceeding. We may therefore now be confident that no arithmetical mistakes have been made in any of the operations by which this equation has been obtained, and consequently that it truly expresses the relation between the given line a , or DF , or $\frac{2c}{2}$, and the unknown line CH . And therefore by resolving the said equation $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$ we may determine the length of the said line CH , and consequently the position of the point A , (in which the said line meets the line FG ;) or the vertex of the triangle ABC , and the length of its two sides AB and AC .

Q. E. I.

*A Method of Reducing the foregoing Equation of the Tenth Order
to a Quadratick Equation.*

ART. 33. By subtracting the absolute, or known, term, $12,625a^{10}$, of this equation from both sides of it, we shall transform it into the following equation, $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} - 12,625a^{10} = 0$, or $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 0$. Now the compound quantity $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ is divisible by the compound quantity $2525a^3 + 360a^2x + 1404a^2x^2 + 1640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 24ax^7 + 3x^8$ without a remainder, and the quotient of such division is the trinomial quantity $5aa - 6ax - 3xx$; as will appear by performing the said division, which will be as follows.

Z z 2

The

The Divisor.

$$- 2155a^8 + 360a^7x + 1404a^6x^2 + 1640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$$

The Quotient.

$$(5ax - 6ax - 3ax$$

The Dividend.

$$\begin{aligned} & - 12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 8,748a^6x^4 - 4,762a^5x^5 - 501a^4x^6 - 972a^3x^7 - 971a^2x^8 \\ & - 12,625a^{10} + 1,800a^9x + 7,070a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,010a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8 \\ & + 15,150a^9x + 5,415a^8x^2 - 9,504a^7x^3 - 14,052a^6x^4 - 10,788a^5x^5 - 5,382a^4x^6 - 1,968a^3x^7 - 516a^2x^8 - 90ax^9 \\ & + 15,150a^9x - 2,160a^8x^2 - 8,424a^7x^3 - 9,840a^6x^4 - 5,868a^5x^5 - 2,448a^4x^6 - 744a^3x^7 - 144a^2x^8 - 18ax^9 \\ & + 7,157a^8x^2 - 1,080a^7x^3 - 4,212a^6x^4 - 4,920a^5x^5 - 2,934a^4x^6 - 1,224a^3x^7 - 372a^2x^8 - 72ax^9 - 9x^{10} \\ & + 7,157a^8x^2 - 1,080a^7x^3 - 4,212a^6x^4 - 4,920a^5x^5 - 2,934a^4x^6 - 1,224a^3x^7 - 372a^2x^8 - 72ax^9 - 9x^{10} \end{aligned}$$

Now, since the compound quantity $- 12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 8,748a^6x^4 - 4,762a^5x^5 - 501a^4x^6 - 972a^3x^7 - 971a^2x^8 - 90ax^9 - 9x^{10}$ is equal to nothing, and the said compound quantity is the product of the multiplication of the compound quantity $- 252a^4 + 360a^3x + 1404a^2x^2 + 1640a^1x^3 + 978a^0x^4 + 408a^{-1}x^5 + 124a^{-2}x^6 + 24a^{-3}x^7 + 3a^{-4}x^8$ and the trinomial quantity $5ax - 6ax - 3ax$, it follows that either one, or both, of the said latter compound quantities (which are the factors of the first compound quantity,) must be equal to nothing likewise; for otherwise the product of their multiplication could not be equal to nothing. We have therefore reason to suppose that the trinomial quantity $5ax - 6ax - 3ax$ is equal to nothing, and consequently that $3ax + 6ax$ is equal to $5ax$, and consequently that $ax + \frac{6ax}{3}$, or $ax + 2ax$, is equal to $\frac{5ax}{3}$, and therefore that the length of the line CH, or x , may be determined by resolving the quadratic equation $ax + 2ax = \frac{5ax}{3}$.

Now

Now this equation may be resolved as follows. Add to both sides of it the quantity aa ; and we shall have $xx + 2ax + aa = \frac{5aa}{3} + aa = \frac{5aa}{3} + \frac{3^2a}{3} = \frac{8aa}{3}$, and consequently, (extracting the square-roots on both sides,) $x + a = a \times \frac{\sqrt{8}}{\sqrt{3}} (= a \times \frac{\sqrt{3} \times \sqrt{8}}{\sqrt{3} \times \sqrt{3}} = a \times \frac{\sqrt{3} \times \sqrt{8}}{3} = a \times \frac{\sqrt{12 \times 8}}{3}) = a \times \frac{\sqrt{24}}{3}$, and $x = a \times \frac{\sqrt{24}}{3} - a = a \times \frac{\sqrt{24}}{3} - \frac{3a}{3} = a \times \frac{\sqrt{24}-3}{3}$; that is, x , or the line CH , will be $= a \times \frac{\sqrt{24}-3}{3}$, or $= FD \times \frac{\sqrt{24}-3}{3}$. Q. E. I.

Art. 34. Since x is $= a \times \frac{\sqrt{24}-3}{3}$, we shall have $xx = aa \times \frac{24-6\sqrt{24}+9}{9}$ ($= aa \times \frac{33-6\sqrt{24}}{9} = aa \times \frac{11-2\sqrt{24}}{3} = aa \times \frac{11-\sqrt{4} \times \sqrt{24}}{3} = aa \times \frac{11-\sqrt{14 \times 24}}{3}$) $= aa \times \frac{11-\sqrt{96}}{3}$, and consequently $aa + xx = aa + aa \times \frac{11-\sqrt{96}}{3} (= \frac{3aa}{3} + aa \times \frac{11-\sqrt{96}}{3} = aa \times \frac{3+11-\sqrt{96}}{3}) = aa \times \frac{14-\sqrt{96}}{3}$; that is, AC^2 will be $= aa \times \frac{14-\sqrt{96}}{3}$. And AB^2 , or $5aa + 4ax + xx$, will be $= 5aa + 4a \times a \times \frac{\sqrt{24}-3}{3} + aa \times \frac{11-\sqrt{96}}{3} (= 5aa + 4aa \times \frac{\sqrt{24}-3}{3} + aa \times \frac{11-\sqrt{96}}{3} = \frac{15aa}{3} + 4aa \times \frac{\sqrt{24}-3}{3} + aa \times \frac{11-\sqrt{96}}{3} = aa \times \frac{15+11-\sqrt{96}}{3} + 4aa \times \frac{\sqrt{24}-3}{3} = aa \times \frac{26-\sqrt{96}}{3} + 4aa \times \frac{\sqrt{24}-3}{3} = aa \times \frac{26-\sqrt{96}}{3} + aa \times \frac{4\sqrt{24}-12}{3} = aa \times \frac{26-\sqrt{96}+4\sqrt{96}-12}{3}) = aa \times \frac{14+\sqrt{96}}{3}$. Therefore AC , or the lesser side of the triangle ABC , will be $= a \times \sqrt{\frac{14-\sqrt{96}}{3}}$, or $a \times \frac{\sqrt{14-\sqrt{96}}}{\sqrt{3}}$, and AB , or the greater side of the said triangle, will be $= a \times \sqrt{\frac{14+\sqrt{96}}{3}}$, or $a \times \frac{\sqrt{14+\sqrt{96}}}{\sqrt{3}}$. Q. E. I.

Art. 35. These two values of AC and AB will answer the condition of the Problem. For the sum of their cubes will be equal to three times the base BC of the triangle ABC ; as may be thus demonstrated.

Since AC is $= a \times \frac{\sqrt{14-\sqrt{96}}}{\sqrt{3}}$, and AB is $= a \times \frac{\sqrt{14+\sqrt{96}}}{\sqrt{3}}$, we shall have

$$\text{have } AC^2 = aa \times \frac{14 - \sqrt{96}}{3} (= aa \times \frac{14 - 2\sqrt{24}}{3} = aa \times \frac{13 - 2\sqrt{24} + 2}{3} =$$

$$aa \times \frac{13 - 2\sqrt{3} \times \sqrt{8} + 2}{3} = aa \times \frac{13 - 2 \times \sqrt{3} \times \sqrt{4} \times \sqrt{2} + 2}{3} = aa \times$$

$$\frac{13 - 2 \times 2 \times \sqrt{3} \times \sqrt{2} + 2}{3} = aa \times \frac{4 \times 1 - 2 \times 2 \sqrt{3} \times \sqrt{2} + 2}{3} = aa \times \frac{2\sqrt{3} - \sqrt{2}}{3},$$

$$\text{and consequently } AC = a \times \frac{2\sqrt{3} - \sqrt{2}}{\sqrt{3}} (= a \times \frac{2\sqrt{3}}{\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}}) = a \times 2 - \frac{\sqrt{2}}{\sqrt{3}},$$

$$\text{and we shall have } AB^2 = aa \times \frac{14 + \sqrt{96}}{3} (= aa \times \frac{14 + 2\sqrt{24}}{3} = aa \times \frac{12 + 2\sqrt{24} + 2}{3}$$

$$= aa \times \frac{12 + 2\sqrt{3} \times \sqrt{8} + 2}{3} = aa \times \frac{13 + 2\sqrt{3} \times \sqrt{4} \times \sqrt{2} + 2}{3} = aa \times$$

$$\frac{13 + 2 \times 2 \times \sqrt{3} \times \sqrt{2} + 2}{3} = aa \times \frac{4 \times 3 + 2 \times 2 \sqrt{3} \times \sqrt{2} + 2}{3} = aa \times \frac{2\sqrt{3} + \sqrt{2}}{3},$$

$$\text{and consequently } AB = a \times \frac{2\sqrt{3} + \sqrt{2}}{\sqrt{3}} (= a \times \frac{2\sqrt{3}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}) = a \times \left(2 + \frac{\sqrt{2}}{\sqrt{3}}\right).$$

$$\text{Therefore } AC^3 \text{ will be } = a^3 \times \left(2 - \frac{\sqrt{2}}{\sqrt{3}}\right)^3 = a^3 \times$$

$$8 - 3 \times 4 \times \frac{\sqrt{2}}{\sqrt{3}} + 3 \times 2 \times \frac{2}{3} - \frac{2\sqrt{2}}{3\sqrt{3}} = a^3 \times \left(8 - \frac{12\sqrt{2}}{\sqrt{3}} + 4 - \frac{2\sqrt{2}}{3\sqrt{3}}\right)$$

$$= a^3 \times 12 - \frac{12\sqrt{2}}{\sqrt{3}} - \frac{2\sqrt{2}}{3\sqrt{3}} = a^3 \times 12 - \frac{36\sqrt{2}}{3\sqrt{3}} - \frac{2\sqrt{2}}{3\sqrt{3}} = a^3 \times \left(12 - \frac{38\sqrt{2}}{3\sqrt{3}}\right),$$

$$\text{and } AB^3 \text{ will be } = a^3 \times \left(2 + \frac{\sqrt{2}}{\sqrt{3}}\right)^3 (= a^3 \times$$

$$8 + 3 \times 4 \times \frac{\sqrt{2}}{\sqrt{3}} + 3 \times 2 \times \frac{2}{3} + \frac{2\sqrt{2}}{3\sqrt{3}} = a^3 \times \left(8 + \frac{12\sqrt{2}}{\sqrt{3}} + 4 + \frac{2\sqrt{2}}{3\sqrt{3}}\right)$$

$$= a^3 \times 12 + \frac{12\sqrt{2}}{\sqrt{3}} + \frac{2\sqrt{2}}{3\sqrt{3}} = a^3 \times \left(12 + \frac{36\sqrt{2}}{3\sqrt{3}} + \frac{2\sqrt{2}}{3\sqrt{3}}\right) = a^3 \times \left(12 + \frac{38\sqrt{2}}{3\sqrt{3}}\right),$$

$$\text{and consequently } AC^3 + AB^3 \text{ will be } = a^3 \times \left(12 - \frac{38\sqrt{2}}{3\sqrt{3}}\right) + a^3 \times \left(12 + \frac{38\sqrt{2}}{3\sqrt{3}}\right)$$

$$= a^3 \times 12 + 12 = a^3 \times 24 = 24a^3 = 3 \times 8a^3 = 3 \times 2a^3 = 3 \times BC^3.$$

Q. E. D.

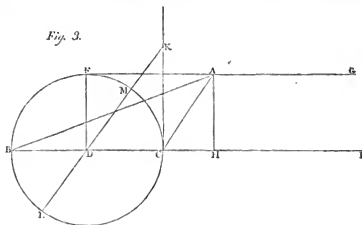
A Remark on the Reduction of the foregoing Equation of the 10th Order to a Quadratic Equation by the long Algebraick Division performed in Art. 33.

Art. 36. We have now completed the solution of Mr. Glenie's Problem, and found the magnitudes of the external line CH and of AC and AB, the two sides of the triangle ABC, (which we were required to describe,) without any

any mention of Mr. Glenie's construction. But I must freely confess that it has not been done without the help of that construction, though no mention has been made of it: for it was by the means of that construction that I discovered that the compound quantity $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ was divisible without a remainder by the trinomial quantity $5aa - 6ax - 3xx$, and that the quotient of that division would be the compound quantity $-2,525a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$; from whence it followed that, if the same compound quantity $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ was to be divided by the compound quantity $-2,525a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, the quotient would be the trinomial quantity $5aa - 6ax - 3xx$, as we have found it to be in art. 33. Nor do I yet know of any method of discovering with certainty, *a priori*, and without the previous knowledge of Mr. Glenie's construction, that the compound quantity $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ is divisible without a remainder by the trinomial quantity $5aa - 6ax - 3xx$. We might, however, have conjectured with a great appearance of probability, that the said compound quantity $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ was divisible by some trinomial quantity of which the first term was $5aa$, and the last term was $3xx$, because $-12,625a^{10}$, (or the first term of the said compound quantity,) is divisible by $5aa$, and $-9x^{10}$, (or the last term of the said compound quantity,) is divisible by $3xx$. But how to discover, or even to form a conjecture, what will be the numeral co-efficient of ax in the middle term of such trinomial quantity, (if there be any such trinomial quantity by which the said compound quantity may be divided without a remainder,) I know not.

Art. 37. The quadratick equation $3xx + 6ax = 5aa$, or $xx + 2ax = \frac{5aa}{3}$, (to which we reduced the foregoing Problem in art. 33 by means of the long algebraick division therein set forth,) may be constructed geometrically, or by drawing only right lines and circles, in the following manner, as well as by the before-mentioned very elegant Construction given us by Mr. Glenie.

A Geometrical Construction of the Quadratick Equation $3xx + 6ax = 5aa$, or $xx + 2ax = \frac{5aa}{3}$, resulting from the foregoing Algebraick Solution of Mr. Glenie's Problem.



With the line BC, or $2a$, or the given base of the proposed triangle, as a diameter, describe the circle BFCB (fig. 3;) and at the point C, or the extremity of the diameter, or base, BC, draw the line CK touching the circle in C, and of such a length that its square shall be equal to $\frac{5aa}{3}$, or five third parts of aa , or DC^2 ; which may be done by Euclid's Elements, Book 2d, Prop. 14th. Then, from the point K draw the line KDL, passing through D, the center of the circle, and meeting the circumference of the circle in the points L and M, of which M lies between K and D. And from the center D of the said circle draw the line DF at right angles to the line BC, or the base of the proposed triangle, and meeting the circumference of the circle in F; which, being a radius of the circle, will be equal to the radius DC, or to half the said base BC; and from the point F draw the line FG parallel to the said base BC. Then let the base BC be prolonged towards the point I to the point H, so that the line CH shall be equal to the line MK. And from the point H draw the right line HA at right angles to CH, and meeting the right line FG (which is parallel to BCH,) in the point A. Lastly, from the point A draw the two lines AB and AC to the two extremities B and C of the given

given base BC. Then will the line CH be equal to the line x in the equation $xx + 2ax = \frac{5aa}{3}$, and the triangle ABC will be the triangle sought, or that of which the perpendicular height will be equal to half the base BC, and of which the sides AB and AC will be of such magnitudes that the sum of their cubes shall be equal to three times the cube of the base BC. Q. E. F.

DEMONSTRATION.

Art. 38. For $CH^2 + 2a \times CH$ is $= CH^2 + BC \times CH = CH^2 + 2DC \times CH =$ (by the construction,) $KM^2 + 2DC \times KM =$ (by the nature of a circle,) $KM^2 + 2DM \times KM = KM^2 + LM \times KM = \overline{KM + LM} \times KM = KL \times KM =$ (by El. 3, 36,) $CK^2 =$ (by the construction,) $\frac{5aa}{3} = xx + 2ax$. Therefore CH must be equal to x .

Q. E. D.

And in the triangle ABC the perpendicular altitude AH is equal to DF, or DC, or half the base BC; which is the first condition of the Problem. And AC^2 is $(= AH^2 + CH^2 =$ (by art. 34,) $aa + aa \times \frac{11 - \sqrt{96}}{3} = \frac{3aa}{3} + aa \times \frac{11 - \sqrt{96}}{3}) = aa \times \frac{14 - \sqrt{96}}{3}$; and AB^2 is $(= AH^2 + BH^2 = aa + \overline{2a + x}^2 = aa + 4aa + 4ax + xx = 5aa + 4ax + xx =$ (by art. 34,) $aa \times \frac{14 + \sqrt{96}}{3}$. Therefore (by what has been shewn above in art. 35,) AC^3 will be $= a^3 \times \sqrt{12 - \frac{38\sqrt{2}}{3\sqrt{3}}}$, and AB^3 will be $= a^3 \times \sqrt{12 + \frac{38\sqrt{2}}{3\sqrt{3}}}$; and consequently $AB^3 + AC^3$ will be $(= a^3 \times \sqrt{12 + \frac{38\sqrt{2}}{3\sqrt{3}}} + a^3 \times \sqrt{12 - \frac{38\sqrt{2}}{3\sqrt{3}}}) = a^3 \times \overline{12 + 12} = a^3 \times 24 = 24a^3 = 3 \times 8a^3 = 3 \times \overline{2a}^3 = 3BC^3$; which is the second condition of the Problem. Therefore the triangle ABC will be the triangle required. Q. E. D.

Observations on the Reduction of the Equation of the 10th Order, (obtained by the foregoing Algebraick Investigation of the Value of x , or CH), to a Quadratick Equation by means of the Operation of Division performed in art. 33.

Art. 39. The reduction of the equation of the 10th order, (obtained in art. 26 and 32, as the result of our investigation of the value of x , or CH), to a quadratick equation by means of the long algebraick division performed in art. 33, is agreeable to the common practice of all modern writers of Algebra from the time of Harriot and Des Cartes, (whose works on Algebra were published in the years 1631 and 1637,) down to the present time. But it nevertheless appears to me to be somewhat obscure and unsatisfactory; because, by bringing all the terms of the equation to the same side of the mark of equality, and thereby making them become equal to nothing, the long compound quantity, consisting of eleven terms, that is to be divided by the other compound quantity consisting of nine terms, is a quantity equal to nothing. Now such a quantity, it may be objected, is no quantity at all, but a mere non-entity, and is consequently incapable of being either multiplied or divided, and no just or certain conclusion can be drawn from any such operation that may seem to have been performed upon it. This objection seems to be at least plausible; and therefore I will now endeavour to strengthen the conclusion obtained by the said operation of Division (which conclusion we have seen to be a true one, and to answer the conditions of the Problem,) namely, "that the quotient, $5aa - 6ax - 3xx$, of the said division is equal to nothing, or that $3xx + 6ax$ is equal to $5aa$, or that the value of the unknown quantity x may be determined by the quadratick equation $3xx + 6ax = 5aa$," by some further and plainer reasonings.

A Proof of the Exactness of the Operation of Division performed above in Art. 33, by multiplying the Divisor in the said Operation into the Quotient that was obtained from it.

Art. 40. Let the compound quantity $-2,525a^2 + 360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, which is the divisor in the aforesaid long operation of division, (and which does not appear from any thing that has been proved in the foregoing articles to be necessarily equal to nothing, but may perhaps be equal to some real quantity, by the superiority, in point of magnitude, of the sum of the eight terms which are connected together by the sign $+$, to the single term $2,525a^2$, which has the sign $-$ prefixed to it,) be multiplied by the trinomial quantity $5aa - 6ax - 3xx$, which was the quotient of the former operation of division. Then, if the said former operation of division was rightly performed, the product of this multiplication ought to be the former dividend, to wit, the compound quantity $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304x^3 - 9,162a^4x^4 - 8,748a^5x^5 - 4,762a^6x^6 - 1,848a^7x^7 - 501a^8x^8 - 90ax^9 - 9x^{10}$. Now this multiplication may be performed as follows:

$$\begin{aligned}
& - 2,525a^3 + 360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 24ax^7 + 3x^8 \\
& \quad 5ax - 6ax - 3ax \\
& - 12,625a^4 + 1,800a^3x + 7,200a^3x^2 + 8,200a^3x^3 + 4,890a^3x^4 + 2,040a^3x^5 + 620a^3x^6 + 120a^3x^7 + 15a^3x^8 \\
& \quad + 15,150a^4x - 8,424a^4x^2 - 9,840a^4x^3 - 5,868a^4x^4 - 2,448a^4x^5 - 744a^4x^6 - 144a^4x^7 - 18a^4x^8 \\
& \quad + 7,575a^4x^2 - 1,080a^4x^3 - 4,920a^4x^4 - 2,593a^4x^5 - 1,224a^4x^6 - 372a^4x^7 - 72a^4x^8 - 9x^9 \\
& - 12,625a^4 + 16,950a^4x + 12,435a^4x^2 - 1,394a^4x^3 - 8,748a^4x^4 - 4,768a^4x^5 - 1,864a^4x^6 - 504a^4x^7 - 90a^4x^8 - 9a^4x^9
\end{aligned}$$

This product agrees exactly in all its terms, and in the signs + and - respectively prefixed to them, with the dividend of the foregoing operation of division, and consequently proves the said foregoing operation of division to have been rightly performed. Now this product we know to be equal to nothing. It therefore follows that, at least, one of the two factors, of which it is the product, must be equal to nothing. Therefore, if the multiplicand $-2,525a^3 + 360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 24ax^7 + 3x^8$ is not equal to nothing, it will follow of necessity that the multiplier $5ax - 6ax - 3ax$ must be equal to nothing, or that $3ax + 6ax$ must be equal to $5ax$, and consequently that $ax + 2ax$ will be equal to $\frac{5ax}{3}$. And, even if the said multiplicand shall be equal to nothing, (which does not yet appear,) it is possible that the multiplier $5ax - 6ax - 3ax$ may be likewise equal to nothing. And therefore there is good reason to suppose that the said multiplier is equal to nothing, and consequently that $ax + 2ax$ is equal to $\frac{5ax}{3}$, and that the value of x , or the line CH , may be determined by resolving the quadratic equation $ax + 2ax = \frac{5ax}{3}$, as, upon trial, we have found that it may.

An Objection that may be made to the foregoing Operation of Multiplication, and to the Conclusion derived from it.

ART. 41. It has been observed that the compound quantity which is the multiplicand of the foregoing operation of multiplication, may, for aught that appears to the contrary, be some real quantity: and, if it is so, it must be capable of being multiplied by any other real quantity. But it may fall be objected, that even in this case it cannot be multiplied by the trinomial quantity $5ax - 6ax - 3ax$, because the said quantity is equal to nothing, and therefore is incapable of either multiplying or being multiplied.

3 A 2

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An Answer to the said Objection.

But this objection is only specious, and may be removed by supposing the multiplicand $-2,525a^3 + 360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 3a^6$, instead of being multiplied by the trinomial quantity $5aa - 6ax - 3xx$, to be multiplied, first, by the single quantity $5aa$, and, secondly, by the binomial quantity $6ax + 3xx$, and then subtracting the product of the latter multiplication from the product of the former multiplication, and comparing the remainder thereby obtained with the compound quantity of eleven terms which was the dividend of the operation of division performed above in art. 33, and which we know to be equal to nothing. For these operations are all evidently very intelligible and practicable, and are equivalent to the single operation of multiplying the said compound multiplicand by the trinomial quantity $5aa - 6ax - 3xx$, and will enable us to draw the same conclusion as was drawn from the said single operation. For, if the said compound multiplicand be a real quantity, (as it is now supposed to be,) and the product of its multiplication by the binomial quantity $6ax + 3xx$, being subtracted from the product of its multiplication by the single quantity $5aa$, leaves a compound quantity that is exactly the same with the compound quantity that was the dividend in the operation of division performed above in art. 33, and is consequently equal to nothing, it is evident that the former product must be equal to the latter product, and consequently that the former multiplier, to wit, $6ax + 3xx$, must be equal to the latter multiplier, to wit, $5aa$, and consequently that the value of x will be determined by the quadratic equation $3xx + 6ax = 5aa$, or $xx + 2ax = \frac{5aa}{3}$. But this will, perhaps, appear more clearly by actually performing those several operations; which may be done as follows:

Art. 42. The multiplication of the compound quantity $-2,525a^3 + 360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 3a^6$ by the single quantity $5aa$ will be as follows:

The Multiplicand.

$$-2,525a^3 + 360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 24ax^7 + 3a^6.$$

The Multiplier.

$5aa$

The Product.

$$-12,625a^{10} + 1,800a^8x + 7,020a^8x^2 + 8,200a^8x^3 + 4,890a^8x^4 + 2,040a^8x^5 + 620a^8x^6 + 120a^8x^7 + 15a^8x^8.$$

This

This product is the very same with the first, or highest, line of terms in the compound product of the multiplication of the same multiplicand by the trinomial quantity $5ax - 6ax - 3xx$, exhibited above in art. 40. Let the multiplicand in this operation of multiplication be, for the sake of brevity, called M, and the multiplier $5aa$ be called A. And then the product of this multiplication, which has been just now obtained, will be denoted by A M.

The multiplication of the same compound quantity $-2,525a^3 + 36a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408ax^5 + 124a^2x^6 + 24ax^7 + 3x^8$ by the binomial quantity $6ax + 3xx$ will be as follows :

The Multiplicand.

$$-2,525a^3 + 36a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 408a^2x^4 + 124a^2x^5 + 24ax^7 + 3x^8$$

The Multiplier.

$$6ax + 3xx$$

The Product.

$$-15,150a^4x + 2,160a^4x^2 + 8,424a^4x^3 + 9,840a^4x^4 + 5,868a^4x^5 + 2,448a^4x^6 + 744a^4x^7 + 144a^4x^8 + 18a^4x^9$$

$$-7,575a^3x^2 + 1,080a^3x^3 + 4,1212a^3x^4 + 4,920a^3x^5 + 2,0934a^3x^6 + 1,224a^3x^7 + 372a^3x^8 + 72ax^9 + 9a^{10}$$

$$-15,150a^4x - 5,415a^4x^2 + 9,504a^4x^3 + 14,051a^4x^4 + 10,788a^4x^5 + 5,982a^4x^6 + 1,968a^4x^7 + 516a^4x^8 + 90ax^9 + 9a^{10}$$

Now let this product be subtracted from the former product AM, or $-12,625a^{10} + 1,800a^9x + 1,800a^8x^2 + 7,020a^7x^3 + 8,200a^6x^4 + 4,890a^5x^5 + 2,040a^4x^6 + 620a^3x^7 + 120a^2x^8 + 15a^2x^9$, obtained by multiplying the same compound multiplicand into the single term $5aa$. This subtraction will be as follows :

$$\text{From } -12,625a^{10} + 1,800a^9x + 1,800a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^9$$

$$\text{Take } -15,150a^4x - 5,415a^4x^2 + 9,504a^4x^3 + 14,051a^4x^4 + 10,788a^4x^5 + 5,982a^4x^6 + 1,968a^4x^7 + 516a^4x^8 + 90ax^9 + 9a^{10}$$

$$\text{Remains } -12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 8,748a^6x^4 - 4,762a^5x^5 - 1,848a^4x^6 - 90ax^9 - 9a^{10}$$

This remainder agrees exactly in all its terms, and in the signs + and - that are prefixed to them, with the compound quantity which was the dividend of the operation of division performed in art. 33, and which was known to

to be equal to nothing. Therefore this remainder must also be equal to nothing, and consequently the product of the multiplication of the compound quantity M by the binomial quantity $6ax + 3xx$, which was subtracted from the other product of the multiplication of the same compound quantity M by the single term $5aa$, must be equal to the said other product; and consequently the binomial quantity $6ax + 3xx$, which was the multiplier of M in one of these multiplications, must be equal to the single quantity $5aa$, which was the multiplier of M in the other multiplication. Q. E. D.

Art. 43. If we denote $6ax$ by the letter B, and $3xx$ by the letter C, this reasoning may be expressed more concisely as follows:

The product of the multiplication of the compound quantity M by the binomial quantity B + C, being subtracted from the product of the multiplication of the same quantity M by the single quantity A, leaves a remainder that consists of eleven terms that are exactly the same, and with the same signs + and - prefixed to them, as the eleven terms of the compound quantity that was the dividend of the operation of division performed above in art. 33, and which we know to be equal to nothing. Therefore the said remainder will also be equal to nothing; that is, the quantity $A \times M - \overbrace{B + C} \times M = 0$. Therefore the quantity $\overbrace{B + C} \times M$ must be = the quantity $A \times M$, and consequently B + C must be = A, or $6ax + 3xx$ must be = $5aa$.

Q. E. D.

This reasoning seems to be liable to no exception.

Art. 44. Hitherto we have obtained the value of the line CH, or x , or of the root of the final equation $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, only by reducing the said equation to the quadratick equation $5aa - 6ax - 3xx$.

$= 0$, or $5aa = 6ax + 3xx$, or $3xx + 6ax = 5aa$, or $xx + 2ax = \frac{5aa}{3}$, by means of the long algebraïck division performed in art. 33; and to that division I confess I was led by the previous knowledge of Mr. Glenie's construction of the Problem, and that I do not know any certain way, independently of that previous knowledge, of discovering the novinomial quantity that was taken for the divisor in it. I will therefore now proceed to shew how we might arrive at a pretty near value of x , or the root of the said final equation, without reducing it to a quadratick equation, and without any reference to, or assistance from, the foregoing construction of the Problem by Mr. Glenie.

A Refo-

A Resolution of the final Equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, (obtained by the foregoing Algebraick Investigation of the Value of the line x , or CH , in fig. 2,) by Mr. Raphson's Method of resolving Equations by Approximation.

Art. 45. It has been shewn above, in art. 21, without any reference to Mr. Glenie's construction of the Problem, that the line AF , in fig. 2, will be greater than the line EF , or that A , the vertex of the triangle sought, will be beyond the point E , and consequently that the line AH drawn from the said vertex at right angles to the line BI , or the base BC produced, and meeting it in H , will lie without the said triangle ABC , and that the line BH will be greater than the base BC , and the line CH will lie without the said triangle. But how much the line BH will exceed the base BC , or how great their difference CH , or x , will be, in comparison of the given line DC , or $\frac{BC}{2}$, or a , in

order to make $AC^2 + AB^2$ be equal to $3BC^2$, according to the second condition of the Problem, is what still remains to be determined, and may be determined by resolving the equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$. We must therefore now endeavour to resolve this equation. Now, in order to resolve this equation by Mr. Raphson's method of approximation, (which I conceive to be the best method of resolving it that can be taken,) it will be expedient to begin our operations by making some conjectures concerning the value of x , that may lead us gradually to a discovery of a tolerably near value of it that may be made the basis of a further approach to its true value by Mr. Raphson's method. And with this view I shall, first, suppose x , or CH , to be equal to a , or DC , or $\frac{BC}{2}$, and compute the values of the lines

AC and AB resulting from that supposition, in order to discover whether the sum of their cubes will be greater than, equal to, or less than, $24a^3$, or $3 \times 8a^3$, or $3 \times 2a^3$, or $3BC^2$, or three times the cube of the given base BC .

Art. 46. Now, if x , or CH , is $= a$, we shall have $AC^2 (= CH^2 + AH^2 = AH^2 + AH^2 = 2AH^2 = 2DC^2) = 2aa$, and consequently $AC = a \times \sqrt{2}$, and $AC^3 (= a^3 \times 2\sqrt{2} = a^3 \times 2 \times 1.414 \&c = a^3 \times 2.828 \&c) = 2.828 \&c \times a^3$; and we shall have $AB^2 (= BH^2 + AH^2 = BC^2 + CH^2 + AH^2 + AH^2)$

+ $AH^2 = \overline{2DC + CH}^2 + DC^2 = \overline{2DC + DC}^2 + DC^2 = \overline{3DC}^2 + DC^2 = 9DC^2 + DC^2 = 10DC^2 = 10aa$, and consequently $AB = a \times \sqrt{10}$, and $AB^2 (= a^2 \times 10 \times \sqrt{10} = a^2 \times 10 \times 3.1622 \text{ \&c}) = a^2 \times 31.622$, &c. Therefore $AC^2 + AB^2$ will be $(= 2.428 \text{ \&c} \times a^2 + 31.622 \text{ \&c} \times a^2) = 34.050$, &c $\times a^2$; which is greater than $24a^2$, or $3BC^2$. Therefore the true value of x , or CH , must be less than a , or AH .

We will therefore, in the second place, suppose x , or CH , to be $= \frac{a}{2}$, or $0.5 \times a$.

Upon this supposition we shall have xx , or $CH^2 = \frac{aa}{4}$, or $0.25 \times aa$. Therefore AC^2 will be $(= CH^2 + AH^2 = 0.25 \times aa + aa) = 1.25 \times aa$; and consequently AC will be $(= \sqrt{1.25 \times a}) = 1.118 \text{ \&c} \times a$, and AC^2 will be $(= \overline{1.118 \text{ \&c}}^2 \times a^2) = 1.397$, &c $\times a^2$; and AB^2 will be $(= BH^2 + AH^2 = \overline{BC + CH}^2 + AH^2 = \overline{2a + 0.5 \times a}^2 + aa = \overline{2.5 \times a}^2 + aa + \overline{2.5}^2 \times aa + aa = 6.25 \times aa + aa) = 7.25 \times aa$; and consequently AB will be $(= \sqrt{7.25 \times a}) = 2.692 \text{ \&c} \times a$, and AB^2 will be $(= AB \times AB^2 = 2.692 \text{ \&c} \times a \times 7.25 \text{ \&c} \times aa = 2.692 \text{ \&c} \times 7.25 \times a^2) = 19.517$, &c $\times a^2$. Therefore $AC^2 + AB^2$ will be $(= 1.397$, &c $\times a^2 + 19.517$, &c $\times a^2) = 20.914$, &c $\times a^2$; which is less than $24a^2$, or $3BC^2$. Therefore the true value of x , or CH , must be greater than $\frac{AH}{2}$, or $\frac{a}{2}$, or $0.5 \times a$.

This last value of $AC^2 + AB^2$, to wit, $20.914 \text{ \&c} \times a^2$, (which results from the supposition that x , or CH , is equal to $\frac{AH}{2}$, or $\frac{a}{2}$, or $0.5 \times a$), is but little greater than $20a^2$, or $\frac{20}{24} \times 24a^2$, or $\frac{5}{6} \times 24a^2$, or 5 sixth parts of the value of $AC^2 + AB^2$ when x is the value sought. We will therefore now suppose, in the 3d place, that $0.5 \times a$ is about $\frac{5}{6}$ sixth parts of the true value of x , and consequently that the said true value is $= 0.6 \times a$.

Now, upon this supposition, we shall have xx , or $CH^2 = 0.36 \times aa$, and $AC^2 (= CH^2 + AH^2 = 0.36 \times aa + aa) = 1.36 \times aa$, and consequently $AC (= \sqrt{1.36 \times a}) = 1.166$, &c $\times a$, and $AC^2 (= 1.36 \times aa \times 1.166$, &c $\times a) = 1.585$, &c $\times a^2$; and we shall have $AB^2 (= BH^2 + AH^2 = \overline{BC + CH}^2 + AH^2 = \overline{2a + 0.6 \times a}^2 + aa = \overline{2.6 \times a}^2 + aa = 6.76 \times aa + aa) = 7.76 \times aa$, and consequently $AB (= \sqrt{7.76 \times a}) = 2.785$, &c

&c $\times a$, and $AB^3 (= 7.76 \times aa \times 2.785, \&c \times a) = 21.611, \&c \times a^3$. Therefore $AC^3 + AB^3$ will be $(= 1.585, \&c \times a^3 + 21.611, \&c \times a^3) = 23.196, \&c \times a^3$; which is somewhat less than $24a^3$, or $3BC^3$. Therefore the true value of x , or CH , must be somewhat greater than $0.6 \times a$. But their difference cannot be great, because $23.196, \&c \times a^3$ differs so little from $24a^3$. We will therefore take $0.6 \times a$ for our first near value of x , or the basis of our approximation to a more exact value of it according to the directions of Mr. Raphson's method.

Art. 47. Since x is somewhat greater than $0.6 \times a$, let us denote their difference by the letter z , or suppose x to be equal to the binomial quantity $0.6 \times a + z$, and let this quantity be substituted instead of x in the terms of the equation $16,950a^6x + 12,435a^5x^2 - 1,304a^4x^3 - 9,162a^3x^4 - 8,748a^2x^5 - 4,762ax^6 - 1,848a^2x^7 - 501a^3x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, omitting all those members of the several powers of the binomial quantity $0.6 \times a + z$ which involve any higher powers of z than its simple power, or z itself, agreeably to Mr. Raphson's directions. This substitution may be made in the following manner:

Since x is $= 0.6 \times a + z$, we shall have

$$\begin{aligned} xx (= 0.6 \times a + z)^2 &= 0.6 \times a)^2 + 2 \times 0.6 \times a \times z + \&c) \\ &= 0.36 \times aa + 1.2 \times az + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^3 (= 0.6 \times a + z)^3 &= 0.6 \times a)^3 + 3 \times 0.6 \times a)^2 \times z + \&c \\ &= 0.216 \times a^3 + 3 \times 0.36 \times a^2z + \&c) \\ &= 0.216 \times a^3 + 1.08 \times a^2z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^4 (= 0.6 \times a + z)^4 &= 0.6 \times a)^4 + 4 \times 0.6 \times a)^3 \times z + \&c \\ &= 0.1296 \times a^4 + 4 \times 0.216 \times a^3z + \&c) \\ &= 0.1296 \times a^4 + 0.864 \times a^3z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^5 (= 0.6 \times a + z)^5 &= 0.6 \times a)^5 + 5 \times 0.6 \times a)^4 \times z + \&c \\ &= 0.077,76 \times a^5 + 5 \times 0.1296 \times a^4z + \&c) \\ &= 0.077,76 \times a^5 + 0.6480 \times a^4z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^6 (= 0.6 \times a + z)^6 &= 0.6 \times a)^6 + 6 \times 0.6 \times a)^5 \times z + \&c \\ &= 0.046,656 \times a^6 + 6 \times 0.077,76 \times a^5z + \&c) \\ &= 0.046,656 \times a^6 + 0.466,56 \times a^5z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^7 (= 0.6 \times a + z)^7 &= 0.6 \times a)^7 + 7 \times 0.6 \times a)^6 \times z + \&c \\ &= 0.027,993,6 \times a^7 + 7 \times 0.046,656 \times a^6z + \&c) \\ &= 0.027,993,6 \times a^7 + 0.326,592 \times a^6z + \&c, \end{aligned}$$

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and

$$\begin{aligned}\text{and } x^8 & (= 0.6 \times a + z)^8 = 0.6 \times a^8 + 8 \times 0.6 \times a^7 \times z + \&c \\ & = 0.016,796,16 \times a^8 + 8 \times 0.027,593,6 \times a^7 z + \&c) \\ & = 0.016,796,16 \times a^8 + 0.223,948,8 \times a^7 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^9 & (= 0.6 \times a + z)^9 = 0.6 \times a^9 + 9 \times 0.6 \times a^8 \times z + \&c \\ & = 0.010,077,696 \times a^9 + 9 \times 0.016,796,16 \times a^8 z + \&c) \\ & = 0.010,077,696 \times a^9 + 0.151,165,44 \times a^8 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^{10} & (= 0.6 \times a + z)^{10} = 0.6 \times a^{10} + 10 \times 0.6 \times a^9 \times z + \&c \\ & = 0.006,046,617,6 \times a^{10} + 10 \times 0.010,077,696 \times a^9 z + \&c) \\ & = 0.006,046,617,6 \times a^{10} + 0.100,776,96 \times a^9 z + \&c.\end{aligned}$$

$$\text{Therefore } 16,950a^8 z \text{ will be } (= 16,950a^8 \times 0.6 \times a + z = 16,950a^9 \times 0.6a + 16,950a^8 \times z) = 10,170.0 \times a^9 + 16,950a^8 z;$$

$$\begin{aligned}\text{and } 12,435a^8 z^2 \text{ will be } & (= 12,435a^8 \times 0.36 \times aa + 1.2 \times az + \&c) \\ & = 12,435a^8 \times 0.36 \times a^2 + 12,435a^8 \times 1.2 \times az + \&c) \\ & = 4,476.66 \times a^{10} + 14,922 \times a^9 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 1,304a^7 z^2 \text{ will be } & (= 1,304a^7 \times 0.216 \times a^2 + 1.08 \times a^2 z + \&c) \\ & = 1,304a^7 \times 0.216a^2 + 1,304a^7 \times 1.08 \times a^2 z + \&c) \\ & = 281.664 \times a^{10} + 1,408.32 \times a^9 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 9,162a^7 z^3 \text{ will be } & (= 9,162a^7 \times 0.1296 \times a^3 + 0.864 \times a^2 z + \&c) \\ & = 9,162a^7 \times 0.1296a^3 + 9,162a^7 \times 0.864 \times a^2 z + \&c) \\ & = 1,187 \times a^{10} + 7,915.968a^9 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 8,748a^7 z^4 \text{ will be } & (= 8,748a^7 \times 0.077,76 \times a^4 + 0.6480 \times a^2 z + \&c) \\ & = 8,748a^7 \times 0.077,76 \times a^4 + 8,748a^7 \times 0.6480 \times a^2 z + \&c) \\ & = 680.244,48 \times a^{10} + 5,668.7040 \times a^9 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 4,762a^4 z^5 \text{ will be } & (= 4,762a^4 \times 0.046,656 \times a^5 + 0.466,56 \times a^2 z + \&c) \\ & = 4,762a^4 \times 0.046,656 \times a^5 + 4,762a^4 \times 0.466,56 \times a^2 z + \&c) \\ & = 222.175,872 \times a^{10} + 2,221.758,72 \times a^9 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 1,848a^3 z^6 \text{ will be } & (= 1,848a^3 \times 0.027,993,6 \times a^6 + 0.326,592 \times a^2 z + \&c) \\ & = 1,848a^3 \times 0.027,993,6 \times a^6 + 1,848a^3 \times 0.326,592 \times a^2 z + \&c) \\ & = 51.731,064 \times a^{10} + 603.542,016 \times a^9 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 501a^2 z^7 \text{ will be } & (= 501a^2 \times 0.016,796,16 \times a^7 + 0.223,948,8a^7 z + \&c) \\ & = 501a^2 \times 0.016,796,16 \times a^7 + 501a^2 \times 0.223,948,8a^7 z + \&c) \\ & = 8.414,876,16 \times a^{10} + 112.198,348,8a^9 z + \&c,\end{aligned}$$

and

and $90ax^9$ will be ($= 90a \times 0.010,077,696 \times a^9 + 0.151,165,44 \times a^9z + \&c$)
 $= 90a \times 0.010,077,696 \times a^9 + 90a \times 0.151,165,44 \times a^9z + \&c$)
 $= 0.906,992,640 \times a^{10} + 13.604,889,60 \times a^9z + \&c,$

and $9x^{10}$ will be ($= 9 \times 0.006,046,617,6 \times a^{10} + 0.100,776,96 \times a^9z + \&c$)
 $= 9 \times 0.006,046,617,6 \times a^{10} + 9 \times 0.100,776,96 \times a^9z + \&c$)
 $= 0.054,419,553,4 \times a^{10} + 0.906,992,64 \times a^9z + \&c.$

Therefore the whole compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ will be equal to the following set of quantities involving the powers of a and z , to wit,

10.170	$\times a^{10} +$	16,950	$\times a^9z$
+ 4,476.66	$\times a^{10} +$	14,922	$\times a^9z + \&c$
- 281.664	$\times a^{10} -$	1,408.32	$\times a^9z - \&c$
- 1,187.000	$\times a^{10} -$	7,915.968	$\times a^9z - \&c$
- 680.244,48	$\times a^{10} -$	5,668.704,0	$\times a^9z - \&c$
- 222.175,872	$\times a^{10} -$	2,221.758,72	$\times a^9z - \&c$
- 51.731,064	$\times a^{10} -$	603.542,016	$\times a^9z - \&c$
- 8.414,876,16	$\times a^{10} -$	112.198,348,8	$\times a^9z - \&c$
- 0.906,992,640	$\times a^{10} -$	13.604,889,60	$\times a^9z - \&c$
- 0.054,419,558,4	$\times a^{10} -$	0.906,992,64	$\times a^9z - \&c,$

that is, to the quadrinomial quantity

$$14,646.660,000,000,0 \times a^{10} + 31,872.000,000,00 \times a^9z + \&c$$

$$- 2,432.191,704,358,4 \times a^{10} - 17,945.002,967,04 \times a^9z - \&c,$$

or to the binomial quantity

$$12,214.468,295,641,6 \times a^{10} + 13,926.997,032,96 \times a^9z + \&c.$$

But the compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ is $= 12,625a^{10}$.

Therefore the binomial quantity $12,214.468,295,641,6 \times a^{10} + 13,926.997,032,96 \times a^9z$ will also be $= 12,625a^{10}$. And consequently $13,926.997,032,96 \times a^9z$ will be ($= 12,625a^{10} - 12,214.468,295,641,6 \times a^{10}$) $= 410.531,704.358,4 \times a^{10}$, and (dividing both sides by a^9 ,) $13,926.997,032,96 \times z$ will be $= 410.531,704,358,4 \times a$, and z will be $= \frac{410.531,704,358,4 \times a}{13,926.997,032,96}$
 $= 0.029 \times a$. Therefore x , or $0.6 \times a + z$, will be $= 0.6 \times a + 0.029 \times a$
 $= 0.629 \times a$, or nearly $0.63 \times a$. Q. E. D.

Art. 48. We will now substitute $0.63 \times a$ instead of x in the compound quantity

quantity $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$, in order to discover whether the value of that compound quantity resulting from such substitution is greater, or less, than the absolute term, $12,625a^{10}$, of the foregoing equation. This substitution may be made as follows:

If x be $= 0.63 \times a$, we shall have

$$\begin{aligned} xx &= 0.396,9 & \times aa, \text{ and } x^2 &= 0.250,047 & \times a^2, \\ \text{and } x^4 &= 0.157,529,61 & \times a^4, \text{ and } x^3 &= 0.099,243,654,3 & \times a^3, \\ \text{and } x^6 &= 0.062,523,502,2 & \times a^6, \text{ and } x^7 &= 0.039,389,806,3 & \times a^7, \\ \text{and } x^8 &= 0.024,815,578,0 & \times a^8, \text{ and } x^9 &= 0.015,633,814,1 & \times a^9, \\ \text{and } x^{10} &= 0.009,849,302,9 & \times a^{10}. \end{aligned}$$

Therefore $16,950a^2x$ will be $(= 16,950a^2 \times 0.63 \times a)$

$$= 10,678.50 \times a^{10};$$

and $12,435a^2x^2$ will be $(= 12,435a^2 \times 0.396,9 \times a^2)$

$$= 4,935.451,5 \times a^{10};$$

and $1,304a^2x^3$ will be $(= 1,304a^2 \times 0.250,047 \times a^3)$

$$= 326.061,188 \times a^{10};$$

and $9,162a^2x^4$ will be $(= 9,162a^2 \times 0.157,529,61 \times a^4)$

$$= 1,443.286,286,82 \times a^{10};$$

and $8,748a^2x^5$ will be $(= 8,748a^2 \times 0.099,243,654,3 \times a^5)$

$$= 868.183,487,816,4 \times a^{10};$$

and $4,762a^2x^6$ will be $(= 4,762a^2 \times 0.062,523,502,2 \times a^6)$

$$= 297.736,917,476,4 \times a^{10};$$

and $1,848a^2x^7$ will be $(= 1,848a^2 \times 0.039,389,806,3 \times a^7)$

$$= 72.792,362,042,4 \times a^{10};$$

and $501a^2x^8$ will be $(= 501a^2 \times 0.024,815,578,0 \times a^8)$

$$= 12.432,604,578,0 \times a^{10};$$

and $90ax^9$ will be $(= 90a \times 0.015,633,814,1 \times a^9)$

$$= 1.407,043,269,0 \times a^{10};$$

and $9x^{10}$ will be $(= 9 \times 0.009,849,302,9 \times a^{10})$

$$= 0.088,643,726,1 \times a^{10}.$$

And consequently the whole compound quantity $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ will be equal to the following ten quantities, to wit,

$$10,678.$$

$$\begin{aligned}
10,678.5000 \times a^{10} &= 326.061,288,000,0 \times a^{10} \\
+ 4,935.4515 \times a^{10} &= 1,443.286,286,820,0 \times a^{10} \\
&= 868.133,487,816,4 \times a^{10} \\
&= 298.112,058,489,6 \times a^{10} \\
&= 72.792,362,042,4 \times a^{10} \\
&= 12.432,604,5; 8,0 \times a^{10} \\
&= 1.407,043,269,0 \times a^{10} \\
&= 0.088,643,736,1 \times a^{10},
\end{aligned}$$

and consequently to the binomial quantity $15,613.9515 \times a^{10} = 3,022.363,764,741,5 \times a^{10}$, or to the single quantity $12,591.587,735,258,5 \times a^{10}$; which is somewhat less than $12,625a^{10}$, or the absolute term of the equation $16,950a^9x + 12,435a^8x^2 = 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$. Therefore $0.63 \times a$ must be less than the true value of x in the said equation. Q. E. I.

Art. 49. In order therefore to obtain a more exact value of x or the root of this equation, let us suppose x to be equal to $0.63 \times a + z$, and let us substitute $0.63 \times a + z$ instead of x in the equation $16,950ax + 12,435a^2x^2 - 1,304a^3x^3 - 9,162a^4x^4 - 8,748a^5x^5 - 4,762a^6x^6 - 1,848a^7x^7 - 501a^8x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$; which may be done in the manner following:

Since x is $= 0.63 + z$, we shall have

$$\begin{aligned}
xx &= (0.63 \times a + z)^2 = \overline{0.63 \times a}^2 + 2 \times 0.63 \times a \times z + \&c \\
&= 0.3969 \times a^2 + .1.26az + \&c,
\end{aligned}$$

$$\begin{aligned}
\text{and } x^3 &= (0.63 \times a + z)^3 = \overline{0.63 \times a}^3 + 3 \times \overline{0.63 \times a}^2 \times z + \&c \\
&= 0.250,047 \times a^3 + 3 \times 0.3969 \times a^2 \times z + \&c \\
&= 0.250,047 \times a^3 + 1.1907 \times a^2z + \&c,
\end{aligned}$$

$$\begin{aligned}
\text{and } x^4 &= (0.63 \times a + z)^4 = \overline{0.63 \times a}^4 + 4 \times \overline{0.63 \times a}^3 \times z + \&c \\
&= 0.157,529,61 \times a^4 + 4 \times 0.250,047 \times a^3 \times z + \&c \\
&= 0.157,529,61 \times a^4 + 1.000,188 \times a^3z + \&c,
\end{aligned}$$

$$\begin{aligned}
\text{and } x^5 &= (0.63 \times a + z)^5 = \overline{0.63 \times a}^5 + 5 \times \overline{0.63 \times a}^4 \times z + \&c \\
&= 0.099,243,654,3 \times a^5 + 5 \times 0.157,529,61 \times a^4 \times z + \&c \\
&= 0.099,243,654,3 \times a^5 + 0.787,648,05 \times a^4z + \&c,
\end{aligned}$$

$$\begin{aligned}
\text{and } x^6 &= (0.63 \times a + z)^6 = \overline{0.63 \times a}^6 + 6 \times \overline{0.63 \times a}^5 \times z + \&c \\
&= 0.062,523,502,2 \times a^6 + 6 \times 0.099,243,654,3 \times a^5 \times z + \&c \\
&= 0.062,523,502,2 \times a^6 + 0.595,461,925,8 \times a^5z + \&c,
\end{aligned}$$

and

$$\begin{aligned}\text{and } x^7 & (= \overline{0.63 \times a + z})^7 = \overline{0.63 \times a})^7 + 7 \times \overline{0.63 \times a})^6 \times z + \&c \\ & = 0.039,389,806,3 \times a^7 + 7 \times 0.062,523,502,2 \times a^6 \times z + \&c) \\ & = 0.039,389,806,3 \times a^7 + 0.437,661,515,4 \times a^6 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^8 & (= \overline{0.63 \times a + z})^8 = \overline{0.63 \times a})^8 + 8 \times \overline{0.63 \times a})^7 \times z + \&c \\ & = 0.024,815,578,0 \times a^8 + 8 \times 0.039,389,806,3 \times a^7 \times z + \&c) \\ & = 0.024,815,578,0 \times a^8 + 0.315,118,450,4 \times a^7 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^9 & (= \overline{0.63 \times a + z})^9 = \overline{0.63 \times a})^9 + 9 \times \overline{0.63 \times a})^8 \times z + \&c \\ & = 0.015,633,814,1 \times a^9 + 9 \times 0.024,815,578,0 \times a^8 z + \&c) \\ & = 0.015,633,814,1 \times a^9 + 0.223,340,202,0 \times a^8 z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^{10} & (= \overline{0.63 \times a + z})^{10} = \overline{0.63 \times a})^{10} + 10 \times \overline{0.63 \times a})^9 \times z + \&c \\ & = 0.009,849,302,9 \times a^{10} + 10 \times 0.015,633,814,1 \times a^9 z + \&c) \\ & = 0.009,849,302,9 \times a^{10} + 0.156,338,141,0 \times a^9 z + \&c.\end{aligned}$$

$$\begin{aligned}\text{Therefore } 16,950a^8x \text{ will be } & (= 16,950a^8 \times \overline{0.63 \times a + z}) \\ & = 16,950a^8 \times 0.63 \times a + 16,950a^8 \times z) \\ & = 10,678.50a^{10} + 16,950a^8z,\end{aligned}$$

$$\begin{aligned}\text{and } 12,435a^8x^2 \text{ will be } & (= 12,435a^8 \times \overline{0.3969 \times a^2 + 1.26 \times az + \&c}) \\ & = 12,435a^8 \times 0.3969a^2 + 12,435a^8 \times 1.26az + \&c) \\ & = 4935.4515a^{10} + 15,668.10a^8z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 1,304a^8x^3 \text{ will be } & (= 1,304a^8 \times \overline{0.250,047 \times a^3 + 1.1907 \times a^2z + \&c}) \\ & = 1,304a^8 \times 0.250,047 \times a^3 + 1,304a^8 \times 1.1907 \times a^2z + \&c) \\ & = 326.061,288a^{10} + 1552.6728a^8z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 9,162a^8x^4 \text{ will be } & (= 9,162a^8 \times \overline{0.157,529,61 \times a^4 + 1.000,188a^3z + \&c}) \\ & = 9,162a^8 \times 0.157,529,61 \times a^4 + 9,162a^8 \times 1.000,188 \times a^3z + \&c) \\ & = 1443.286,286,82 \times a^{10} + 9163.722,456 \times a^8z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 8,748a^8x^5 \text{ will be } & (= 8,748a^8 \times \overline{0.099,243,654,3 \times a^5 + 0.787,648,05a^4z + \&c}) \\ & = 8,748a^8 \times 0.099,243,654,3 \times a^5 + 8,748a^8 \times 0.787,648,05 \times a^4z + \&c) \\ & = 868.183,487,816,4a^{10} + 6890.345,141,40 \times a^8z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 4,762a^8x^6 \text{ will be } & (= 4,762a^8 \times \overline{0.062,523,502,2 \times a^6 + 0.595,461,925,8 \times a^5z + \&c}) \\ & = 4,762a^8 \times 0.062,523,502,2 \times a^6 + 4,762a^8 \times 0.595,461,925,8 \times a^5z + \&c) \\ & = 297.736,917,476,4a^{10} + 2,835.589,690,659,6 \times a^8z + \&c,\end{aligned}$$

and

and $1,848a^2x'$ will be $(= 1848a^2 \times$

$$\begin{aligned} & 0.039,389,806,3 \times a^2 + 0.437,664,515,4a^2z + \&c = \\ & 1,848a^2 \times 0.039,389,806,3 \times a^2 + 1,848a^2 \times 0.437,664,515,4a^2z + \&c) \\ & = 72,792,362,042,4 \times a^{20} + 808,804,024,459,2 \times a^2z + \&c, \end{aligned}$$

and $501a^2x'$ will be $(= 501a^2 \times$

$$\begin{aligned} & 0.024,815,578,0 \times a^2 + 0.315,118,450,4 \times a^2z + \&c = \\ & 501a^2 \times 0.024,815,578,0 \times a^2 + 501a^2 \times 0.315,118,450,4 \times a^2z + \&c), \\ & = 12,432,604,578,0 \times a^{20} + 157,874,343,650,4 \times a^2z + \&c, \end{aligned}$$

and $90a^2$ will be $(= 90a^2 \times$

$$\begin{aligned} & 0.015,633,814,1 \times a^2 + 0.223,340,202,0 \times a^2z + \&c = \\ & 90a^2 \times 0.015,633,814,1 \times a^2 + 90a^2 \times 0.223,340,202,0 \times a^2z + \&c) \\ & = 1,407,043,269,0 \times a^{20} + 20,100,618,180,0 \times a^2z + \&c, \end{aligned}$$

and $9x^{20}$ will be $(= 9 \times 0.009,849,302,9 \times a^{20} + 0.156,338,141,0 \times a^2z + \&c$
 $= 9 \times 0.009,849,302,9 \times a^{20} + 9 \times 0.156,338,141,0 \times a^2z + \&c)$
 $= 0.088,643,726,1 \times a^{20} + 1.407,043,269,0 \times a^2z + \&c.$

And consequently the whole compound quantity $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90a^2x^9 - 9x^{20}$ will be equal to the following compound quantity, consisting of twenty terms, to wit,

$$\begin{array}{ll} 10,678.50 \times a^{20} & + 16,950 \times a^2x \\ + 4,935.4515 \times a^{20} & + 15,668.10 \times a^2z + \&c \\ - 326.061,288 \times a^{20} & - 1,552.6728 \times a^2x - \&c \\ - 1443.286,286,82 \times a^{20} & - 9,163.722,456 \times a^2x - \&c \\ - 868.183,487,816,4 \times a^{20} & - 6,890.345,141,40 \times a^2x - \&c \\ - 297.736,917,476,4 \times a^{20} & - 2,835.589,690,659,6 \times a^2x - \&c \\ - 72.792,362,042,4 \times a^{20} & - 808.804,024,459,2 \times a^2x - \&c \\ - 12.432,604,578,0 \times a^{20} & - 157.874,343,650,4 \times a^2x - \&c \\ - 1.407,043,269,0 \times a^{20} & - 20,100,618,180,0 \times a^2x - \&c \\ - 0.088,643,726,1 \times a^{20} & - 1.407,043,269,0 \times a^2x - \&c, \end{array}$$

that is, to the quadrinomial quantity

$$\begin{aligned} & 15,613.9515 \times a^{20} + 32,618.100,000,000,0 \times a^2z + \&c \\ & - 3,021.988,633,728,3 \times a^{20} - 21,430.516,117,718,2 \times a^2x - \&c, \end{aligned}$$

or to the binomial quantity

$$12,591.962,866,271,7 \times a^{20} + 11,187.583,882,281,8 \times a^2z.$$

But

But the said compound quantity $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 + 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^{10} - 9x^{10}$ is $= 12,625a^{10}$.

Therefore the binomial quantity $12,591,962,866,271,7 \times a^{10} + 11,187,583,882,281,8 \times a^2x$ will also be $= 12,625a^{10}$. And consequently $11,187,583,882,281,8 \times a^2x$ will be $= 12,625a^{10} - 12,591,962,866,271,7 \times a^{10} = 33,037,133,728,3 \times a^{10}$, and consequently x will be $= \frac{33,037,133,728,3 \times a^{10}}{11,187,583,882,281,8 \times a^2} = \frac{33,037,133,728,3 \times a}{11,187,583,882,281,8} = 0.002,95 \times a$. Therefore x , or $0.63 \times a + x$, will be $= 0.63 \times a + 0.002,95 \times a + 0.632,95 \times a$. Q. E. I.

The four first figures, 0.6329 , of this value of x are exact, its true value being $a \times \frac{\sqrt{24-3}}{3} = a \times \frac{4.898,979,485,5 \text{ \&c} - 3}{3} = a \times \frac{1.898,979,485,5 \text{ \&c}}{3} = a \times 0.632,993,161,8 \text{ \&c}$, or $0.632,993,161,8 \text{ \&c} \times a$, as has been shewn above in art. 33.

Art. 50. That $0.632,95 \times a$ is pretty nearly equal to the true value of x in the foregoing equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^{10} - 9x^{10} = 12,625a^{10}$ will appear by substituting it instead of x in the compound quantity that forms the left-hand side of that equation. This substitution may be made as follows:

If x is $= 0.632,95 \times a$, we shall have

$$\begin{aligned} xx &= 0.400,625,702,5 \times a^2, \text{ and } x^3 = 0.253,576,038 \times a^3, \\ \text{and } x^4 &= 0.160,500,953 \times a^4, \text{ and } x^5 = 0.101,589,078 \times a^5, \\ \text{and } x^6 &= 0.064,300,806 \times a^6, \text{ and } x^7 = 0.040,699,195 \times a^7, \\ \text{and } x^8 &= 0.025,760,555 \times a^8, \text{ and } x^9 = 0.016,305,143 \times a^9, \\ \text{and } x^{10} &= 0.010,320,342 \times a^{10}. \end{aligned}$$

Therefore $16,950a^2x$ will be $(= 16,950a^2 \times 0.632,95 \times a)$
 $= 10,728,502,50 \times a^{10}$,

and $12,435a^3x^2$ will be $(= 12,435a^3 \times 0.400,625,702,5 \times a^2)$
 $= 4,981,780,610,587 \times a^{10}$,

and $1,304a^4x^3$ will be $(= 1,304a^4 \times 0.253,576,038 \times a^3)$
 $= 330,663,153,552 \times a^{10}$,

and $9,162a^5x^4$ will be $(= 9,162a^5 \times 0.160,500,953 \times a^4)$
 $= 1,470,509,731,386 \times a^{10}$,

and $8,748a^6x^5$ will be $(= 8,748a^6 \times 0.101,589,078 \times a^5)$
 $= 888,701,254,344 \times a^{10}$.

and

and $4,762a^4x^8$ will be $(= 4,762a^4 \times 0.064,300,806 \times a^{10})$
 $= 306.200,438,172 \times a^{10},$

and $1,848a^3x^7$ will be $(= 1,848a^3 \times 0.040,699,195 \times a^7)$
 $= 75.212,112,360 \times a^{10},$

and $501a^2x^6$ will be $(= 501a^2 \times 0.025,760,555 \times a^6)$
 $= 12.906,037,055 \times a^{10},$

and $90ax^5$ will be $(= 90a \times 0.016,305,143 \times a^5)$
 $= 1.467,462,870 \times a^{10},$

and $9x^{10}$ will be $(= 9 \times 0.010,320,340 \times a^{10})$
 $= 0.092,883,060 \times a^{10}.$

Therefore the whole compound quantity $16,950a^5x + 12,435a^4x^2 - 1,304a^3x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ will be =

$$\left\{ \begin{array}{rcl} 10,728,502,50 \times a^{10} & - & 330.663,153,552 \times a^{10} \\ + 4.981.780,610,587 \times a^{10} & - & 1,470.509,731,386 \times a^{10} \\ & - & 888.701,254,344 \times a^{10} \\ & - & 306.200,438,172 \times a^{10} \\ & - & 75.212,112,360 \times a^{10} \\ & - & 12.906,037,055 \times a^{10} \\ & - & 1.467,462,870 \times a^{10} \\ & - & 0.092,883,060 \times a^{10} \end{array} \right\}$$

$$= 15,710.283,110,587 \times a^{10} - 3,085.753,072,799 \times a^{10}$$

$= 12,624.530,037,788 \times a^{10}$; which differs by less than a^{10} , or $1a^{10}$, from $12.625 \times a^{10}$, or the absolute term of the equation $16,950a^5x + 12,435a^4x^2 - 1,304a^3x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$. Therefore $0.632,95 \times a$ must be very nearly equal to the value of x in the said equation. Q. E. D.

Art. 51. If we continue this approximation to the value of x one step further, by supposing x to be equal to $0.632,95 \times a + z$, and substituting this binomial quantity $0.632,95 \times a + z$ instead of x in the terms of the equation $16,950a^5x + 12,435a^4x^2 - 1,304a^3x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, we shall find z to be $= 0.000,043,15 \times a$, and consequently x , or $0.632,95 \times a + z$, to be $= 0.632,95 \times a + 0.000,043,15 \times a$, or $0.632,993,15 \times a$; which is exact in the first seven figures $0.632,993,1$, the more accurate value of it being $\frac{\sqrt{24}-3}{3} \times a$, (or $\frac{4.898,979,485,5 \&c - 3}{3} \times a$, or $\frac{1.898,979,485,5 \&c}{3} \times a$), or

$0.632,993,161,8 \&c \times a$. This substitution may be made as follows:

If x is $= 0.632,95 \times a + z$, we shall have

$$\begin{aligned} ax (= 0.632,95 \times a + z)^1 &= 0.632,95 \times a^1 + 2 \times 0.632,95 + a \times z + \&c) \\ &= 0.400,625,702,5aa + 1.265,90 \times az + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^2 (= 0.632,95 \times a + z)^2 &= 0.632,95 \times a^2 + 3 \times 0.632,95 \times a^1 \times z + \&c) \\ &= 0.253,576,038a^2 + 3 \times 0.400,625,702,5aa \times z + \&c) \\ &= 0.253,576,038a^2 + 1.201,877,107,5aaa + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^4 (= 0.632,95 \times a + z)^4 &= 0.632,95 \times a^4 + 4 \times 0.632,95 \times a^3 \times z + \&c) \\ &= 0.160,500,953a^4 + 4 \times 0.253,576,038a^3 \times z + \&c) \\ &= 0.160,500,953a^4 + 1.014,304,152a^3z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^5 (= 0.632,95 \times a + z)^5 &= 0.632,95 \times a^5 + 5 \times 0.632,95 \times a^4 \times z + \&c) \\ &= 0.101,589,078a^5 + 5 \times 0.160,500,953 \times a^4 \times z + \&c) \\ &= 0.101,589,078a^5 + 0.802,504,765a^4z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^6 (= 0.632,95 \times a + z)^6 &= 0.632,95 \times a^6 + 6 \times 0.632,95 \times a^5 \times z + \&c) \\ &= 0.064,300,806a^6 + 6 \times 0.101,589,078a^5 \times z + \&c) \\ &= 0.064,300,806a^6 + 0.609,534,468a^5z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^7 (= 0.632,95 \times a + z)^7 &= 0.632,95 \times a^7 + 7 \times 0.632,95 \times a^6 \times z + \&c) \\ &= 0.040,699,195a^7 + 7 \times 0.064,300,806 \times a^6 \times z + \&c) \\ &= 0.040,699,195a^7 + 0.450,105,642a^6z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^8 (= 0.632,95 \times a + z)^8 &= 0.632,95 \times a^8 + 8 \times 0.632,95 \times a^7 \times z + \&c) \\ &= 0.025,760,555a^8 + 8 \times 0.040,699,195a^7 \times z + \&c) \\ &= 0.025,760,555a^8 + 0.325,593,560a^7z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^9 (= 0.632,95 \times a + z)^9 &= 0.632,95 \times a^9 + 9 \times 0.632,95 \times a^8 \times z + \&c) \\ &= 0.016,305,143a^9 + 9 \times 0.025,760,555a^8 \times z + \&c) \\ &= 0.016,305,143a^9 + 0.231,844,995a^8z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^{10} (= 0.632,95 \times a + z)^{10} &= 0.632,95 \times a^{10} + 10 \times 0.632,95 \times a^9 \times z + \&c) \\ &= 0.010,320,340a^{10} + 10 \times 0.016,305,143a^9 \times z + \&c) \\ &= 0.010,320,340a^{10} + 0.163,051,430a^9z + \&c. \end{aligned}$$

Therefore $16,950a^9$ will be $(= 16,950a^9 \times 0.632,95 \times a + 2 = 16,950a^9 \times 0.632,95 \times a + 16,950a^9 \times z) = 10,728.02,50a^{10} + 16,950a^9z;$

and $12,435a^8$ will be $(= 12,435a^8 \times 0.400,625,702,5aa + 1.265,90 \times az + \&c) = 12,435a^8 \times 0.400,625,7025aa + 12,435a^8 \times 1.265,90 \times az + \&c) = 4,981.780,610,587a^{10} + 15,741.466,50 \times a^9z + \&c;$

and

and $1,304a^7x^3$ will be $(= 1,304a^7 \times$

$$\begin{aligned} & \frac{0.253,576,038a^3 + 1.201,877,107,5aaz + \&c}{=} \\ & = 1,304a^7 \times 0.253,576,038a^3 + 1,304a^7 \times 1.201,877,107,5aaz + \&c) \\ & = 330.663,153,552a^{10} + 1,567.247,748,180a^9z + \&c; \end{aligned}$$

and $9,162a^6x^4$ will be $(= 9,162a^6 \times$

$$\begin{aligned} & \frac{0.160,500,953a^4 + 1.014,304,152a^3z + \&c}{=} \\ & = 9,162a^6 \times 0.160,500,953a^4 + 9,162a^6 \times 1.014,304,152a^3z + \&c) \\ & = 1,470.509,731,386a^{10} + 9,293.054,640,624a^9z + \&c; \end{aligned}$$

and $8,748a^5x^5$ will be $(= 8,748a^5 \times$

$$\begin{aligned} & \frac{0.101,589,078a^5 + 0.802,504,765a^4z + \&c}{=} \\ & = 8,748a^5 \times 0.101,589,078a^5 + 8,748a^5 \times 0.802,504,765a^4z + \&c) \\ & = 888.701,254,344a^{10} + 7,020.311,684,220a^9z + \&c; \end{aligned}$$

and $4,762a^4x^6$ will be $(= 4,762a^4 \times$

$$\begin{aligned} & \frac{0.064,300,806a^6 + 0.609,534,468a^5z + \&c}{=} \\ & = 4,762a^4 \times 0.064,300,806a^6 + 4,762a^4 \times 0.609,534,468a^5z + \&c) \\ & = 306.200,438,172a^{10} + 2,902.603,136,616a^9z + \&c; \end{aligned}$$

and $1,848a^3x^7$ will be $(= 1,848a^3 \times$

$$\begin{aligned} & \frac{0.040,699,195a^7 + 0.450,105,642a^6z + \&c}{=} \\ & = 1,848a^3 \times 0.040,699,195a^7 + 1,848a^3 \times 0.450,105,642a^6z + \&c) \\ & = 75.214,112,360a^{10} + 831.795,226,416a^9z + \&c; \end{aligned}$$

and $501a^2x^8$ will be $(= 501a^2 \times$

$$\begin{aligned} & \frac{0.025,760,555a^8 + 0.325,593,560a^7z + \&c}{=} \\ & = 501a^2 \times 0.025,760,555a^8 + 501a^2 \times 0.325,593,560a^7z + \&c) \\ & = 12.906,037,055a^{10} + 163.122,373,560a^9z + \&c; \end{aligned}$$

and $90ax^9$ will be $(= 90a \times 0.016,305,143a^9 + 0.231,844,995a^8z + \&c)$

$$\begin{aligned} & = 90a \times 0.016,305,143a^9 + 90a \times 0.231,844,995a^8z + \&c) \\ & = 1.467,462,870a^{10} + 20.866,049,550a^9z + \&c; \end{aligned}$$

and $9x^{10}$ will be $(= 9 \times 0.010,320,340a^{10} + 0.163,051,430a^9z + \&c)$

$$\begin{aligned} & = 9 \times 0.010,320,340a^{10} + 9 \times 0.163,051,430a^9z + \&c) \\ & = 0.092,883,660a^{10} + 1.467,462,870a^9z + \&c. \end{aligned}$$

Therefore the whole compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3$
 $- 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9$
 $- 9x^{10}$ will be $=$

$$\begin{aligned}
 & \left\{ \begin{array}{l} 10,718.502,500,000a^{10} + 16,950.000,000,000a^8z + \&c \\ + 4,981.780,610,587a^{10} + 15,741.466,500,000a^6z + \&c \\ - 330.663,153,552a^{10} - 1,567.247,748,180a^8z - \&c \\ - 1,470.509,731,386a^{10} - 9,293.054,640,624a^6z - \&c \\ - 888.701,154,344a^{10} - 7,020.311,684,220a^8z - \&c \\ - 306.200,438,172a^{10} - 2,902.603,136,616a^6z - \&c \\ - 75.212,112,360a^{10} - 831.795,226,416a^8z - \&c \\ - 12.906,037,055a^{10} - 163.122,373,560a^6z - \&c \\ - 1.467,462,870a^{10} - 20.866,049,550a^8z - \&c \\ - 0.092,883,060a^{10} - 1.457,462,870a^6z - \&c \end{array} \right\} \\
 &= \left\{ \begin{array}{l} 15,710.283,110,587a^{10} + 32,691.466,500,000a^8z + \&c \\ - 3,085.753,072,799a^{10} - 21,800.468,322,036a^6z - \&c \end{array} \right\} \\
 &= 12,624.530,037,788a^{10} + 10,890.998,177,964a^8z.
 \end{aligned}$$

But that compound quantity is equal to $12,625a^{10}$.

Therefore $12,624.530,037,788a^{10} + 10,890.998,177,964a^8z$ will also be equal to $12,625a^{10}$. And consequently $10,890.998,177,964a^8z$ will be $= 12,625a^{10} - 12,624.530,037,788a^{10} = 0.469,962,212a^{10}$, and $10,890.998,177,964z$ will be $= 0.469,962,212 \times a$, and z will be $= \frac{0.469,962,212}{10,890.998,177,964} \times a = 0.000,043,15 \times a$; and x , or $0.632,95 \times a + z$, will be $= 0.632,95 \times a + 0.000,043,15 \times a = 0.632,993,15 \times a$. Q. E. D.

Art. 52. Having now found the value of x , or CH , to a considerable degree of exactness by resolving the equation $16,950a^8x + 12,435a^4x^2 = 1,304a^4x^3 - 9,162a^8x^4 - 8,748a^4x^3 - 4,762a^4x^4 - 1,848a^4x^2 - 501a^4x - 90ax^2 - 9x^{10} = 12,625a^{10}$ itself, without reducing it (as at first,) to a quadratic equation by the above-mentioned operation of division, we will compute, from the said value of x , the corresponding values of the lines AC and AB , or the sides of the proposed triangle ABC , and then find the cubes of the said values, and add the said cubes together, in order that it may appear that their sum is very nearly equal to $24a^3$, (or $3 \times 8a^3$, or $3 \times 2a^3$, or $3BC^3$), or three times the cube of the given base BC , agreeably to the second condition of the foregoing Problem.

Now we have seen above that AC^2 is $(= AH^2 + CH^2) = aa + xx$, and that AB^2 is $(= AH^2 + BH^2) = 5aa + 4ax + xx$, and consequently that AC is $= \sqrt{aa + xx}$, and that AB is $= \sqrt{5aa + 4ax + xx}$. We must therefore compute the values of these expressions $\sqrt{aa + xx}$ and $\sqrt{5aa + 4ax + xx}$, upon a supposition that x is $= 0.632,993,15 \times a$; which may be done as follows:

Since

Since x is $= 0.632,993,15 \times a$, we shall have $xx (= 0.632,993,15)^2 \times aa$
 $= 0.400,680,327,946,922,5 \times aa$, and $4ax (= 4a \times 0.632,993,15 \times a) =$
 $2.531,972,60 \times aa$, and consequently $aa + xx (= aa + 0.400,680,327,946,$
 $922,5 \times aa) = 1.400,680,327,946,922,5 \times aa$, and $5aa + 4ax + xx (= 5aa$
 $+ 2.531,972,60 \times aa + 0.400,680,327,946,922,5 \times aa) = 7.932,652,927,946,$
 $922,5 \times aa$. Therefore AC , or $\sqrt{aa + xx}$, will be $(= \sqrt{1.400,680,327,946,922,5 \times a}) =$
 $1.183,503,41 \times a$, and AB , or $\sqrt{5aa + 4ax + xx}$, will be $(= \sqrt{7.932,652,$
 $927,946,922,5 \times a}) = 2.816,496,569 \times a$. Therefore AC^2 will be $(=$
 $1.183,503,41)^2 \times a^2 = 1.657,709,893 \times a^2$, and AB^2 will be $(=$
 $2.816,496,569)^2 \times a^2 = 22.342,289,740 \times a^2$, and consequently $AC^2 + AB^2$
will be $(= 1.657,709,893 \times a^2 + 22.342,289,740 \times a^2) = 23.999,999,633$
 $\times a^2$; which is very nearly equal to $24a^2$, or $3BC^2$, agreeably to the conditions
of the Problem. Therefore $1.183,503,41 \times a$ and $2.816,496,569 \times a$ will be
very nearly equal to the true values of the sides AC and AB of the triangle
 ABC , which we were required to describe. Q. E. D.

We have therefore now completely solved the foregoing Problem, proposed
above in art. 20, without the help of Mr. Glenie's Construction.

*Further Observations on the Equation $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 -$
 $9,162a^2x^4 - 8,478a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 -$
 $9x^{10} = 12,625a^{10}$, obtained by the foregoing Algebraick Investigation of the
Value of the line x , or CH .*

Art. 53. The value of x that has been obtained in the preceeding articles,
to wit, $0.632,993,15 \times a$, is not, however, the only root of the equation
 $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,478a^2x^5 - 4,762a^2x^6$
 $- 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, to which we have
been led by the foregoing algebraick investigation of this Problem. But it
will have another root that will be greater than $0.632,993,15 \times a$, and will be
nearly equal to $0.79 \times a$; as we shall now proceed to shew.

If we suppose the line x to increase gradually from 0 *ad infinitum*, it is evi-
dent that all the terms of the compound quantity $16,950a^2x + 12,435a^2x^2 -$
 $1,304a^2x^3 - 9,162a^2x^4 - 8,478a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 -$
 $90ax^9 - 9x^{10}$ will increase gradually from 0 *ad infinitum*; and it is also evi-
dent that, in the beginning of such increase, and while x is only a small part
of a , the two first terms, $16,950a^2x$ and $12,435a^2x^2$, of the said compound
quantity

quantity will be greater than all the following terms of it, and that they will exceed them in a very great proportion, inasmuch that the sum of the said two first terms will be greater than the sum of all the following eight terms, and in a very great proportion; and that, the smaller x is taken, the greater will be the proportion in which the sum of the said two first terms will exceed the sum of the said following eight terms. But this superiority, in point of magnitude, of the sum of the said two first terms over the sum of the following eight terms, will not always continue during the increase of x from 0 *ad infinitum*; but the sum of the said eight terms will gradually gain ground upon the sum of the said two first terms, and will, at some point of time during the increase of x , become equal to the sum of the said two first terms, and afterwards will be greater than the said sum, and will continue to exceed it ever after, and will exceed it in a ratio that will continually grow greater and greater as x increases *ad infinitum*. These things being admitted, (which seem to be too evident to need a proof,) the particular magnitude of the line x at the instant that the sum of the last eight terms of the said compound quantity, (which, in the foregoing equation, are all marked with the sign —, or are subtracted from the two first terms, $16,950a^2x$ and $12,435a^2x^2$, of the said compound quantity,) becomes equal to the sum of the two first terms of it, may be found by resolving the equation $16,950a^2x + 12,435a^2x^2 = 1,304a^2x^2 + 9,162a^2x^2 + 8,748a^2x^2 + 4,762a^2x^2 + 1,848a^2x^2 + 501a^2x^2 + 90ax^2 + 9x^2$, or (dividing all the terms by x ;) the equation $16,950a + 12,435ax = 1,304ax^2 + 9,162ax^2 + 8,748ax^2 + 4,762ax^2 + 1,848ax^2 + 501ax^2 + 90ax + 9x$, or (subtracting $12,435ax$ from both sides,) the equation $9x^2 + 90ax + 501ax^2 + 1,848ax^2 + 4,762ax^2 + 8,748ax^2 + 9,162ax^2 + 1,304ax^2 - 12,435ax = 16,950a$. Now the root of this equation will be somewhat greater than a . For, if we suppose x to be equal to a , the compound quantity $9x^2 + 90ax + 501ax^2 + 1,848ax^2 + 4,762ax^2 + 8,748ax^2 + 9,162ax^2 + 1,304ax^2 - 12,435ax$ will be $(= 9a^2 + 90a^2 + 501a^2 + 1,848a^2 + 4,762a^2 + 8,748a^2 + 9,162a^2 + 1,304a^2 - 12,435a^2 = 26,424a^2 - 12,435a^2) = 13,989a^2$; which is somewhat less than $16,950a^2$, or the absolute term of the said equation, and consequently a will be somewhat less than the true value of x in the said equation. And, if we seek further for the value of x in this equation, we shall find it to be a little less than $1.031 \times a$. For, if $1.031 \times a$ be substituted instead of x in the said compound quantity $9x^2 + 90ax + 501ax^2 + 1,848ax^2 + 4,762ax^2 + 8,748ax^2 + 9,162ax^2 + 1,304ax^2 - 12,435ax$, it will make the said compound quantity be equal to $17,004.514,200,729 \times a^2$, which exceeds $16,950a^2$, or the absolute term of the said equation, by less than $55a^2$, or the 308th part of the said absolute term. I shall therefore consider the root

of the said equation as being equal to $1.030 \times a$, or $1.03 \times a$, or $a + \frac{3}{100} \times a$.

And consequently, when x is equal, or nearly equal, to $a + \frac{3}{100} \times a$, the compound quantity $9x^2 + 90ax + 501ax^2 + 1,848ax^2 + 4,762ax^2 + 8,748ax^2 + 9,162ax^2 + 1,304ax^2 - 12,435ax$ will be equal to $16,950a^2$, and therefore (multiplying both sides into x) the compound quantity $9x^3 + 90a^2x$

$50ax^9 + 501a^2x^8 + 1,848a^3x^7 + 4,762a^4x^6 + 8,748a^5x^5 + 9,162a^6x^4 + 1,304a^7x^3$ will be $= 16,950a^6x$, and (adding $12,435a^4x^3$ to both sides,) the quantity $9x^{10} + 90ax^9 + 501a^2x^8 + 1,848a^3x^7 + 4,762a^4x^6 + 8,748a^5x^5 + 9,162a^6x^4 + 1,304a^7x^3$ will be $= 16,950a^6x + 12,435a^4x^3$, and (subtracting $9x^{10} + 90ax^9 + 501a^2x^8 + 1,848a^3x^7 + 4,762a^4x^6 + 8,748a^5x^5 + 9,162a^6x^4 + 1,304a^7x^3$ from both sides,) the compound quantity $16,950a^6x + 12,435a^4x^3 - 1,304a^7x^3 - 9,162a^6x - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 5x^{10}$ will be $= 0$. It appears therefore that, while x increases from 0 to $1.03 \times a$, or $a + \frac{3a}{100}$, the said compound quantity will

have increased from 0 (to which it is equal when x is equal to 0,) to some certain quantity, and have decreased again from that quantity to 0. And hence it follows that the successive increments of the sum of the two first terms, $16,950a^6x$ and $12,435a^4x^3$, of the said compound quantity, (from which two terms all the following eight terms of it are subtracted,) will, during the increase of x from 0 to $1.03 \times a$, or $a + \frac{3}{100} \times a$, be, first, greater, and, afterwards, less, than the contemporary increments of the sum of the said following eight terms, $1,304a^7x^3$, $9,162a^6x^4$, $8,748a^5x^5$, $4,762a^4x^6$, $1,848a^3x^7$, $501a^2x^8$, $90ax^9$, and $9x^{10}$. And the greatest magnitude of the whole compound quantity $16,950a^6x + 12,435a^4x^3 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$, or of the excess of the sum of the two first terms of the said quantity above the sum of the eight following terms of it, will be that which it has when the infinitely small increment of the latter sum is equal to the contemporary increment of the former sum.

Now, if $\dot{x}$, or the letter x with a point placed over it, be put for the infinitely small increment of x , the contemporary increment of the former sum, or the binomial quantity $16,950a^6x + 12,435a^4x^3$, will be $(16,950a^6 \times \dot{x} + 12,435a^4 \times 3x^2\dot{x})$, or $16,950a^6\dot{x} + 24,870a^4x^2\dot{x}$; and the contemporary increment of the latter sum, or of the octinomial quantity $1,304a^7x^3 + 9,162a^6x^4 + 8,748a^5x^5 + 4,762a^4x^6 + 1,848a^3x^7 + 501a^2x^8 + 90ax^9 + 9x^{10}$, will be $(1,304a^7 \times 3x^2\dot{x} + 9,162a^6 \times 4x^3\dot{x} + 8,748a^5 \times 5x^4\dot{x} + 4,762a^4 \times 6x^5\dot{x} + 1,848a^3 \times 7x^6\dot{x} + 501a^2 \times 8x^7\dot{x} + 90a \times 9x^8\dot{x} + 9 \times 10x^9\dot{x}) = 3,912a^7x^2\dot{x} + 36,648a^6x^3\dot{x} + 43,740a^5x^4\dot{x} + 28,572a^4x^5\dot{x} + 12,936a^3x^6\dot{x} + 4,008a^2x^7\dot{x} + 810ax^8\dot{x} + 90a^2x^9\dot{x}$. Therefore, in order to find the above-mentioned greatest magnitude of the foregoing compound quantity, we must suppose the increment $3,912a^7x^2\dot{x} + 36,648a^6x^3\dot{x} + 43,740a^5x^4\dot{x} + 28,572a^4x^5\dot{x} + 12,936a^3x^6\dot{x} + 4,008a^2x^7\dot{x} + 810ax^8\dot{x} + 90a^2x^9\dot{x}$ to be equal to the increment $16,950a^6\dot{x} + 24,870a^4x^2\dot{x}$.

Now let all the terms of this equation be divided by $\dot{x}$. And we shall have $3,912a^7x^2 + 36,648a^6x^3 + 43,740a^5x^4 + 28,572a^4x^5 + 12,936a^3x^6 + 4,008a^2x^7 + 810ax^8 + 90a^2x^9 = 16,950a^6 + 24,870a^4x^2$. Therefore (subtracting $24,870a^4x^2$ from both sides,) we shall have $3,912a^7x^2 + 36,648a^6x^3 + 43,740a^5x^4 + 28,572a^4x^5 + 12,936a^3x^6 + 4,008a^2x^7 + 810ax^8 + 90a^2x^9 = 24,870a^4x^2$.

$24,870a^2x = 16,950a^2$, or (placing the terms that involve the higher powers of x before those which involve its lower powers,) $90x^9 + 810ax^8 + 4,008a^2x^7 + 12,936a^3x^6 + 28,572a^4x^5 + 43,740a^5x^4 + 36,648a^6x^3 + 3,912a^7x^2 - 24,870a^8x = 16,950a^2$; and (dividing all the terms by 6,) we shall have $15x^9 + 135ax^8 + 668a^2x^7 + 2,156a^3x^6 + 4,762a^4x^5 + 7,290a^5x^4 + 6,108a^6x^3 + 652a^7x^2 - 4,145a^8x = 2,825a^2$.

Therefore the compound quantity $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 = 9x^{10}$ will have arrived at its greatest magnitude when x is equal to the root of the equation $15a^9 + 135ax^8 + 668a^2x^7 + 2,156a^3x^6 + 4,762a^4x^5 + 7,290a^5x^4 + 6,108a^6x^3 + 652a^7x^2 - 4,145a^8x = 2,825a^2$.

Art. 54. Now the root of this equation will be found to be nearly equal to, but somewhat less than, $0.7177 \times a$. For, if x is supposed to be equal to $0.7177 \times a$, the compound quantity $15x^9 + 135ax^8 + 668a^2x^7 + 2,156a^3x^6 + 4,762a^4x^5 + 7,290a^5x^4 + 6,108a^6x^3 + 652a^7x^2 - 4,145a^8x$ will be found to be $(= 5,803.788,796 \times a^9 - 2,974.866,500,000 \times a^9) = 2,828.921,296,899 \times a^9$, which is less than $2,829 \times a^9$, and therefore exceeds the quantity $2,825a^2$, or the absolute term of the said equation, by less than $4a^9$, or less than the 706th part of the said absolute term. And consequently $0.7177 \times a$ must be very little greater than the true value of x in the said equation. Therefore (by what has been shewn in art. 53,) $0.7177 \times a$ will be very nearly equal to the value of x at the time that the compound quantity $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 = 9x^{10}$ (during the increase of x from 0 to $1.03 \times a$, or $a + \frac{3}{100} \times a$) becomes a *maximum*, or arrives at its greatest magnitude; and consequently the greatest magnitude of that compound quantity will be that value of it which will arise from the substitution of $0.7177 \times a$ instead of x in the several terms that compose it.

Art. 55. Now, if we substitute $0.7177 \times a$ instead of x in the several terms of the compound quantity $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 = 9x^{10}$, we shall have

$$16,950a^2x (= 16,950a^2 \times 0.7177 \times a) = 12,165.0150 \times a^3,$$

$$\text{and } 12,435a^3x^2 = 12,435a^3 \times 0.7177^2 \times a^2 = 12,435a^3 \times 0.515,093,29 \times a^2 = 6,405.185,061,15 \times a^{10},$$

$$\text{and } 1,304a^4x^3 (= 1,304a^4 \times 0.7177^3 \times a^3) = 1,304a^4 \times 0.369,582,454 \times a^3 = 481.935,520,016 \times a^{10},$$

$$\text{and } 9,162a^5x^4 (= 9,162a^5 \times 0.7177^4 \times a^4) = 9,162a^5 \times 0.265,249,327 \times a^4 = 2,430.214,333,974 \times a^{10},$$

and

$$\text{and } 8,748a^4x^4 (= 8,748a^4 \times \overline{0.7177})^4 \times a^4 = 8,748a^4 \times 0.190,369,441 \times a^4) \\ = 1,665,351,869,868 \times a^{10},$$

$$\text{and } 4,762a^4x^6 (= 4,762a^4 \times \overline{0.7177})^6 \times a^6 = 4,762a^4 \times 0.136,628,147 \times a^6) \\ = 650,623,236,014 \times a^{10},$$

$$\text{and } 1,848a^4x^7 (= 1,848a^4 \times \overline{0.7177})^7 \times a^7 = 1,848a^4 \times 0.098,058,021 \times a^7) \\ = 181,201,222,808 \times a^{10},$$

$$\text{and } 501a^4x^8 (= 501a^4 \times \overline{0.7177})^8 \times a^8 = 501a^4 \times 0.070,376,241 \times a^8) \\ = 35,258,496,741 \times a^{10},$$

$$\text{and } 90ax^9 (= 90a \times \overline{0.7177})^9 \times a^9 = 90a \times 0.050,509,028 \times a^9) \\ = 4,545,812,520 \times a^{10},$$

$$\text{and } 9x^{10} (= 9 \times \overline{0.7177})^{10} \times a^{10} = 9 \times 0.036,250,329 \times a^{10}) \\ = 0,326,252,961 \times a^{10};$$

and consequently the whole compound quantity $16,950a^4x + 12,435a^4x^2 - 1,304a^4x^3 - 9,162a^4x^4 - 8,748a^4x^5 - 4,762a^4x^6 - 1,848a^4x^7 - 501a^4x^8 - 90ax^9 - 9x^{10}$ will be =

$$\begin{array}{rcl} 12,165,0150 & \times a^{10} & - 481,935,520,016 \times a^{10} \\ + 6,405,185,061,15 \times a^{10} & & - 2,430,214,333,974 \times a^{10} \\ & & - 1,665,351,869,868 \times a^{10} \\ & & - 650,623,236,014 \times a^{10} \\ & & - 181,201,222,808 \times a^{10} \\ & & - 35,258,496,741 \times a^{10} \\ & & - 4,545,812,520 \times a^{10} \\ & & - 0,326,252,961 \times a^{10} \end{array} \quad \left. \vphantom{\begin{array}{rcl} 12,165,0150 & \times a^{10} & - 481,935,520,016 \times a^{10} \\ + 6,405,185,061,15 \times a^{10} & & - 2,430,214,333,974 \times a^{10} \\ & & - 1,665,351,869,868 \times a^{10} \\ & & - 650,623,236,014 \times a^{10} \\ & & - 181,201,222,808 \times a^{10} \\ & & - 35,258,496,741 \times a^{10} \\ & & - 4,545,812,520 \times a^{10} \\ & & - 0,326,252,961 \times a^{10} \end{array}} \right\}$$

$$= 18,570,200,061,150 \times a^{10} - 5,449,456,744,902 \times a^{10}$$

$$= 13,120,743,316,248 \times a^{10}, \text{ or, very nearly, } 13,121a^{10}.$$

We may therefore consider $13,121a^{10}$ as the *maximum*, or greatest magnitude of the compound quantity $16,950a^4x + 12,435a^4x^2 - 1,304a^4x^3 - 9,162a^4x^4 - 8,748a^4x^5 - 4,762a^4x^6 - 1,848a^4x^7 - 501a^4x^8 - 90ax^9 - 9x^{10}$, to which it will have increased from 0 while x increased from 0 to $0.7177 \times a$, and from which it will decrease again to 0 while x increases further from $0.7177 \times a$

to $1.03 \times a$, or $a + \frac{3}{100} \times a$.

Q. E. I.

Art. 56. Since, while x increases from $0.7177 \times a$ to $1.03 \times a$, the compound quantity $16,950a^4x + 12,435a^4x^2 - 1,304a^4x^3 - 9,162a^4x^4 - 8,748a^4x^5 - 4,762a^4x^6 - 1,848a^4x^7 - 501a^4x^8 - 90ax^9 - 9x^{10}$ decreases

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from

from $13,121a^{10}$ to 0, it follows that it must, at some one instant of time during its said decrease, become equal to any quantity whatsoever that is less than $13,121a^{10}$, and consequently that it must, at some one instant of time during its said decrease, become equal to the quantity $12,625a^{10}$, which is the absolute term of the equation $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$; and therefore there will be some value of x , greater than $0.7177 \times a$, that, being substituted instead of x in the compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$, will make the said compound quantity be equal to $12,625a^{10}$, or the absolute term of the said equation, or, in other words, will be a root of the said equation. We may therefore conclude that the said equation will have two roots, of which the lesser will be equal to $0.632,993,15 \times a$, (as has been shewn in the foregoing articles,) and the greater will be greater than $0.7177 \times a$, but less than $1.03 \times a$. This root we will now proceed to investigate.

Art. 57. Now, if the greater of these roots were as much greater than the middle quantity $0.7177 \times a$ as the said middle quantity is greater than the lesser root $0.632,993,15 \times a$, or (nearly) $0.6330 \times a$, the said greater root would be $(= 2 \times 0.7177 \times a - 0.6330 \times a = 1.4354 \times a - 0.6330 \times a) = 0.8024 \times a$. But when x is become greater than $0.7177 \times a$, the increment of the octinomial quantity $1,304a^7x^3 + 9,162a^6x^4 + 8,748a^5x^5 + 4,762a^4x^6 + 1,848a^3x^7 + 501a^2x^8 + 90ax^9 + 9x^{10}$ is greater than the contemporary increment of the binomial quantity $16,950a^9x + 12,435a^8x^2$, and is continually increasing while x increases from $0.7177 \times a$ to $1.03 \times a$. Therefore, while x increases from $0.7177 \times a$ to $0.8024 \times a$, or receives an increment equal to the excess of $0.7177 \times a$ above $0.6330 \times a$, or the lesser root of the equation, the compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$, or the excess of the binomial quantity $16,950a^9x + 12,435a^8x^2$ above the octinomial quantity $1,304a^7x^3 + 9,162a^6x^4 + 8,748a^5x^5 + 4,762a^4x^6 + 1,848a^3x^7 + 501a^2x^8 + 90ax^9 + 9x^{10}$, will be diminished by a greater quantity than that by which the quantity $13,121a^{10}$ (which is the greatest possible magnitude of the said compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$, and is its value at the instant that x is equal to $0.7177 \times a$), exceeds the quantity $12,625a^{10}$, or the absolute term of the equation $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, and therefore will be less than $12,625a^{10}$, or the absolute term of the said equation. Therefore the value of x at the instant of time at which the said compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ is become a second time equal to the quantity $12,625a^{10}$, must be less than $0.8024 \times a$; or, in other words, the greater root of the equation $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$ must be less

less than $0.8024 \times a$. We will therefore suppose it to be equal to $0.79 \times a$, and will now proceed to substitute $0.79 \times a$ instead of x in the said compound quantity $16,950a^9x + 12,435a^8x^2 + 1,304a^7x^3 + 9,162a^6x^4 + 8,748a^5x^5 + 4,762a^4x^6 + 1,848a^3x^7 + 501a^2x^8 + 90ax^9 + 9x^{10}$, in order to discover how nearly the value of the said compound quantity resulting from the said substitution will approach to the absolute term, $12,625a^{10}$, of the foregoing equation, and whether it will be greater, or less, than the said absolute term: after which we may make further approaches to the true value of x , or the greater root of the said equation, in the manner directed by Mr. Raphson.

Art. 58. Now, if we suppose x to be $= 0.79 \times a$, we shall have

$$\begin{aligned} xx & (= 0.79 \times a)^2 = 0.79^2 \times a^2 = 0.6241 \times a^2, \\ \text{and } x^3 & (= 0.79 \times a)^3 = 0.79^3 \times a^3 = 0.493,039 \times a^3, \\ \text{and } x^4 & (= 0.79 \times a)^4 = 0.79^4 \times a^4 = 0.389,500,81 \times a^4, \\ \text{and } x^5 & (= 0.79 \times a)^5 = 0.79^5 \times a^5 = 0.307,705,639,9 \times a^5, \\ \text{and } x^6 & (= 0.79 \times a)^6 = 0.79^6 \times a^6 = 0.243,087,455 \times a^6, \\ \text{and } x^7 & (= 0.79 \times a)^7 = 0.79^7 \times a^7 = 0.192,039,089 \times a^7, \\ \text{and } x^8 & (= 0.79 \times a)^8 = 0.79^8 \times a^8 = 0.151,710,880 \times a^8, \\ \text{and } x^9 & (= 0.79 \times a)^9 = 0.79^9 \times a^9 = 0.119,851,595 \times a^9, \\ \text{and } x^{10} & (= 0.79 \times a)^{10} = 0.79^{10} \times a^{10} = 0.094,682,760 \times a^{10}. \end{aligned}$$

Therefore $16,950a^9x$ will be $(= 16,950a^9 \times 0.79 \times a$

$$= 16,950 \times 0.79 \times a^{10}) = 13,390.50 \times a^{10};$$

and $12,435a^8x^2$ will be $(= 12,435a^8 \times 0.6241 \times a^2$

$$= 12,435 \times 0.6241 \times a^{10}) = 7760.6835 \times a^{10};$$

and $1,304a^7x^3$ will be $(= 1,304a^7 \times 0.493,039 \times a^3$

$$= 1,304 \times 0.493,039 \times a^{10}) = 642.922,856 \times a^{10};$$

and $9,162a^6x^4$ will be $(= 9,162a^6 \times 0.389,500,81 \times a^4$

$$= 9,162 \times 0.389,500,81 \times a^{10}) = 3,568.606,421,22 \times a^{10};$$

and $8,748a^5x^5$ will be $(= 8,748a^5 \times 0.307,705,639,9 \times a^5$

$$= 8,748 \times 0.307,705,639,9 \times a^{10}) = 2,691.808,937,845 \times a^{10};$$

and $4,762a^4x^6$ will be $(= 4,762a^4 \times 0.243,087,455 \times a^6$

$$= 4,762 \times 0.243,087,455 \times a^{10}) = 1,157.582,460,710 \times a^{10};$$

and $1,848a^3x^7$ will be $(= 1,848a^3 \times 0.192,039,089 \times a^7$

$$= 1,848 \times 0.192,039,089 \times a^{10}) = 354.888,236,472 \times a^{10};$$

and $501a^2x^8$ will be $(= 501a^2 \times 0.151,710,880 \times a^8$

$$= 501 \times 0.151,710,880 \times a^{10}) = 76.007,150,880 \times a^{10};$$

3 D 2

and

and $90ax^9$ will be $(= 90a \times 0.119,851,595 \times a^9)$

$$= 90 \times 0.119,851,595 \times a^{10} = 10.786,643,550 \times a^{10};$$

and $9x^{10}$ will be $(= 9 \times 0.094,682,760 \times a^{10}) = 0.852,144,840 \times a^{10}$.

Therefore the whole compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ will be =

$$\left\{ \begin{array}{l} 13,390.50 \times a^{10} \\ + 7,760.683,5 \times a^{10} \end{array} \right. \quad \left\{ \begin{array}{l} 642.922,856,000 \times a^{10} \\ - 3,568.606,421,220 \times a^{10} \\ - 2,691.808,937,845 \times a^{10} \\ - 1,157.582,460,710 \times a^{10} \\ - 354.888,237,472 \times a^{10} \\ - 76.007,150,880 \times a^{10} \\ - 10.786,643,550 \times a^{10} \\ - 0.852,144,840 \times a^{10} \end{array} \right.$$

$$= 21,151.183,5 \times a^{10} \quad - 8,503.454,851.517 \times a^{10}$$

$= 12,647.728,648.483 \times a^{10}$; which is a little greater than $12,625a^{10}$, or the absolute term of the equation $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$. Therefore (by what has been shewn in art. 53, 54, and 55,) $0.79 \times a$ will be a little less than the true value of the greater root of that equation.

Art. 59. In order therefore to find a nearer value of this root, we will now suppose $0.79 \times a + z$ to be equal to it, and will substitute $0.79 \times a + z$ instead of x in the terms of the said equation, and will then resolve the transformed equation resulting from this substitution, in the manner directed by Mr. Raphion. This may be done as follows :

Since x is $= 0.79 \times a + z$, we shall have

$$\begin{aligned} xx & (= \overline{0.79 \times a + z})^2 = \overline{0.79 \times a}^2 + 2 \times 0.79 \times a \times z + \&c \\ &= 0.79^2 \times a^2 + 1.58 \times az + \&c) \\ &= 0.6241 \times a^2 + 1.58az + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^3 & (= \overline{0.79 \times a + z})^3 = \overline{0.79 \times a}^3 + 3 \times \overline{0.79 \times a}^2 \times z + \&c \\ &= 0.79^3 \times a^3 + 3 \times 0.6241a^2 \times z + \&c) \\ &= 0.493,039 \times a^3 + 1.8723a^2z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^4 & (= \overline{0.79 \times a + z})^4 = \overline{0.79 \times a}^4 + 4 \times \overline{0.79 \times a}^3 \times z + \&c \\ &= 0.79^4 \times a^4 + 4 \times 0.493,039a^3 \times z + \&c) \\ &= 0.389,500,81a^4 + 1.972,156a^3z + \&c, \end{aligned}$$

and

$$\begin{aligned}\text{and } x^5 & (= \overline{0.79 \times a + z})^5 = \overline{0.79 \times a}^5 + 5 \times \overline{0.79 \times a}^4 \times z + \&c \\ & = \overline{0.79}^5 \times a^5 + 5 \times 0.389,500,81a^4 \times z + \&c) \\ & = 0.307,705,639,9a^5 + 1.947,504,05a^4z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^6 & (= \overline{0.79 \times a + z})^6 = \overline{0.79 \times a}^6 + 6 \times \overline{0.79 \times a}^5 \times z + \&c \\ & = \overline{0.79}^6 \times a^6 + 6 \times 0.307,705,639,9a^5 \times z + \&c) \\ & = 0.243,087,455a^6 + 1.846,233,839,4a^5z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^7 & (= \overline{0.79 \times a + z})^7 = \overline{0.79 \times a}^7 + 7 \times \overline{0.79 \times a}^6 \times z + \&c \\ & = \overline{0.79}^7 \times a^7 + 7 \times 0.243,087,455a^6 \times z + \&c) \\ & = 0.192,039,089a^7 + 1.701,612,185a^6z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^8 & (= \overline{0.79 \times a + z})^8 = \overline{0.79 \times a}^8 + 8 \times \overline{0.79 \times a}^7 \times z + \&c \\ & = \overline{0.79}^8 \times a^8 + 8 \times 0.192,039,089a^7 \times z + \&c) \\ & = 0.151,710,880a^8 + 1.536,312,712a^7z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^9 & (= \overline{0.79 \times a + z})^9 = \overline{0.79 \times a}^9 + 9 \times \overline{0.79 \times a}^8 \times z + \&c \\ & = \overline{0.79}^9 \times a^9 + 9 \times 0.151,710,880a^8 \times z + \&c) \\ & = 0.119,851,595 \times a^9 + 1.365,398,020a^8z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } x^{10} & (= \overline{0.79 \times a + z})^{10} = \overline{0.79 \times a}^{10} + 10 \times \overline{0.79 \times a}^9 \times z + \&c \\ & = \overline{0.79}^{10} \times a^{10} + 10 \times 0.119,851,595a^9 \times z + \&c) \\ & = 0.094,682,760a^{10} + 1.198,515,950a^9z + \&c.\end{aligned}$$

$$\begin{aligned}\text{Therefore } 16,950a^9x \text{ will be } & (= 16,950a^9 \times \overline{0.79 \times a + z}) \\ & = 16,950a^9 \times 0.79 \times a + 16,950a^9 \times z = 16,950 \times 0.79 \times a^{10} \\ & + 16,950a^9z = 13,390.50 \times a^{10} + 16,950a^9z;\end{aligned}$$

$$\begin{aligned}\text{and } 12,435a^8x^5 \text{ will be } & (= 12,435a^8 \times \overline{0.6241a^5 + 1.58az} + \&c) \\ & = 12,435a^8 \times 0.6241a^5 + 12,435a^8 \times 1.58az + \&c) \\ & = 7,770.6835a^{10} + 18,647.30 \times a^8z + \&c;\end{aligned}$$

$$\begin{aligned}\text{and } 1,304a^7x^3 \text{ will be } & (= 1,304a^7 \times \overline{0.493,039 \times a^3 + 1.8723a^2z} + \&c) \\ & = 1,304a^7 \times 0.493,039a^3 + 1,304a^7 \times 1.8723a^2z + \&c) \\ & = 642.922,856a^{10} + 2441.4792a^9z + \&c;\end{aligned}$$

$$\begin{aligned}\text{and } 9,162a^6x^2 \text{ will be } & (= 9,162a^6 \times \overline{0.389,500,81a^2 + 1.972,156a^1z} + \&c) \\ & = 9,162a^6 \times 0.389,500,81a^2 + 9,162a^6 \times 1.972,156a^1z + \&c) \\ & = 3,568.606,421,22a^{10} + 18,068.893,272a^9z + \&c,\end{aligned}$$

$$\begin{aligned}\text{and } 8,748a^5x^5 \text{ will be } & (= 8,748a^5 \times \overline{0.307,705,639,9a^1 + 1.947,504,05a^0z} + \&c) \\ & = 8,748a^5 \times 0.307,705,639,9a^1 + 8,748a^5 \times 1.947,504,05a^0z + \&c) \\ & = 2,691,808,937,845a^{10} + 17,036.765,429,400a^9z + \&c;\end{aligned}$$

and

and $4,762a^4x^8$ will be $(= 4,762a^4 \times$

$$\begin{aligned} & \frac{0.243,087,455a^8 + 1.840,233,839,4a^8z + \&c}{=} \\ & = 4,762a^4 \times 0.243,087,455a^8 + 4,762a^4 \times 1.846,233,839,4a^8z + \&c) \\ & = 1,157,582,460,710a^{10} + 8,791,765,543,222,8a^8z + \&c; \end{aligned}$$

and $1,848a^3x^7$ will be $(= 1,848a^3 \times$

$$\begin{aligned} & \frac{0.192,039,089a^7 + 1.701,612,185a^7z + \&c}{=} \\ & = 1,848a^3 \times 0.192,039,089a^7 + 1,848a^3 \times 1.701,612,185a^7z + \&c) \\ & = 354,888,236,472a^{10} + 3,144,579,317,880a^5z + \&c; \end{aligned}$$

and $501a^2x^6$ will be $(= 501a^2 \times$

$$\begin{aligned} & \frac{0.151,710,880a^6 + 1.536,312,712a^6z + \&c}{=} \\ & = 501a^2 \times 0.151,710,880a^6 + 501a^2 \times 1.536,312,712a^6z + \&c) \\ & = 76,007,150,880a^{10} + 769,692,668,712a^5z + \&c; \end{aligned}$$

and $90ax^5$ will be $(= 90a \times 0.119,851,595a^5 + 1.365,398,020a^5z + \&c$

$$\begin{aligned} & = 90a \times 0.119,851,595a^5 + 90a \times 1.365,398,020a^5z + \&c) \\ & = 10,786,643,550a^{10} + 122,885,821,800a^5z + \&c; \end{aligned}$$

and $9x^{10}$ will be $(= 9 \times 0.094,682,760a^{10} + 1.198,515,950a^5z + \&c$

$$\begin{aligned} & = 9 \times 0.094,682,760a^{10} + 9 \times 1.198,515,950a^5z + \&c) \\ & = 0,852,144,840a^{10} + 10,786,643,550a^5z + \&c. \end{aligned}$$

Therefore the whole compound quantity $16,950a^8x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ will be =

$$\begin{aligned} & \left\{ \begin{array}{ll} 13,390.50 \times a^{10} & + 16,950 \times a^8z \\ + 7,760.683,5 \times a^{10} & + 18,647.30 \times a^8z + \&c \\ - 642.922,856 \times a^{10} & - 2,441,479,2 \times a^8z - \&c \\ - 3,568.606,421,22 \times a^{10} & - 18,068,893,272 \times a^8z - \&c \\ - 2,691.808,937,845 \times a^{10} & - 17,036.765,429,40 \times a^8z - \&c \\ - 1,157,582,460,710 \times a^{10} & - 8,791,765,543,222 \times a^8z - \&c \\ - 354,888,236,472 \times a^{10} & - 3,144,579,317,880 \times a^8z - \&c \\ - 76,007,150,880 \times a^{10} & - 769,692,668,712 \times a^8z - \&c \\ - 10,786,643,550 \times a^{10} & - 122,885,821,800 \times a^8z - \&c \\ - 0,852,144,840 \times a^{10} & - 10,786,643,550 \times a^8z - \&c \end{array} \right\} \\ & = \left\{ \begin{array}{ll} 21,151.183,500 \times a^{10} & + 35,597,300,000,000 \times a^8z + \&c \\ - 8,503,454,851,517 \times a^{10} & - 50,386,847,896,564 \times a^8z - \&c \end{array} \right\} \\ & = 12,647,728,648,483 \times a^{10} - 15,289,547,896,564 \times a^8z + \&c. \end{aligned}$$

But

But the said compound quantity is $= 12,625a^{10}$.

Therefore the binomial quantity $12,647,728,648,483a^9 - 15,289,547,896,564 \times a^8$ will also be $= 12,625a^{10}$. And consequently (adding $15,289,547,896,564a^8$ to both sides,) $12,647,728,648,483 \times a^{10}$ will be $= 12,625a^{10} + 15,289,547,896,564a^8$; and (subtracting $12,625a^{10}$ from both sides) $15,289,547,896,564a^8$ will be $= 22,728,648,483 \times a^{10}$; and (dividing both sides by a^8) $15,289,547,896,654 \times z$ will be $= 22,728,648,483 \times a$, and z will be $= \frac{22,728,648,483 \times a}{15,289,547,896,654} = 0,001,48 \times a$. Therefore x , or $0,79 \times a + z$, will be $(= 0,79 \times a + 0,001,48 \times a) = 0,791,48 \times a$, or the greater root of the equation $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$ will be nearly equal to $0,791,48 \times a$. Q. E. I.

Art. 60. Now let $0,791,48 \times a$ be substituted instead of x in the compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$, in order to discover whether the value of it resulting therefrom will be nearly equal to $12,625a^{10}$, or the absolute term of the foregoing equation, and consequently whether $0,791,48 \times a$ is nearly equal to the greater root of the said equation, and to discover also whether the said resulting value will be greater, or less, than the said absolute term, and consequently whether $0,791,48 \times a$ is less, or greater, than the true value of the said greater root. This substitution may be made as follows:

If x is $= 0,791,48 \times a$, we shall have

$$\begin{aligned}
 & 9x^{10} (= \frac{0,791,48 \times a}{10})^{10} = \frac{0,791,48^{10} \times a^{10}}{10} = 0,626,440,590 \times a^{10}, \\
 & \text{and } 90ax^9 (= \frac{0,791,48 \times a}{9})^9 = \frac{0,791,48^9 \times a^9}{9} = 0,495,815,198 \times a^9, \\
 & \text{and } 901a^2x^8 (= \frac{0,791,48 \times a}{901})^8 = \frac{0,791,48^8 \times a^8}{901} = 0,392,427,812 \times a^8, \\
 & \text{and } 4,762a^4x^6 (= \frac{0,791,48 \times a}{4,762})^6 = \frac{0,791,48^6 \times a^6}{4,762} = 0,310,598,764 \times a^6, \\
 & \text{and } 8,748a^5x^5 (= \frac{0,791,48 \times a}{8,748})^5 = \frac{0,791,48^5 \times a^5}{8,748} = 0,245,832,709 \times a^5, \\
 & \text{and } 9,162a^6x^4 (= \frac{0,791,48 \times a}{9,162})^4 = \frac{0,791,48^4 \times a^4}{9,162} = 0,194,571,672 \times a^4, \\
 & \text{and } 1,304a^7x^3 (= \frac{0,791,48 \times a}{1,304})^3 = \frac{0,791,48^3 \times a^3}{1,304} = 0,153,999,586 \times a^3, \\
 & \text{and } 12,435a^8x^2 (= \frac{0,791,48 \times a}{12,435})^2 = \frac{0,791,48^2 \times a^2}{12,435} = 0,121,887,592 \times a^2, \\
 & \text{and } 16,950a^9x (= \frac{0,791,48 \times a}{16,950}) = \frac{0,791,48 \times a^9}{16,950} = 0,096,471,591 \times a^{10}.
 \end{aligned}$$

Therefore $16,950a^9x$ will be $(= 16,950a^9 \times 0,791,48 \times a)$

$$= 13,415,586,00 \times a^{10};$$

and $12,435a^8x^2$ will be $(= 12,435a^8 \times 0,626,440,590 \times a^2)$

$$= 7,789,788,736,650 \times a^{10};$$

and

and $1,304a^7x^3$ will be $(= 1,304a^7 \times 0.495,815,198 \times a^4)$

$$= 646,543,018,192 \times a^{10};$$

and $9,162a^4x^4$ will be $(= 9,162a^4 \times 0.392,427,812 \times a^6)$

$$= 3,595,423,613,544 \times a^{10};$$

and $8,748a^3x^3$ will be $(= 8,748a^3 \times 0.310,598,764 \times a^7)$

$$= 2,717,117,987,472 \times a^{10};$$

and $4,762a^2x^5$ will be $(= 4,762a^2 \times 0.245,832,709 \times a^8)$

$$= 1,170,655,360,258 \times a^{10};$$

and $1,848a^2x^7$ will be $(= 1,848a^2 \times 0.194,571,672a^7)$

$$= 359,568,449,856 \times a^{10};$$

and $501a^2x^3$ will be $(= 501a^2 \times 0.153,999,586a^4)$

$$= 77,153,792,586 \times a^{10};$$

and $90ax^9$ will be $(= 90a \times 0.121,887,592 \times a^9)$

$$= 10,969,883,280 \times a^{10};$$

and $9x^{10}$ will be $(= 9 \times 0.096,471,591a^{10})$

$$= 0,868,244,319 \times a^{10}.$$

Therefore the whole compound quantity $16,950a^5x + 12,435a^3x^2 - 1,304a^7x^3 - 9,162a^4x^4 - 8,748a^3x^3 - 4,762a^2x^5 - 1,848a^2x^7 - 501a^2x^3 - 90ax^9 - 9x^{10}$ will be =

$$\left\{ \begin{array}{l} 13,415,586,000,000 \times a^{10} - 646,543,018,192 \times a^{10} \\ + 7,789,788,736,650 \times a^{10} - 3,595,423,613,544 \times a^{10} \\ - 2,717,117,987,472 \times a^{10} \\ - 1,170,655,360,258 \times a^{10} \\ - 359,568,449,856 \times a^{10} \\ - 77,153,792,586 \times a^{10} \\ - 10,969,883,280 \times a^{10} \\ - 0,868,244,319 \times a^{10} \end{array} \right\}$$

$$= 21,205,374,736,650 \times a^{10} - 8,578,300,349,507 \times a^{10}$$

$= 12,627,074,387,143 \times a^{10}$; which is very nearly equal to $12,625a^{10}$, which it exceeds by only the small quantity $2,074,387,143 \times a^{10}$, or the 6086th part of $12,625a^{10}$.

Therefore $0.791,48 \times a$ must be very nearly equal to the true value of the greater root of the equation $16,950a^5x + 12,435a^3x^2 - 1,304a^7x^3 - 9,162a^4x^4 - 8,748a^3x^3 - 4,762a^2x^5 - 1,848a^2x^7 - 501a^2x^3 - 90ax^9 - 9x^{10} = 12,625a^{10}$, but (by what is observed in art. 53, 54, and 55,) will be somewhat less than the said true value.

Q. E. I.

Art. 61.

Art. 61. In order therefore to obtain a still nearer value of this greater root of the said equation, we will now suppose it to be equal to $0.791,48 \times a + z$, and will substitute the said binomial quantity $0.791,48 \times a + z$ instead of x in the terms of the said equation, and then will resolve the transformed equation thence arising in the manner directed by Mr. Raphson. This may be done in the manner following :

Since x is $= 0.791,48 \times a + z$, we shall have

$$\begin{aligned} xx &= (0.791,48 \times a + z)^2 = 0.791,48 \times a^2 + 2 \times 0.791,48 \times a \times z + \&c \\ &= 0.791,48^2 \times a^2 + 1.582,96 \times az + \&c \\ &= 0.626,440,590a^2 + 1.582,96az + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^3 &= (0.791,48 \times a + z)^3 = 0.791,48 \times a^3 + 3 \times 0.791,48 \times a^2 \times z + \&c \\ &= 0.791,48^3 \times a^3 + 3 \times 0.626,440,590a^2 \times z + \&c \\ &= 0.495,815,198a^3 + 1.879,321,770a^2z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^4 &= (0.791,48 \times a + z)^4 = 0.791,48 \times a^4 + 4 \times 0.791,48 \times a^3 \times z + \&c \\ &= 0.791,48^4 \times a^4 + 4 \times 0.495,815,198a^3 \times z + \&c \\ &= 0.392,427,812a^4 + 1.983,260,792a^3z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^5 &= (0.791,48 \times a + z)^5 = 0.791,48 \times a^5 + 5 \times 0.791,48 \times a^4 \times z + \&c \\ &= 0.791,48^5 \times a^5 + 5 \times 0.392,427,812a^4 \times z + \&c \\ &= 0.310,598,764a^5 + 1.962,139,060a^4z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^6 &= (0.791,48 \times a + z)^6 = 0.791,48 \times a^6 + 6 \times 0.791,48 \times a^5 \times z + \&c \\ &= 0.791,48^6 \times a^6 + 6 \times 0.310,598,764a^5 \times z + \&c \\ &= 0.245,832,709a^6 + 1.863,592,584a^5z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^7 &= (0.791,48 \times a + z)^7 = 0.791,48 \times a^7 + 7 \times 0.791,48 \times a^6 \times z + \&c \\ &= 0.791,48^7 \times a^7 + 7 \times 0.245,832,709a^6 \times z + \&c \\ &= 0.194,571,672a^7 + 1.720,828,963a^6z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^8 &= (0.791,48 \times a + z)^8 = 0.791,48 \times a^8 + 8 \times 0.791,48 \times a^7 \times z + \&c \\ &= 0.791,48^8 \times a^8 + 8 \times 0.194,571,672a^7 \times z + \&c \\ &= 0.153,999,586a^8 + 1.556,573,376a^7z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^9 &= (0.791,48 \times a + z)^9 = 0.791,48 \times a^9 + 9 \times 0.791,48 \times a^8 \times z + \&c \\ &= 0.791,48^9 \times a^9 + 9 \times 0.153,999,586a^8 \times z + \&c \\ &= 0.121,887,592a^9 + 1.385,996,274a^8z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } x^{10} &= (0.791,48 \times a + z)^{10} = 0.791,48 \times a^{10} + 10 \times 0.791,48 \times a^9 \times z + \&c \\ &= 0.791,48^{10} \times a^{10} + 10 \times 0.121,887,592a^9 \times z + \&c \\ &= 0.096,471,591a^{10} + 1.218,875,920a^9z + \&c. \end{aligned}$$

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Therefore

$$\text{Therefore } 16,950a^x \text{ will be } (= 16,950a^x \times 0.791,48 \times a + z = 16,950a^x \\ \times 0.791,48 \times a + 16,950a^x \times z) = 13,415,586,00a^{10} + 16,950a^x z;$$

$$\text{and } 12,435a^x \text{ will be } (= 12,435a^x \times 0.626,440,590a^x + 1.582,96az + \&c \\ = 12,435a^x \times 0.626,440,590a^x + 12,435a^x \times 1.582,96az + \&c) \\ = 7,789.788,736,650a^{10} + 19,684.107,60a^x z + \&c;$$

$$\text{and } 1,304a^x \text{ will be } (= 1,304a^x \times \\ 0.495,815,198a^x + 1.879,321,770a^x z + \&c \\ = 1,304a^x \times 0.495,815,198a^x + 1,304a^x \times 1.879,321,770a^x z + \&c) \\ = 646.543,018,192a^{10} + 2,450.635,588,080a^x z + \&c;$$

$$\text{and } 9,162a^x \text{ will be } (= 9,162a^x \times \\ 0.392,427,812a^x + 1.983,260,792a^x z + \&c \\ = 9,162a^x \times 0.392,427,812a^x + 9,162a^x \times 1.983,260,792a^x z + \&c) \\ = 3,595.423,613,544a^{10} + 18,170.635,376,304a^x z + \&c;$$

$$\text{and } 8,748a^x \text{ will be } (= 8,748a^x \times \\ 0.310,598,764a^x + 1.962,139,060a^x z + \&c \\ = 8,748a^x \times 0.310,598,764a^x + 8,748a^x \times 1.962,139,060a^x z + \&c) \\ = 2,717.117,987,472a^{10} + 17,164.792,496,880a^x z + \&c;$$

$$\text{and } 4,762a^x \text{ will be } (= 4,762a^x \times \\ 0.245,832,709a^x + 1.863,592,584a^x z + \&c \\ = 4,762a^x \times 0.245,832,709a^x + 4,762a^x \times 1.863,592,584a^x z + \&c) \\ = 1,170.655,360,238a^{10} + 8,874.427,885,008a^x z + \&c;$$

$$\text{and } 1,848a^x \text{ will be } (= 1,848a^x \times \\ 0.194,571,672a^x + 1.720,828,963a^x z + \&c \\ = 1,848a^x \times 0.194,571,672a^x + 1,848a^x \times 1.720,828,963a^x z + \&c) \\ = 359.568,449,856a^{10} + 3,180.091,923,624a^x z + \&c;$$

$$\text{and } 501a^x \text{ will be } (= 501a^x \times \\ 0.153,999,586a^x + 1.556,573,376a^x z + \&c \\ = 501a^x \times 0.153,999,586a^x + 501a^x \times 1.556,573,376a^x z + \&c) \\ = 77.153,792,586a^{10} + 779.843,261,376a^x z + \&c;$$

$$\text{and } 90a^x \text{ will be } (= 90a^x \times 0.121,887,592a^x + 1.385,996,274a^x z + \&c \\ = 90a^x \times 0.121,887,592a^x + 90a^x \times 1.385,996,274a^x z + \&c) \\ = 10,969,883,280a^{10} + 124.739,664,660a^x z + \&c;$$

and

and $9x^{10}$ will be $(= 9 \times 0.096,471,591a^{10} + 1.218,875,920a^2z + \&c$
 $= 9 \times 0.096,471,591a^{10} + 9 \times 1.218,875,920a^2z + \&c)$
 $= 0.868,244,319a^{10} + 10.969,883,280 \times a^2z + \&c.$

Therefore the whole compound quantity $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3$
 $- 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 -$
 $9x^{10}$ will be =

$$\left\{ \begin{array}{ll} 13,415,586,00 \times a^{10} & + 16,950,000,00 \times a^2z + \&c \\ + 7,789,788,736,650 \times a^{10} & + 19,684,107,60 \times a^2z + \&c \\ - 646,543,018,192 \times a^{10} & - 2,450,635,588,080 \times a^2z - \&c \\ - 3,595,423,613,544 \times a^{10} & - 18,170,635,376,304 \times a^2z - \&c \\ - 2,717,117,987,472 \times a^{10} & - 17,164,792,496,880 \times a^2z - \&c \\ - 1,170,655,360,258 \times a^{10} & - 8,874,427,885,008 \times a^2z - \&c \\ - 359,568,449,856 \times a^{10} & - 3,180,091,923,624 \times a^2z - \&c \\ - 77,153,792,586 \times a^{10} & - 779,843,261,376 \times a^2z - \&c \\ - 10,969,883,280 \times a^{10} & - 124,739,664,660 \times a^2z - \&c \\ - 0.868,244,319 \times a^{10} & - 10.969,883,280 \times a^2z - \&c \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} 21,205,374,736,650 \times a^{10} & + 36,634,107,600,000 \times a^2z + \&c \\ - 8,578,300,349,507 \times a^{10} & - 50,756,136,079,212 \times a^2z - \&c \end{array} \right\}$$

$$= 12,627,074,387,143 \times a^{10} - 14,122,028,479,212 \times a^2z \&c.$$

But the said compound quantity is equal to $12,625a^{10}$.

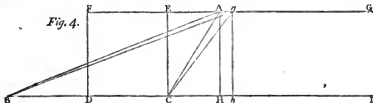
Therefore the binomial quantity $12,627,074,387,143 \times a^{10} - 14,122,028,479,212 \times a^2z$ will also be $= 12,625a^{10}$. And consequently (adding $14,122,028,479,212 \times a^2z$ to both sides,) we shall have $12,627,074,387,143 \times a^{10} = 12,625a^{10} + 14,122,028,479,212 \times a^2z$; and (subtracting $12,625a^{10}$ from both sides,) we shall have $2,074,387,143 \times a^{10} = 14,122,028,479,212 \times a^2z$; and (dividing both sides by a^2) $2,074,387,143 \times a = 14,122,028,479,212 \times z$,

and $z = \frac{2,074,387,143 \times a}{14,122,028,479,212} = 0.000,146,8 \times a$. Therefore x , or $0.791,48 \times a + z$, will be $(= 0.791,48 \times a + 0.000,146,8 \times a = 0.791,626,8 \times a$; or $0.791,626,8 \times a$ will be a still nearer value of the greater root of the equation $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$ than that we had before obtained, to wit, $0.791,48 \times a$. Q. E. I.

This value, $0.791,626,8 \times a$, of the greater root of this equation is, probably, exact in the first six figures, $0.791,626$, and, perhaps, also in the last figure, 8 . But to determine this with certainty, it would be necessary to carry our approximation to the true value of this root one step further; which, after the great length this solution has already run into, seems hardly worth our while.

Of the Relation of the greater Root, $0.791,626,8 \times a$, just now found, of the long Equation of the 10th Order resulting from the Algebraick Investigation of Mr. Glenie's Problem, to the Solution of another Problem that bears a great Resemblance to Mr. Glenie's Problem.

Art. 62. This greater root of the equation $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$ is fitted to solve another Problem that has a considerable analogy to that which we have been here considering, namely, a Problem in which it should be required to determine the vertex of another triangle upon the same base BC as the triangle ABC in fig. 2, and of the same perpendicular altitude DF, but of which the sides should be so much greater than the sides of the triangle ABC, that the difference of their cubes (instead of their sum,) shall be equal to $24a^3$, (or $3 \times 8a^3$, or $3 \times 2a]^3$, or $3 \times BC^3$), or three times the cube of the base BC. For, if in the line BCI we take the line Cb (as in fig. 4,) equal to this greater root, $0.791,626 \times a$, of the said equation, and from the point b draw the line ba at right angles to Cb, or BC, and meeting the line FG in the point a, and from the point a we draw the right lines aB and aC to the extremities of the given base BC, the excess of the cube of aB, the greater side of the new triangle aBC, above the cube of aC, its lesser side, will be equal to three times the cube of the base BC. This may be shewn in the manner following :



Art. 63. Let the perpendicular altitude ab , or FD, of this triangle, which is, as before, equal to half the base BC, be denoted, as before, by the letter a , and consequently the base BC by $2a$; and let Cb, or the portion of the line BCI intercepted between the point C and the right line ab , which is drawn from a , the vertex of the new triangle aBC (of which the sides aB and aC are of such magnitudes that $aB^3 - aC^3$ is equal to $3BC^3$) at right angles to BCI and meeting it in the point b, be denoted by the letter x . Then will aC^2 be $(= ab^2 + Cb^2) = aa + xx$, and aB^2 will be $(= ab^2 + Bb^2 =$

ab^2

$$aB^2 + \overline{BC + Cb}^2 = aa + \overline{2a + x}^2 = aa + 4aa + 4ax + xx = 5aa + 4ax + xx.$$

Now let aB be put $= y$, and aC be put $= x$.

Then, by the conditions of the present Problem, we shall have $y^2 - z^2 = 24a^2$. Therefore (squaring both sides,) we shall have $y^2 - 2y^2z^2 + z^2 = 576a^2$, and (adding $2y^2z^2$ to both sides,) $y^2 + z^2 = 576a^2 + 2y^2z^2$, and (subtracting $576a^2$ from both sides,) $2y^2z^2 = y^2 + z^2 - 576a^2$, and (by squaring both sides,) $4y^2z^2 = y^{12} + z^{12} + 2y^6z^6 - 1152a^6y^6 - 1152a^6z^6 + 331,776a^{12}$, and (by subtracting $2y^2z^2$ from both sides,) $2y^2z^2 = y^{12} + z^{12} - 1152a^6y^6 - 1152a^6z^6 + 331,776a^{12}$, and (by adding $1152a^6y^6 + 1152a^6z^6$ to both sides,) $2y^2z^2 + 1152a^6y^6 + 1152a^6z^6 = y^{12} + z^{12} + 331,776a^{12}$, and (by subtracting $y^{12} + z^{12}$ from both sides,) $2y^2z^2 + 1152a^6y^6 + 1152a^6z^6 - y^{12} - z^{12} = 331,776a^{12}$; which is the very same equation we derived in art. 28, from the former Problem. Therefore, if we substitute $aa + xx$ instead of zz , and $5aa + 4ax + xx$ instead of yy , in this equation, as was done above in art. 29 and 30, we shall ultimately come to the same equation as was obtained in art. 32, by means of those substitutions, to wit, the equation $16,950a^8x + 12,435a^6x^2 - 13,044a^4x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^4x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, the greater root of which will give us the length of the line Cb , as we have already seen that the lesser root of it has given us the length of the line CH . Q. E. D.

A Proof of the Relation of the said greater Root, $0.791,626,8 \times a$, to the Solution of the said Second Problem.

Art. 64. Since aC^2 is $= aa + xx$, and aB^2 is $= 5aa + 4ax + xx$, it follows that aC will be $= \sqrt{aa + xx}$, and aB will be $= \sqrt{5aa + 4ax + xx}$. Now we have found that x is nearly equal to $0.791,626,8 \times a$. Therefore $4ax$ will be $(= 4a \times 0.791,626,8 \times a = 4 \times 0.791,626,8 \times a^2) = 3.166,507,2 \times a^2$, and xx will be $(= \overline{0.791,626,8 \times a}^2 = \overline{0.791,626,8}^2 \times a^2) = 0.626,672,990,478,24a^2$. Therefore aC^2 , or $aa + xx$, will be $(= aa + 0.626,672,990,478,24 \times a^2) = 1.626,672,990,478,24 \times a^2$, and aB^2 , or $5aa + 4ax + xx$, will be $(= 5aa + 3.166,507,2 \times aa + 0.626,672,990,478,24 \times aa) = 8.793,180,190,478,24 \times aa$. Therefore aC will be $(= a \times \sqrt{1.626,672,990,478,24}) = a \times 1.275,410,9$, or $1.275,410,9 \times a$; and aB will be $(= a \times \sqrt{8.793,180,190,478,24}) = a \times 2.965,329,6$, or $2.965,329,6 \times a$; and consequently aC

aC^3 will be $(= aC^2 \times aC = 1.626,672,990,478,24 \times aa \times 1.275,410,9 \times a)$
 $= 2.074,676,462 \times a^3$, and aB^3 will be $(= aB^2 \times aB = 8.793,180,190,$
 $478,24 \times aa \times 2.965,329,6 \times a) = 26.074,677,496 \times a^3$. Therefore aB^3
 $- aC^3$ will be $= 26.074,677,496 \times a^3 - 2.074,676,462 \times a^3 = 24.000,$
 $001,034 \times a^3$; which is very nearly equal to $24a^3$, (or $3 \times 8a^3$, or $3 \times 2a^3$),
 or $3BC^3$; and consequently the sides aB and aC of the triangle aBC have the
 property required by the second Problem. Therefore the greater value of x
 in the equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5$
 $- 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$ now ap-
 pears to be the value of the line Cb , which determines the vertex a of the se-
 cond triangle aBC , which is required to be described in the second Problem,
 and in which the difference of the cubes of the sides aB and aC is equal to
 three times the cube of the base BC , just as the lesser value of x in the same
 equation was the value of the line CH , which determines the vertex A of the
 first triangle ABC , which was required to be described in the first Problem,
 and in which the sum of the cubes of the sides AB and AC was equal to three
 times the cube of the same base BC . And thus the Problem is now completely
 solved by Algebra and Arithmetick, and all the difficulties relating to the final
 equation produced by it are satisfactorily answered, without the help of Mr.
 Glenie's Construction. But still I am at a loss to conjecture by what artifice
 Mr. Glenie could discover that the said final equation was capable of being re-
 duced by an operation of Division from an equation of the tenth order to a
 quadratick equation, and thereby made capable of being constructed geometri-
 cally, or by drawing only right lines and circles.

And further, I very much doubt whether the second Problem, (which relates
 to the difference of the cubes of the sides of the proposed triangle,) can be con-
 structed in the same manner as Mr. Glenie has constructed the first Problem,
 or by drawing only right lines and circles.

Farther Observations, in Addition to those contained above in Art. 39, 40, 41, 42, and 43, on the Reduction of the foregoing Equation of the 10th Order (that was obtained by the Algebraick Investigation of the Value of the line CH), to a Quadratick Equation, by means of the Operation of Division performed in Art. 33, page 356.

Art. 65. It was observed above in art. 39, that a plausible objection might be made to the division of the long compound quantity $- 12,625a^{10} + 16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7$

$-501a^2x^9 - 90ax^9 - 9x^{10}$ by the compound quantity $-2,525a^3 + 360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, on account of the said dividend's being equal to nothing, and consequently being incapable of being either multiplied or divided. And hence, it was observed, a doubt might be entertained concerning the truth of the conclusion derived from it, to wit, that the trinomial quantity $5aa - 6ax - 3xx$, (which was the quotient of the said division,) would be equal to nothing, or that $5aa$ would be equal to $6ax + 3xx$, or that the quadratic equation $3xx + 6ax = 5aa$ would be a true proposition. And therefore some further and plainer reasonings were made use of in art. 41, 42, and 43, to strengthen the said conclusion and establish its truth; which were grounded on the multiplication of the said long divisor, first, by the single quantity $5aa$, and, secondly, by the binomial quantity $6ax + 3xx$, and on the subtraction of the second of the products thereby obtained from the first product; all which operations, being performed upon real quantities, or such as had a finite magnitude, were evidently practicable. And the conclusion obtained in art. 42 and 43, by means of these reasonings, to wit, that $5aa$ was equal to $6ax + 3xx$, seems to have been fairly and justly proved. But I have since thought of another method of proving the truth of the same conclusion without having recourse to those multiplications, and without ever mentioning any dividend that is equal to nothing, or of which the subtracted members are, together, equal to those from which they are subtracted, and consequently without giving any handle for making the above-mentioned objection. This method I will now endeavour to explain:

Art. 66. The final equation obtained by the foregoing algebraick investigation of the value of the line CH, or x , was $16,950a^5x + 12,435a^5x^2 - 1,304a^5x^3 - 9,162a^5x^4 - 8,748a^5x^5 - 4,762a^5x^6 - 1,848a^5x^7 - 501a^5x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$.

Add $90ax^9 + 9x^{10}$ to both sides of this equation; and we shall have $16,950a^5x + 12,435a^5x^2 - 1,304a^5x^3 - 9,162a^5x^4 - 8,748a^5x^5 - 4,762a^5x^6 - 1,848a^5x^7 - 501a^5x^8 = 12,625a^{10} + 90ax^9 + 9x^{10}$.

Now let $12,625a^{10}$ be subtracted from both sides of this equation; and we shall then have $-12,625a^{10} + 16,950a^5x + 12,435a^5x^2 - 1,304a^5x^3 - 9,162a^5x^4 - 8,748a^5x^5 - 4,762a^5x^6 - 1,848a^5x^7 - 501a^5x^8 = 90ax^9 + 9x^{10}$.

From the compound quantity $-12,625a^{10} + 16,950a^5x + 12,435a^5x^2 - 1,304a^5x^3 - 9,162a^5x^4 - 8,748a^5x^5 - 4,762a^5x^6 - 1,848a^5x^7 - 501a^5x^8$, (which forms the left-hand side of this last equation, and which, for the sake of brevity, we will call A,) let us now subtract the compound quantity $-12,625a^{10} + 1,800a^5x + 7,020a^5x^2 + 8,200a^5x^3 + 4,890a^5x^4 + 2,040a^5x^5 + 620a^5x^6 + 120a^5x^7 + 15a^5x^8$, which is the product of the multiplication of the foregoing long divisor $-2,525a^3 + 360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, (which, for the sake of brevity, we will call D,) into $5aa$, and which is consequently equal to $5aaD$; and the remainder will be the compound quantity $15,150a^5x + 5,415a^5x^2 - 9,504a^5x^3 - 14,052a^5x^4$.

$14,052a^6x^4 - 10,788a^5x^3 - 5,382a^4x^2 - 1,968a^3x - 516a^2x^2$, which we will call R. This remainder, together with the compound quantity that is equal to $5aaD$, by the subtraction of which from the compound quantity $-12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8$, or A, it was produced, must, from the nature of the operation of subtraction, be equal to the said compound quantity A, from which the subtraction was made, or to the left-hand side of the last equation. They may therefore be substituted instead of that compound quantity in the said equation. And, if they are so substituted, the equation will be as follows, to wit, the compound quantity $-12,625a^{10} + 1,800a^9x + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8$ together with the compound quantity $15,150a^9x + 5,415a^8x^2 - 9,504a^7x^3 - 14,052a^6x^4 - 10,788a^5x^5 - 5,382a^4x^6 - 1,968a^3x^7 - 516a^2x^8$ will be $= 90ax^9 + 9x^{10}$.

Now let the compound quantity $15,150a^9x + 5,415a^8x^2 - 9,504a^7x^3 - 14,052a^6x^4 - 10,788a^5x^5 - 5,382a^4x^6 - 1,968a^3x^7 - 516a^2x^8$ be subtracted from both sides of the equation. And we shall then have the compound quantity $-12,625a^{10} + 1,800a^9x + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8 =$ the compound quantity $-15,150a^9x - 5,415a^8x^2 + 9,504a^7x^3 + 14,052a^6x^4 + 10,788a^5x^5 + 5,382a^4x^6 + 1,968a^3x^7 + 516a^2x^8 + 90ax^9 + 9x^{10}$, or $5aaD = -R + 90ax^9 + 9x^{10}$.

Art. 67. Now both the sides of this last equation may be divided by the compound quantity $-2,525a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, or D, without a remainder; and the quotient of the division of the first, or left-hand, side of this equation, to wit, the compound quantity $-12,625a^{10} + 1,800a^9x + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8$, or $5aaD$, by the said divisor D, will be the single quantity $5aa$; and the quotient of the division of the second, or right-hand, side of this equation, to wit, the compound quantity $-15,150a^9x - 5,415a^8x^2 + 9,504a^7x^3 + 14,052a^6x^4 + 10,788a^5x^5 + 5,382a^4x^6 + 1,968a^3x^7 + 516a^2x^8 + 90ax^9 + 9x^{10}$, by the same divisor D will be the binomial quantity $6ax + 3xx$. But, when two equal quantities are divided by the same divisor, and there are no remainders to the divisions, it is evident that the quotients of such divisions must be equal to each other. Therefore the second quotient $6ax + 3xx$ must be equal to the first quotient $5aa$, or the quadratic equation $6ax + 3xx = 5aa$, or $3xx + 6ax = 5aa$, must be a true Proposition. Q. E. D.

Art. 68. The several operations of Multiplication, Subtraction, and Division, that have been mentioned in the two foregoing articles, may be performed as follows:

The Multiplication of the Compound Quantity — $2,525a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, or D, into $5aa$.

The Multiplicand.

$$- 2,525a^8 + 360a^7x + 1,404a^6x^2 + 978a^5x^3 + 408a^4x^4 + 124a^3x^5 + 24ax^7 + 3x^8, \text{ which is } = D.$$

The Multiplier.

$5aa$.

The Product.

$$- 12,625a^{10} + 1,800a^9x + 8,200a^8x^2 + 4,890a^7x^3 + 2,040a^6x^4 + 620a^5x^5 + 120a^4x^6 + 15a^3x^7 = 5aaD.$$

The Subtraction of the Compound Quantity — $12,625a^{10} + 1,800a^9x + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8$, from the Compound Quantity — $12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 8,748a^6x^4 - 4,762a^5x^5 - 1,848a^4x^6 - 501a^3x^7 - 501a^2x^8$, or A.

$$\begin{aligned} & - 12,625a^{10} + 16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 8,748a^6x^4 - 4,762a^5x^5 - 1,848a^4x^6 - 501a^3x^7 = A. \\ & - 12,625a^{10} + 1,800a^9x + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8 = 5aaD; \\ & \quad + 15,150a^8x + 5,415a^7x^2 - 9,504a^6x^3 - 14,052a^5x^4 - 10,788a^4x^5 - 5,132a^3x^6 - 1,968a^2x^7 - 516a^2x^8 = R. \end{aligned}$$

Therefore $5aaD + R$ will be $= A$, and consequently $= 90ax^8 + 9a^{10}$.

Therefore

Therefore $\zeta a^2 D$ will be $= -R + 90ax^9 + 9x^{10}$; or $-12,625a^{10} + 1,800a^9x + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8 + 15a^2x^9 + 9,504a^2x^{10} + 14,052a^2x^{11} + 10,788a^2x^{12} + 5,382a^2x^{13} + 1,968a^2x^{14} + 516ax^{15} + 90ax^{16} + 9x^{17}$.

The Division of the Compound Quantity — $12,625a^{10} + 1,800a^9x + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8$, by the *Compound Quantity* — $2,525a^3 + 360a^2x + 1,404a^2x^2 + 978a^2x^3 + 124a^2x^4 + 24ax^5 + 3x^6$, or D .

The Divisor.

$$- 2,525a^3 + 360a^2x + 1,404a^2x^2 + 978a^2x^3 + 124a^2x^4 + 24ax^5 + 3x^6$$

(The Quotient, ζa^2 .)

The Dividend.

$$\begin{aligned} & - 12,625a^{10} + 1,800a^9x + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8 \\ & - 12,925a^{10} + 1,800a^9x + 7,020a^8x^2 + 8,200a^7x^3 + 4,890a^6x^4 + 2,040a^5x^5 + 620a^4x^6 + 120a^3x^7 + 15a^2x^8 \end{aligned}$$

The Division of the Compound Quantity — $15,150a^8x - 5,415a^7x^2 + 9,504a^6x^3 + 14,052a^5x^4 + 10,788a^4x^5 + 5,388a^3x^6 + 1,968a^2x^7 + 516a^1x^8 + 90ax^9 + 9x^{10}$, or $R - R + 90ax^9 + 9x^{10}$, by the same Divisor — $2,523a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, or D .

The Divisor.

$$- 2,523a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$$

(The Quotient, $6ax + 3ax$.)

The Dividend.

$$- 15,150a^8x - 5,415a^7x^2 + 9,504a^6x^3 + 14,052a^5x^4 + 10,788a^4x^5 + 5,388a^3x^6 + 1,968a^2x^7 + 516a^1x^8 + 90ax^9 + 9x^{10}$$

$$- 15,150a^8x + 2,160a^7x^2 + 8,424a^6x^3 + 9,840a^5x^4 + 5,868a^4x^5 + 2,448a^3x^6 + 744a^2x^7 + 144a^1x^8 + 18ax^9$$

$$- 7,575a^8x + 1,080a^7x^2 + 4,212a^6x^3 + 4,920a^5x^4 + 2,934a^4x^5 + 1,224a^3x^6 + 372a^2x^7 + 72ax^8 + 9x^{10}$$

$$- 7,575a^8x + 1,080a^7x^2 + 4,212a^6x^3 + 4,920a^5x^4 + 2,934a^4x^5 + 1,224a^3x^6 + 372a^2x^7 + 72ax^8 + 9x^{10}$$

Therefore the quotient $6ax + 3ax$ will be equal to the quotient $5aa$.

$Q_2 E, D.$

ART. 69. The foregoing long divisor, — $2,523a^8 + 360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, or D , will, if we suppose it to be equal to 0, enable us to solve the second Problem above mentioned, which relates to a triangle, in which the difference of the cubes of the sides, (instead of their sum,) is equal to three times the cube of the given base BC . For, if that quantity be $= 0$, we shall have $360a^7x + 1,404a^6x^2 + 1,640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8 = 2,523a^8$, which is an equation that evidently can have but one real and affirmative root. And that root, if we were to resolve the equation in the same manner as we resolved the equation $16,950a^8x + 12,443a^7x^2 - 1,304a^6x^3 - 9,162a^5x^4 - 8,748a^4x^5 - 4,762a^3x^6 - 1,848a^2x^7 - 501ax^8 - 9x^9 = 12,625a^9$ in the foregoing articles, would be found to be

||

$= 0.791,626,8 \times a$. And that this quantity will be the root of this equation will be evident, without resolving the equation, by substituting the said quantity instead of x in the compound quantity $360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 24ax^7 + 3x^8$, which forms the left-hand side of the said equation. For we shall find, upon making this substitution, that the value of the said compound quantity resulting from it will be very nearly equal to $2,525a^2$, or the absolute term of the said equation. This substitution may be made as follows:

If x be $= 0.791,626,8 \times a$, we shall have

$$\begin{aligned} xx & (= \overline{0.791,626,8 \times a})^2 = \overline{0.791,626,8}^2 \times a^2 = 0.626,672,990 \times a^2, \\ \text{and } x^2 & (= \overline{0.791,626,8 \times a})^2 = \overline{0.791,626,8}^2 \times a^2 = 0.496,091,133 \times a^2, \\ \text{and } x^4 & (= \overline{0.791,626,8 \times a})^4 = \overline{0.791,626,8}^4 \times a^4 = 0.392,719,036 \times a^4, \\ \text{and } x^3 & (= \overline{0.791,626,8 \times a})^3 = \overline{0.791,626,8}^3 \times a^3 = 0.310,886,913 \times a^3, \\ \text{and } x^6 & (= \overline{0.791,626,8 \times a})^6 = \overline{0.791,626,8}^6 \times a^6 = 0.246,106,412 \times a^6, \\ \text{and } x^7 & (= \overline{0.791,626,8 \times a})^7 = \overline{0.791,626,8}^7 \times a^7 = 0.194,824,431 \times a^7, \\ \text{and } x^8 & (= \overline{0.791,626,8 \times a})^8 = \overline{0.791,626,8}^8 \times a^8 = 0.154,228,240 \times a^8. \end{aligned}$$

Therefore $360a^2x$ will be $(= 360a^2 \times 0.791,626,8 \times a = 360 \times 0.791,626,8 \times a^3) = 284,985,648,0 \times a^3$;

and $1,404a^2x^2$ will be $(= 1,404a^2 \times 0.626,672,990 \times a^2 = 1,404 \times 0.626,672,990 \times a^4) = 879,848,877,960 \times a^4$;

and $1,640a^2x^3$ will be $(= 1,640a^2 \times 0.310,886,913 \times a^3 = 1,640 \times 0.310,886,913 \times a^5) = 813,589,458,120 \times a^5$;

and $978a^2x^4$ will be $(= 978a^2 \times 0.392,719,036 \times a^4 = 978 \times 0.392,719,036 \times a^6) = 384,079,217,208 \times a^6$;

and $408a^2x^5$ will be $(= 408a^2 \times 0.310,886,913 \times a^5 = 408 \times 0.310,886,913 \times a^7) = 126,841,860,504 \times a^7$;

and $124a^2x^6$ will be $(= 124a^2 \times 0.246,106,412 \times a^6 = 124 \times 0.246,106,412 \times a^8) = 30,517,195,088 \times a^8$;

and $24ax^7$ will be $(= 24a \times 0.194,824,431 \times a^7 = 24 \times 0.194,824,431 \times a^8) = 4,675,786,344 \times a^8$;

and $3x^8$ will be $(= 3 \times 0.154,228,240 \times a^8) = 0.462,684,720 \times a^8$.

Therefore the whole compound quantity $360a^2x + 1,404a^2x^2 + 1,640a^2x^3 + 978a^2x^4 + 408a^2x^5 + 124a^2x^6 + 24ax^7 + 3x^8$ will be =

284.

$$\left\{ \begin{array}{l} 284.985,648,000 \times a^5 \\ + 879.848,877,960 \times a^4 \\ + 813.589,458,120 \times a^3 \\ + 384.079,217,208 \times a^2 \\ + 126.841,860,504 \times a \\ + 30.517,195,088 \times a^0 \\ + 4.675,786,344 \times a^0 \\ + 0.462,684,720 \times a^0 \end{array} \right\}$$

= $2525.000,727,944 \times a^5$; which is very nearly equal to the absolute term $2,525a^5$ of the equation $360a^5x + 1,404a^4x^2 + 1,640a^3x^3 + 978a^2x^4 + 408ax^5 + 124a^4x^6 + 24ax^7 + 3x^8 = 2525a^5$. And consequently the quantity $0.791,626,8 \times a$ must be very nearly equal to the root of that equation.

Q. E. D.

A S C H O L I U M.

Art. 70. We have now gone through the solution of Mr. Glenie's Problem by two different methods; namely, first, by reducing the long equation of the 10th order resulting from the algebraick investigation of the value of the line CH, or x , to a quadratick equation by the division of its terms, (brought all to the same side of the mark of equality, and consequently made to be equal to 0,) by the long divisor D, and then resolving the said quadratick equation, and likewise giving a geometrical construction of it; and, secondly, by resolving at once the said long equation of the 10th order by the help of Mr. Raphson's method of approximation, without any previous reduction of it to a lower order. And both these methods of resolution have undergone as full an examination and explanation as I have been able to give them. And now, upon the whole matter, I must beg leave to observe that, though the second method is much more laborious than the former, on account of the many tedious substitutions of different quantities instead of x , which it is necessary to make in the terms of the said long equation of the 10th power, in the several processes that occur in it, before we arrive at the accurate value of x , or CH, which we at last obtained, to wit, $0.632,993,15 \&c \times a$, yet I think it deserves to be preferred to the former method, on account of the greater simplicity and perspicuity of the reasonings made use of in it than of those which are employed in the first method. For, in the said second method there is no obscurity whatsoever that attends our ideas and reasonings in any part of it; but we, first, discover, (by a few easy trials suggested by the conditions of the Problem,) that

that the value of x which we are seeking, or which is equal to the line CH in fig. 2, must be somewhat greater than $0.6 \times a$, and then, by gradual approaches to it's more accurate value, we find it, first, to be nearly equal to, but somewhat greater than $0.63 \times a$, and, secondly, to be somewhat greater than $0.632,95 \times a$, and, lastly, to be very nearly equal to $0.632,993,15 \times a$; and afterwards, by clear and plain reasonings on the course of the increase of the compound quantity of ten terms which forms the left-hand side of the said long equation, while x increases from $0.632,993,15 \times a$ to $1.03 \times a$, (contained in art. 53, 54, 55, 56, and 57,) we discover that the said long equation must have also another root that is greater than $0.632,993,15 \times a$, and which must also be greater than $0.7177 \times a$, but less than $1.03 \times a$; and, lastly, by the application of Mr. Raphson's method of approximation, we find the said greater root to be very nearly $= 0.791,626,8 \times a$. And in all these reasonings and processes, though they are attended with a great deal of labour and calculation, there is no subtlety or ambiguity whatsoever: but in the former method of resolving the said long equation, by reducing it by a division to a quadratic equation, there is a sort of mysterious obscurity that runs through it, and which I am afraid I have not been able in irely to remove, notwithstanding the endeavours I have used to do so in some of the foregoing articles; at least I must confess I feel a cloud of that sort still hanging upon my own mind. And therefore, as Clearness and Peripicuity of Conception form, in my opinion, the great merit and beauty of the Mathematical Sciences, I am forced to give a preference to the second method of resolving the said long equation above the former method of resolving it. It is, however, often expedient, in solving Problems of difficulty, to have recourse to methods of different kinds, obscure ones as well as clear ones, to enable us to discover the value of the quantity we are seeking. But clear methods alone can be justly considered as *scientific*.

A Recapitulation of the Processes of the foregoing Resolution of the final Equation
 $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6$
 $- 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, (resulting from the
 foregoing *Algebraick Investigation of Mr. Glenie's Problem*), by the *Second Method*,
 or that which proceeds directly to determine the Value of x in the said Equation
 itself, without previously reducing it to an Equation of a lower Order by a Division.

Art. 71. As I have declared in the foregoing article that I prefer the second, or direct, method of resolving the long equation $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, (resulting from the algebraick investigation of Mr. Glenie's Problem,) to the other method of resolving it, which is grounded
 on

on a reduction of it to a quadrarick equation by a division of it by a long compound quantity consisting of nine terms; and, as the processes used in this second, or preferable, method of resolution are many and tedious, and intermixed with the long numbers resulting from the several arithmetical computations employed in it, it will, I conceive, be agreeable to the reader, that, before I conclude this discourse, I should recapitulate and set down the several processes of this resolution in a short and summary manner, and so as to exhibit them all, as much as possible, in one view. This may be done in the manner following:

This independent resolution of the foregoing long equation of the 10th power is begun in art. 45, page 367; and the first step taken in it towards discovering the length of the line CH in fig. 2, (which is here denoted by x ;) is to suppose it to be equal to a , or DF, the given altitude of the triangle ABC, and to try whether the value of the sum of the cubes of the two sides of the said triangle, which results from such a supposition will be greater, or will be less, than $24a^3$, or three times the cube of the given base BC; and the result is, that the sum of the cubes of the two sides of the said triangle resulting from this supposition will be $= 34.050 \times a^3$, which is greater than $24a^3$, or $3BC^2$. Therefore a must be greater than CH.

We then supposed, in the second place, (in art. 46, page 368,) that CH, or x , was equal to $\frac{a}{2}$; and we tried what would be the value of the sum of the cubes of the two sides of the required triangle ABC upon this supposition. And we found that the value of the said sum upon this supposition would be only $20.914 \&c \times a^3$; which is less than $24a^3$, or $3BC^2$. And thence we concluded that $\frac{a}{2}$, or $0.5 \times a$, would be less than CH, or that CH would be greater than $0.5 \times a$.

We then, in the 3d place, supposed x to be equal to $\frac{6}{10} \times a$, or $0.6 \times a$, and tried what would be the value of the sum of the cubes of the two sides of the said triangle resulting from this supposition: and we found it to be $23.196 \&c \times a^3$; which is but little less than $24a^3$. And hence we concluded that $0.6 \times a$ would be but little less than the true value of CH, or x , and would therefore be a proper quantity to be made the basis of a further approach to the true value of CH, or x , in the foregoing long equation, by the rules of Mr. Raphson's excellent method of resolving equations by approximation.

We then, in the 4th place, (in art. 47, pages 369, 370, and 371,) substituted the binomial quantity $0.6 \times a + z$ instead of x in the terms of the said long equation, and found the new equation resulting from such substitution, (after omitting all the terms of it that involved any higher powers of z than it's simple power, or z itself,) to be $13,926.997,032,96 \times a^2 z = 410.531,704.358,4$

$358,4 \times a^{10}$; by resolving which we found z to be $= \frac{420,537,704,358,4 \times a^{10}}{13,926,997,032,96 a^9}$, or $0.029 \times a$; and consequently x , or $0.6 \times a + z$, to be $= 0.629 \times a$, or, nearly $0.63 \times a$.

We then, in the 5th place, (in art. 48, page 372,) substituted $0.63 \times a$ instead of x in the compound quantity $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$, which forms the first, or left-hand, side of the foregoing equation of the 10th order, that we might thereby discover whether the value of the said compound quantity resulting from such substitution would be greater, or less, than $12,625a^{10}$, or the absolute term of the said equation: and we found that the said result was $= 12,591.587,735,258,5 \times a^{10}$, which is somewhat less than the said absolute term $12,625a^{10}$. And thence we concluded that $0.63 \times a$ must be somewhat less than the true value of x in the said equation.

We then, in the 6th place, (in art. 49, pages 373, 374, 375, and 376,) supposed x to be equal to the binomial quantity $0.63 \times a + z$, and substituted the said binomial quantity instead of x in the terms of the foregoing long equation, omitting all those members of the several powers of the said binomial quantity which involved in them any higher powers of z than the simple power, or z itself; and we found the new equation resulting from such substitution to be $11,187,583,882,281,8 \times a^2z = 33,037,133,728,3 \times a^{10}$; by resolving which equation we found z to be $(= \frac{33,037,133,728,3 \times a^{10}}{11,187,583,882,281,8 \times a^2} = \frac{33,037,133,728,3 \times a^8}{11,187,583,882,281,8}) = 0.002,95 \times a$. And hence it followed that x , or $0.63 \times a + z$, must be $= 0.63 \times a + 0.002,95 \times a = 0.632,95 \times a$; which is a pretty near value of x , the four first figures of it, to wit, 0.6329 , being exact.

We then, in the 7th place, (in art. 50, pages 376 and 377,) substituted $0.632,95 \times a$ instead of x in the terms of the compound quantity $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$, which forms the first, or left-hand, side of the foregoing equation of the 10th order, that we might thereby discover whether the value of the said compound quantity resulting from such substitution would be greater, or would be less, than $12,625a^{10}$, or the absolute term of the said equation: and we found that the said result was $= 12,624,530,037,788 \times a^{10}$; which differs by less than a^{10} , or $1a^{10}$, from the said absolute term $12,625a^{10}$. And we thence concluded that $0.632,95 \times a$ would be somewhat, though but very little, less than the true value of x in the said equation.

And we then, in the 8th and last place, (in art. 51, pages 377, 378, 379, and 380,) supposed x to be equal to the binomial quantity $0.632,95 \times a + z$, and substituted the said binomial quantity instead of x in the terms of the foregoing long equation, omitting all those members of the powers of the said binomial

nomial quantity which involved in them any higher powers of z than its simple power, or z itself; and we found the new equation arising from such substitution to be $10,890.998,177,964 a^2 z = 0.469,962,212 a^2$; by resolving which equation we found z to be $(= \frac{0.469,962,212}{10,890.998,177,964} \times a) = 0.000,043,15 \times a$.

And hence it followed that x , or $0.632,95 \times a + z$, must be $(= 0.632,95 \times a + 0.000,043,15 \times a) = 0.632,993,15 \times a$; which is a very near value of x , the seven first figures of it, to wit, $0.632,993,1$, being exact. And, if we were to carry this process of approximation to the true value of x one step further, it would give us the said value exact to about fifteen figures.

We then (in art. 52, pages 380, and 381,) derived from this value, $0.632,993,15 \times a$, of the line x , or CH , the corresponding values of $\sqrt{ax} + xx$ and $\sqrt{5aa + 4ax + xx}$, or of the sides AC and AB of the triangle ABC , which we were required to describe; and found them to be equal to $a \times \sqrt{1.400,680,327,946,922,5}$ and $a \times \sqrt{7.932,652,927,946,922,5}$ respectively, or to $1.183,503,41 \times a$ and $2.816,496,569 \times a$ respectively; the cubes of which are $1.657,709,893 \times a^3$ and $22.342,289,740 \times a^3$. And these cubes, being added together, were found to be equal to $23.999,999,633 \times a^3$; which is very nearly equal to $24a^3$, or $3BC^3$, agreeably to the conditions of the Problem. And we thence concluded that $1.183,503,41 \times a$ and $2.816,496,569 \times a$ were very nearly equal to the true values of the sides AC and AB of the triangle ABC , which we were required to describe. Q. E. D.

And thus the Problem that had been proposed before in art. 20, page 345, was completely solved without the help of Mr. Glenie's Construction, and without reducing the equation of the 10th order resulting from the algebraick investigation of the said Problem, to an equation of a lower order.

Art. 72. We then proceeded to make some further observations upon the long equation $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, which we had before resolved, and of which we had found the quantity $0.632,993,15 \times a$ to be a root. And we shewed that this equation must have also another root that would be greater than the former, and greater also than $0.7177 \times a$, but less than $a + \frac{3}{100} \times a$, or $1.03 \times a$. This is shewn by just reasonings on the necessary increase of the compound quantity $16,950a^9x + 12,435a^8x^2 - 1,304a^7x^3 - 9,162a^6x^4 - 8,748a^5x^5 - 4,762a^4x^6 - 1,848a^3x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$ from nothing to a certain magnitude, and the subsequent decrease of it from that magnitude to nothing, while x increases from 0 to $1.03 \times a$, which is its magnitude when the sum of the eight terms $1,304a^7x^3, 9,162a^6x^4, 8,748a^5x^5, 4,762a^4x^6, 1,848a^3x^7, 501a^2x^8, 90ax^9$, and $9x^{10}$ of the said compound quantity, (which are marked with the sign $-$, or are subtracted from the two first terms $16,950a^9x$ and $12,435a^8x^2$, of the said

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compound quantity,) is become equal to the sum of the said two first terms $16,950a^2x$ and $12,435a^2x^2$, from which they are subtracted. And it is further shewn that the greatest magnitude to which the said compound quantity will have attained during the said increase of x from 0 to $1.03 \times a$, is that which it has when the sum of the infinitely small increments of the eight terms $1,304a^2x^3$, $9,162a^2x^4$, $8,748a^2x^5$, $4,762a^2x^6$, $1,848a^2x^7$, $501a^2x^8$, $90ax^9$, and $9x^{10}$ is become equal to the sum of the contemporary infinitely small increments of the two terms $16,950a^2x$ and $12,435a^2x^2$; at which time x is, nearly, equal to $0.7177 \times a$, and consequently the said compound quantity is equal to $(13,120.743,316,248 \times a^{10})$, or, nearly, $13,121a^{10}$. These things are shewn in art. 53, 54, 55, and 56, pages 381, 382, 383, 384, 385, and 386. And from these articles it follows, that, while the said compound quantity decreases from $13,121a^{10}$ (which is greater than $12,625a^{10}$, or the absolute term of the foregoing long equation,) to 0, or the quantity x increases from $0.7177 \times a$ to $1.03 \times a$, the said compound quantity must become a second time equal to $12,625a^{10}$, or the absolute term of the foregoing equation; and it follows likewise that the value of x at that time must be greater than $0.7177 \times a$, but less than $1.03 \times a$; that is, in other words, the equation $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$ must have a second root besides that already found, or, $0.632,993,15 \&c \times a$, and that the said second root must be greater than $0.7177 \times a$, but less than $1.03 \times a$. And we then proceeded to investigate the magnitude of the said second root in the following manner:

Art. 73. By reasoning on the comparative magnitudes of the increments of the binomial quantity $16,950a^2x + 12,435a^2x^2$ and the octinomial quantity $1,304a^2x^3 + 9,162a^2x^4 + 8,748a^2x^5 + 4,762a^2x^6 + 1,848a^2x^7 + 501a^2x^8 + 90ax^9 + 9x^{10}$ before and after x has attained the magnitude of $0.7177 \times a$, (which is it's value when the said increments are equal to each other,) we discovered, that, when the compound quantity $16,950a^2x + 12,435a^2x^2 - 1,304a^2x^3 - 9,162a^2x^4 - 8,748a^2x^5 - 4,762a^2x^6 - 1,848a^2x^7 - 501a^2x^8 - 90ax^9 - 9x^{10}$, after having been equal to $13,121a^{10}$ when x was equal to $0.7177 \times a$, has decreased to $12,625a^{10}$, which was it's magnitude when x was equal to $0.632,993,15 \&c \times a$, or has undergone a decrement equal to it's increment during the increase of x from $0.632,993,15 \&c \times a$ to $0.7177 \times a$, the increment of x during the time of the said decrease of the said compound quantity will be less than the former increment of x during the increase of the said compound quantity from $12,625a^{10}$ to $13,121a^{10}$. But that former increment of x was from $0.632,993,15 \&c \times a$, to $0.7177 \times a$, and consequently was equal to the excess of $0.7177 \times a$ above $0.632,993,15 \&c \times a$, or (nearly) above $0.6330 \times a$, that is, to $0.0847 \times a$. Therefore the increment of x during the time of the decrease of the said compound quantity from $13,121a^{10}$ to $12,625a^{10}$ will be less than $0.0847 \times a$; and consequently the magnitude of x at the time that the said compound quantity becomes a second time equal to $12,625a^{10}$ will be less than $0.7177 \times a + 0.0847 \times a$, or than $0.8024 \times a$, or (neglecting the two last figures,) than $0.80 \times a$; or, in other words, the greater

greater root of the equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$ will be less than $0.80 \times a$. We therefore conjectured that it would be nearly equal to $0.79 \times a$.

And, secondly, when we had thus pitched upon $0.79 \times a$ for our first conjectural value of the greater root of the said equation, we tried the truth, or justness, of our conjecture, by substituting $0.79 \times a$ instead of x in the terms of the compound quantity $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10}$, which forms the first, or left-hand, side of the said equation, in order to see how near the value of the said compound quantity would approach to $12,625a^{10}$, or the absolute term of the said equation, and to discover whether it would be greater than the said absolute term, or less than it. And we found that the value of the said compound quantity resulting from this substitution was $12,647.728, 648, 483 \times a^{10}$; which is a little greater than $12,625a^{10}$, or the absolute term of the said equation. And hence, as it had been before shewn, that while x increases from $0.7177 \times a$ to $1.03 \times a$, the said compound quantity decreases continually from $13,121a^{10}$ to 0, we concluded that $0.79 \times a$ would be somewhat less than the true value of the greater root of the said equation. This substitution was made in art. 58, pages 387 and 388.

We therefore, in the 3d place, (in art. 59, pages 388, 389, 390, and 391,) supposed x to be equal to the binomial quantity $0.79 \times a + z$, and substituted the said binomial quantity instead of x in the terms of the equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, omitting all those members of the several powers of the binomial quantity $0.79 \times a + z$ which involved any higher powers of z than its simple power, or z itself; and we found the new equation resulting from such substitution to be $15,289,547, 896, 564 a^2z = 22,728,648,483 a^{10}$; whence it followed that z was $= \frac{22,728,648,483}{15,289,547,896,564} \times a$, or $0.001,48 \times a$. Therefore x , or $0.79 \times a + z$, will be $(= 0.79 \times a + 0.001,48 \times a) = 0.791,48 \times a$, or the greater root of the foregoing equation will be nearly equal to $0.791,48 \times a$. Q. E. I.

We then, in the 4th place, (in art. 60, pages 391, and 392,) substituted $0.791,48 \times a$ instead of x in the compound quantity $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10}$, in order to discover whether the value of the said compound quantity resulting from such substitution would be greater, or would be less, than $12,625a^{10}$, or the absolute term of the foregoing equation, and consequently whether $0.791,48 \times a$ was less, or was greater, than the true value of the greater root of the said equation: and we found that the value of the said compound quantity resulting from that substitution was $12,627.074,387,143 \times a^{10}$; which is greater than $12,625a^{10}$, or the absolute term of the foregoing equation, but

exceeds it by only the small quantity $2.074,387,143 \times a^{10}$, or the 6086th part of $12,625a^{10}$. And we thence concluded that $0.791,48 \times a$ must be somewhat, though a very little, less than the true value of x , or the greater root of the foregoing equation.

We then, in the 5th place, (in art. 61, pages 393, 394, and 395,) supposed x to be equal to the binomial quantity $0.791,48 \times a + z$, and substituted the said binomial quantity, instead of x , in the terms of the equation $16,950a^2x + 12,435a^3x^2 - 1,304a^4x^3 - 9,162a^5x^4 - 8,748a^6x^5 - 4,762a^7x^6 - 1,848a^8x^7 - 501a^9x^8 - 90ax^9 - 9x^{10} = 12,625a^{10}$, omitting all those members of the several powers of the binomial quantity $0.791,48 \times a + z$ which involved in them any higher powers of z than it's simple power, or z itself; and we found that the new equation resulting from such substitution was $14,122.028,479,212 \times a^2z = 2.074,387,143 \times a^{10}$; whence it followed that z was $(= \frac{2.074,387,143 \times a}{14,122.028,479,212}) = 0.000,146,8 \times a$. Therefore x , or $0.791,48 \times a + z$, will be $(= 0.791,48 \times a + 0.000,146,8 \times a) = 0.791,626,8 \times a$; that is, the greater root of the foregoing equation will be very nearly equal to $0.791,626,8 \times a$. Q. E. I.

And this value of the said greater root of the foregoing equation is, probably, exact in the first six figures $0.791,626$, and, perhaps, also in the last figure 8. And if we were to carry this process of approximation to the true value of this greater root of the said equation one step further, by supposing x to be equal to the binomial quantity $0.791,626,8 \times a + z$, and substituting the said binomial quantity instead of x in the terms of the said equation, it would give us the said value exact to about thirteen figures.

These investigations of the values of the two roots of the foregoing long equation of the 10th order seem to be excellent illustrations of the utility of Mr. Raphson's method of resolving equations of high orders by approximation.

N. B. The foregoing Problem of Mr. Glenie had been proposed in *The Ladies' Diary* for the year 1794, page 48, in the words following:

In the Palace of one of the Persian Kings, it is said there was a triangular area, such, that the cubes on two of the sides were, together, equal to thrice the cube on the third side, (which was 200 feet in length,) and that the area itself contained 10,000 superficial feet. Supposing this to have been really the case, it is required to construct the triangle by common, or plane, Geometry.

And in the *Ladies' Diary* for the year 1795, there were published Mr. Glenie's own Construction of this Problem, (which has been given above in art. 1, page 335,) and two other Constructions of it by other persons.

End of the Examination and Investigation of Mr. GLENIE's Problem.

AN
IMPORTANT PROPOSITION

CONCERNING

The Asymptotick Areas of an Equilateral Hyperbola;

FROM WHICH IT FOLLOWS,

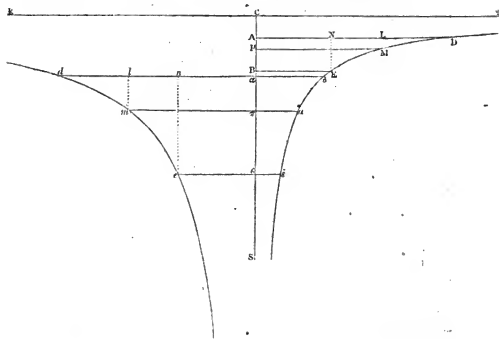
That they are Logarithms, or Measures, of the Ratios of the Ordinates
that bound them.

*Demonstrated by the late Dr. WILLIAM RIDLINGTON, L. L. D. formerly
Fellow of Trinity-Hall in the University of Cambridge, and Professor of Civil Law
in the same University.*

OF THE
ASYMPTOTICK AREAS OF AN HYPERBOLA.

A P R O P O S I T I O N.

LET $DME\delta\mu$ be the curve of an Equilateral Hyperbola, of which C is the center, and $CAPB\alpha\tau\epsilon$ is an asymptote. And let CA , CB , $C\alpha$, and $C\epsilon$ be four portions of the said asymptote, of which the two latter, $C\alpha$ and $C\epsilon$, are to each other in the same proportion as the two former, CA and CB . And from the points A , B , α , and ϵ , let the four right lines AD , BE , $\alpha\delta$, and $\epsilon\tau$ be drawn at right angles to the said asymptote $CAB\alpha\epsilon$, and produced till they meet the curve of the hyperbola in the points D , E , δ , and τ . Then will the mixtilinear, or asymptotick, area $\alpha\delta\tau\epsilon$, intercepted between the two latter ordinates $\alpha\delta$ and $\epsilon\tau$, be equal to the mixtilinear, or asymptotick, area $ADEB$, intercepted between the two former ordinates AD and BE .



D E M O N.

DEMONSTRATION.

Let the ordinates ad and ce be produced on the other side of the asymptote $CA\alpha\epsilon$ to the points d and e , which are at such distances that ad shall be to the ordinate ad , and ce to the ordinate ce , in the same proportion as $C\alpha$ is to CA . Then will the points d and e be situated in another equilateral hyperbola having the same center C , and the same asymptote $CA\alpha\epsilon$, as the former hyperbola, but with a transverse axis that will be greater than the transverse axis of the former hyperbola in the proportion of $C\alpha$ to CA . Let dme be the curve of this second hyperbola.

Let P be any point in the asymptote $CA\alpha\epsilon$, situated between the points A and B , and let PM be an ordinate drawn from the point P , and meeting the curve of the first hyperbola in M . And from the points M and E draw the right lines ML and EN parallel to the asymptote $CA\alpha\epsilon$, and meeting the ordinate AD in the points L and N .

Then, in the first place, since, by the construction, ad is to ad as $C\alpha$ is to CA ; and, by the nature of an hyperbola, AD is to ad as $C\alpha$ is to CA ; it follows that ad will be to ad in the same proportion as AD is to ad . And consequently ad will be equal to AD .

In the line ad let dl be taken equal to DL ; and from the point l let the right line lm be drawn parallel to the asymptote $CA\alpha\epsilon$, and meeting the curve dme of the second hyperbola in m . And from the point e in the hyperbolic curve dme draw the right line en parallel to the asymptote $CA\alpha\epsilon$, and meeting the right line ad in n . And from the point m in the said curve draw the right line $m\pi\mu$ at right angles to the asymptote $CA\alpha\epsilon$, and meeting the said asymptote in the point π and the curve of the first hyperbola in the point μ .

Then, since, by the hypothesis, CA is to CB as $C\alpha$ is to $C\epsilon$, it follows, *permutando*, that CA will be to $C\alpha$ as CB is to $C\epsilon$, and, *invertendo*, that $C\alpha$ will be to CA as $C\epsilon$ is to CB . But, by the nature of the hyperbola, $C\epsilon$ is to CB as BE is to ce . Therefore BE will be to ce as $C\alpha$ is to CA . But, by the construction, ce is to ce as $C\alpha$ is to CA . Therefore ce will be to ce as BE is to ce . And consequently ce will be equal to BE .

But na is equal to ce , because it is the opposite side of a parallelogram; and, for the same reason, AN will be equal to BE . Therefore na will be equal to AN .

But it has been shewn that ad is equal to AD .

Therefore the excess of ad above na will be equal to the excess of AD above AN ; that is, dn will be equal to DN ; and consequently the two
mixtilinear

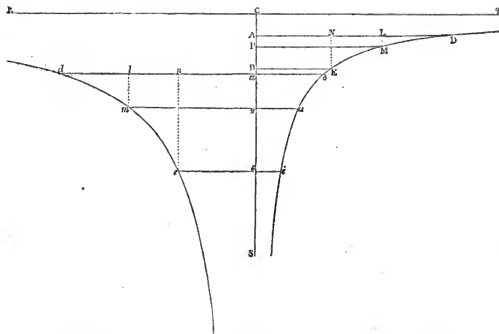
mixtilinear figures dne and DNE will have the equal lines dn and DN for their bases.

Further, since $C\alpha$ is to $C\epsilon$ as CA is to CB , and, consequently, *invertendo*, $C\epsilon$ is to $C\alpha$ as CB is to CA , it follows, *dividendo*, that $C\epsilon - C\alpha$ will be to $C\alpha$ as $CB - CA$ is to CA , that is, $\alpha\epsilon$ will be to $C\alpha$ as AB is to CA . Therefore, *permutando*, $\alpha\epsilon$ will be to AB as $C\alpha$ is to CA .

But ne is equal to $\alpha\epsilon$, and NE is equal to AB .

Therefore ne will be to NE as $C\alpha$ is to CA ; that is, the greatest ordinate of the mixtilinear figure dne will be to the greatest ordinate of the mixtilinear figure DNE as $C\alpha$ is to CA .

Further, the ordinate lm of the mixtilinear figure dne corresponding to the absciss dl of the base dn of the said figure, will be to the ordinate LM of the mixtilinear figure DNE corresponding to the absciss DL of the base DN of the said figure, (which absciss is equal to the absciss dl of the other figure,) in the same proportion of $C\alpha$ to CA . For, since the point m is situated in the second hyperbola, of which all the ordinates to the asymptote $CA\alpha\epsilon$ are greater



than those of the first hyperbola in the same proportion of $C\alpha$ to CA , we shall have $m\pi$ to $\pi\mu$ as $C\alpha$ to CA . But $m\pi$ is equal to la , being the opposite side of the parallelogram $lm\pi\alpha$. Therefore la will be to $\pi\mu$ as $C\alpha$ is to CA . But la is $= ad - dl = AD - DL = AL$; and AL is equal to PM , because it is the opposite side of the parallelogram $ALMP$. Therefore PM will be equal to la , and consequently will be to $\pi\mu$ as $C\alpha$ is to CA . But, by the nature of the hyperbola, $C\pi$ is to CP as PM is to $\pi\mu$. Therefore $C\pi$ is to CP as $C\alpha$ is to CA . Therefore, *permutando*, $C\pi$ is to $C\alpha$ as CP to CA ; and, *dividendo*, $C\pi - C\alpha$ is to $C\alpha$ as $CP - CA$ is to CA ; that is, $\alpha\pi$ is to $C\alpha$ as AP is to CA ; and consequently, *permutando*, $\alpha\pi$ is to AP as $C\alpha$ is to CA . But lm is equal to $\alpha\pi$, as being the opposite side of the parallelogram $lm\pi\alpha$; and LM is equal to AP , as being the opposite side of the parallelogram $ALMP$. Therefore lm will be to LM as $C\alpha$ is to CA .

And the same thing might be shewn of every other ordinate in the figure *dne* and the corresponding ordinate, or ordinate belonging to an equal absciss of the base, in the figure *DNE*.

Therefore the whole area of the mixtilinear figure *dne* will be to the whole area of the mixtilinear figure *DNE* in the same proportion of $C\alpha$ to CA .

Further, because the side $e\epsilon$ of the rectangular parallelogram $\pi a \epsilon e$ is equal to the side BE of the rectangular parallelogram $ANEB$, it follows that the former parallelogram will be to the latter as the side $a\epsilon$ of the former is to the side AB of the latter. But $a\epsilon$ is $= C\epsilon - C\alpha$, and AB is $= CB - CA$. Therefore the parallelogram $\pi a \epsilon e$ is to the parallelogram $ANEB$ as $C\epsilon - C\alpha$ is to $CB - CA$. But $C\epsilon - C\alpha : CB - CA :: C\alpha : CA$. Therefore the parallelogram $\pi a \epsilon e$ will be to the parallelogram $ANEB$ as $C\alpha$ is to CA .

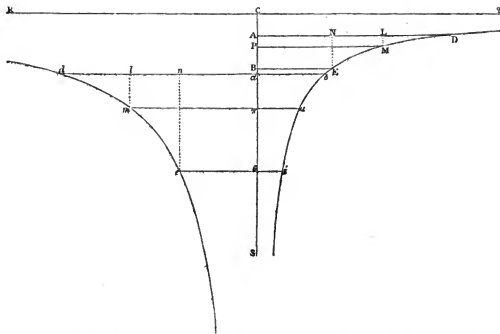
Therefore the mixtilinear area *dne* is to the mixtilinear area *DNE* as $C\alpha$ is to CA , and the parallelogram $\pi a \epsilon e$ is to the parallelogram $ANEB$ in the same proportion of $C\alpha$ to CA . Consequently, by El. 5, 12, the sum of the mixtilinear area *dne* and parallelogram $\pi a \epsilon e$ will be to the sum of the mixtilinear area *DNE* and the parallelogram $ANEB$ in the same proportion of $C\alpha$ to CA ; that is, the whole figure, or asymptotick area, *d a \epsilon e*, of the second hyperbola, will be to the whole figure, or asymptotick area, *DABE*, of the first hyperbola, in the same proportion of $C\alpha$ to CA .

But the said asymptotick area *d a \epsilon e* of the second hyperbola will also be to the asymptotick area *d a \epsilon e* of the first hyperbola in the same proportion of $C\alpha$ to CA , because these two areas have the line $a\epsilon$ for their common base, and all the ordinates $d\alpha$, $m\pi$, $e\epsilon$, of the former area are, by the construction, greater than the corresponding ordinates αd , $\pi\mu$, ϵe , of the latter area in the said proportion of $C\alpha$ to CA .

Therefore

Therefore the said asymptotick area $dace$ of the second hyperbola will be greater than each of the two asymptotick areas $DABE$ and $dace$ of the first hyperbola in the same proportion of Ca to CA . Therefore, by El. 5, 8, those two asymptotick areas of the first hyperbola must be equal to each other.

Q. E. D.



Dr. Ridlington added the following note to the foregoing demonstration ; to wit, " N. B. This is a particular case of a more general Proposition, which is, that, if the abscissæ of one curve are taken in a given ratio to the abscissæ of another, and their ordinates be at the same time in the reciprocal of that given ratio, the areas of the two curves will be equal. And this Proposition is itself but a particular case of a still more general Proposition."

Dr. Ridlington was one of the most learned and acute mathematicians I ever was acquainted with ; I mean, with respect to pure, or abstract, Mathematicks. For he never would meddle with rational mechanicks, or physical astronomy, or the resistance of fluids, or the like subjects, on account of the subtleties and difficulties arising from the idea of *force*. He died at Nice in July, 1770, at about the age of 54 years. He never published any thing upon the Mathema-

ticks, though he would have been capable of explaining many parts of them, and particularly the Doctrine of Fluxions and the Quadrature of Curvilinear Figures, with great clearness and elegance.

January 12, 1797.

F. MASERES.

This Proposition is demonstrated in a different manner, and without the intervention of a second Hyperbola, in my Elements of Plane Trigonometry, art. 186, 187, and 188, pages 211, 212, 213, 214, 215, and 216; and some useful conclusions relating to Logarithms are deduced from it in the following articles, 189, 190, 191, 192, and 193, pages 216, 217, 218, 219, and 220, of the same book. The first Demonstration of it was given by Gregory Saint Vincent, in the year 1647. As this Proposition contains the very foundation of the connection that subsists between these Hyperbolick Areas and Logarithms, I have sometimes been surpris'd to find that it is wholly omitted in several respectable treatises on the Conick Sections.

TWO PROBLEMS

CONCERNING

The Variations of the Ratio, or Proportion, of the Trinomial Quantity

$aa + 2ax + xx$ to the Trinomial Quantity $aa - ax + xx$,

Upon a Supposition that a denotes a Line of a given Magnitude, and that x denotes a variable Line that increases continually from nothing *ad infinitum*.

By FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF THE COURT OF EXCHEQUER.

T W O P R O B L E M S

Concerning the Variations of the Ratio, or Proportion, of the Trinomial Quantity $aa + 2ax + xx$ to the Trinomial Quantity $aa - ax + xx$, upon a Supposition that a denotes a Line of a given Magnitude, and that x denotes a variable Line that increases continually from nothing ad infinitum.

P R O B L E M I.

Art. 1. **L**ET a be a line of any given magnitude; and let x be another line that increases continually from nothing to greater and greater magnitudes without limit. It is required to determine whether the ratio, or proportion, of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$ will likewise increase at the same time, without limit, or so as to become greater than any ratio of majority how great soever, or whether it will only increase to some certain ratio of majority, and will afterwards decrease.

Observations preparatory to the Solution of this 1st Problem.

Art. 2. Before we enter on the solution of this Problem, it will be necessary to premise the following observations.

O B S E R V A T I O N I.

The trinomial quantity $aa + 2ax + xx$ is always a possible quantity, whatever be the magnitude of the variable line x .

For, whatever be the magnitude of the variable line x , it is evidently always possible to add its square, xx , and the rectangle, $2ax$, contained under it and twice the given line a , to aa , or the square of the given line a . And the sum of these three quantities is the trinomial quantity $aa + 2ax + xx$; which therefore is always possible.

Q. E. D.

O B S E R -

OBSERVATION II.

The second trinomial quantity $aa - ax + xx$ is likewise always a possible quantity, whatever be the magnitude of x .

For, in the 1st place, when x is less than the given line a , ax will be less than aa , and, *a fortiori*, will be less than $aa + xx$, and consequently may be subtracted from $aa + xx$; by which subtraction we shall obtain the quantity $aa + xx - ax$, or $aa - ax + xx$. Therefore the said quantity $aa - ax + xx$ will, in this case, be a possible quantity.

2dly, When x is equal to a , ax will be equal to aa , and consequently will be less than $aa + xx$, and therefore may be subtracted from it; by which subtraction we shall obtain the quantity $aa + xx - ax$, or $aa - ax + xx$. Therefore, when x is equal to a , as well as when x is less than a , the trinomial quantity $aa - ax + xx$ will be a possible quantity.

And, 3dly, when x is greater than a , xx will be greater than ax ; and consequently, *a fortiori*, $xx + aa$ will be greater than ax . Therefore ax may be subtracted from $xx + aa$; by which subtraction we shall obtain the trinomial quantity $xx + aa - ax$, or $aa - ax + xx$. Therefore in this case, as well as when x was less than the given line a or equal to it, the said trinomial quantity $aa - ax + xx$ will be a possible quantity. Therefore, whatever be the magnitude of the variable line x , the trinomial quantity $aa - ax + xx$ will always be possible.

Q. E. D.

OBSERVATION III.

The first trinomial quantity $aa + 2ax + xx$ will always be greater than the second trinomial quantity $aa - ax + xx$, and their difference will be equal to $3ax$. So that, when x is of any magnitude, how small; or how great, soever, the ratio of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$ will always be a ratio of majority.

OBSERVATION IV.

But, though the ratio of the first trinomial quantity, $aa + 2ax + xx$, to the second trinomial quantity, $aa - ax + xx$, is always a ratio of majority, yet, when

when x is very small in comparison of the given line a , it will be a very small ratio of majority, that will differ very little from a ratio of equality. For, when x is almost equal to nothing, or 0, the ratio of $aa + 2ax + xx$ to $aa - ax + xx$ will differ very little from the ratio of $aa + 2a \times 0 + 0 \times 0$ to $aa - a \times 0 + 0 \times 0$, or of $aa + 0 + 0$ to $aa - 0 + 0$, or of aa to aa , which is a ratio of equality. And x may be taken of so small a magnitude in comparison of a that the ratio of the trinomial quantity $aa + 2ax + xx$ to $aa - ax + xx$ shall come as near as we please to the ratio of aa to aa , or a ratio of equality.

From these observations it appears that the ratio of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity, $aa - ax + xx$ will always be a ratio of majority, and that it will begin to increase from a ratio of equality, as it's lowest limit, and will successively become equal to different finite ratios of majority, as x increases from 0 to different finite magnitudes. And the business of the present Problem is to determine whether it will increase *ad infinitum*, or so as to become greater than any finite ratio of majority, how great soever, while x increases from 0 *ad infinitum*, or whether there is some finite ratio of majority than which it can never become greater.

S O L U T I O N.

Art. 3. Now, in order to determine this question, we must observe that, when x is much greater than a , as, for example, a million times as great, $2ax$ will be very much greater than aa , and xx very much greater than $2ax$, and even than $2ax + aa$, so that aa and ax and $2ax$ may be considered as almost equal to 0 in comparison of xx . Therefore in this case the ratio of $xx + 2ax + aa$ to $xx - ax + aa$ will be very nearly equal to the ratio of $xx + 0 + 0$ to $xx - 0 + 0$, or of xx to xx , that is, to a ratio of equality. And x may be taken of so great a magnitude in comparison of a , that the ratio of $xx + 2ax + aa$ to $xx - ax + aa$, or of $aa + 2ax + xx$ to $aa - ax + xx$, shall come as near as we please to the ratio of xx to xx , or to a ratio of equality. Therefore, while the variable line x increases from 0 *ad infinitum*, the variable ratio of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$ will not increase continually *ad infinitum*, or so as to become greater than any finite ratio, how great soever, but will, first, increase from a ratio of equality, as it's lowest limit, (or that to which it is equal when x is infinitely small, or equal to 0,) to some finite ratio of majority, than which it can never be greater, and will then decrease from the said finite ratio of majority, while x increases further, and will gradually approach to a ratio of equality, and may be made to come as near as we please to such ratio, by increasing the magnitude of the said variable line x ; which ratio of equality will therefore be it's highest limit, as well as it's lowest limit.

Q. E. D.

VOL. IV.

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PROBLEM

P R O B L E M II.

Art. 4. To find the magnitude of the variable line x , when the ratio of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$ is the greatest possible.

Observations preparatory to the Solution of this 2nd Problem.

To solve this Problem it will be necessary to premise the following observations.

O B S E R V A T I O N I.

If there be two quantities of the same kind, A and B, whereof the first, A, is either equal to, or greater than, the second, B; and there be two other quantities, C and D, of the same kind as the two former, that are to each other in the same proportion as A to B; the sum of the first and third quantities will be to the sum of the second and fourth quantities in the same proportion as the first quantity is to the second; that is, $A + C$ will be to $B + D$ as A is to B.

This is demonstrated in the 12th proposition of the 5th book of Euclid's Elements.

O B S E R V A T I O N II.

If the proportion of C to D be not restrained to be the same as that of A to B, but be either the same, or any other proportion whatsoever, and D be less than B, and be subtracted from it, and C be added to A; the ratio of $A + C$ to $B - D$ will be greater than the ratio of A to B.

For the ratio of $A + C$ to B is evidently a greater ratio of majoriis than the ratio of A alone to B; and the ratio of $A + C$ to $B - D$ is greater than the ratio of $A + C$ to B. Therefore, *à fortiori*, the ratio of $A + C$ to $B - D$ must be greater than the ratio of A to B. Q. E. D.

O B S E R.

OBSERVATION III.

If C be greater than D, and in a greater proportion than A is greater than B, the sum of A and C will be to the sum of B and D in a greater proportion than A is to B.

For, if the proportion of C to D be greater than that of A to B, the proportion of some quantity less than C to D will be equal to the proportion of A to B. Let that quantity be $C - c$. Then will A be to B as $C - c$ is to D. Therefore, (by Observation 1st.) $A + C - c$ will be to $B + D$ as A is to B. But the proportion of $A + C$ to $B + D$ is greater than the proportion of $A + C - c$ to $B + D$. Therefore the proportion of $A + C$ to $B + D$ will be greater than the proportion of A to B.

Q. E. D.

OBSERVATION IV.

If C be greater than D, but in a less proportion than A is greater than B, the sum of A and C will be to the sum of B and D in a less proportion than that of A to B.

For, if the proportion of C to D be less than the proportion of A to B, the proportion of some quantity greater than C to D will be equal to the proportion of A to B. Let that quantity be $C + c$. Then will A be to B as $C + c$ is to D. Therefore (by Observation 1st.) $A + C + c$ will be to $B + D$ as A is to B. But the proportion of $A + C$ to $B + D$ is less than the proportion of $A + C + c$ to $B + D$. Therefore the proportion of $A + C$ to $B + D$ will be less than the proportion of A to B.

Q. E. D.

SOLUTION.

Art. 5. These Observations being premised, we must, in the next place, examine the variations of the two trinomial quantities $aa + 2ax + xx$ and $aa - ax + xx$, and of the proportion of the former of them to the latter, while x increases continually from 0 *ad infinitum*.

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Let us therefore suppose that x increases from 0 *ad infinitum* by the continual addition of some very small increment, as, for instance, of a line that shall be only the ten-thousand millionth part of the given line a ; which increment we will denote by $\dot{x}$, or the letter x with a point placed over it. Then it is evident, that, while x increases from any particular value of it to $x + \dot{x}$, or it's next greater value, the rectangle ax will increase from ax to $a \times x + \dot{x}$ or $ax + a\dot{x}$, and xx , or the square of x , will increase from xx to $x + \dot{x}$, or to $xx + 2x\dot{x} + \dot{x}\dot{x}$, or (neglecting $\dot{x}\dot{x}$ on account of it's extream smallness in comparison of $2x\dot{x}$.) to $xx + 2x\dot{x}$. Therefore the trinomial quantity $aa + 2ax + xx$ will at the same time increase to the quinquinomial quantity $aa + 2ax + 2x\dot{x} + xx + 2x\dot{x}$, or to $aa + 2ax + xx + 2a\dot{x} + 2x\dot{x}$; and the trinomial quantity $aa - ax + xx$ will, at the same time, become equal to the quinquinomial quantity $aa - ax - a\dot{x} + xx + 2x\dot{x}$, or $aa - ax + xx - a\dot{x} + 2x\dot{x}$. But this change in the magnitude of this latter quantity will not always be an increase of it, but will sometimes be a diminution of it. For, while x is less than $\frac{a}{2}$, $2x\dot{x}$ will be less than $a\dot{x}$, and consequently $aa - ax + xx - a\dot{x} + 2x\dot{x}$ will be less than $aa - ax + xx$: but, when x is greater than $\frac{a}{2}$, $2x\dot{x}$ will be greater than $a\dot{x}$, and consequently $aa - ax + xx - a\dot{x} + 2x\dot{x}$ will be greater than $aa - ax + xx$.

Therefore, if, when x is less than $\frac{a}{2}$, we denote the trinomial quantity $aa + 2ax + xx$ by A, and, it's increment, $2a\dot{x} + 2x\dot{x}$ by C, and the trinomial quantity $aa - ax + xx$ by B, and the binomial quantity $a\dot{x} - 2x\dot{x}$ by D, the quinquinomial quantity $aa + 2ax + xx + 2a\dot{x} + 2x\dot{x}$ will be equal to A + C, and the quinquinomial quantity $aa - ax + xx - a\dot{x} + 2x\dot{x}$ will be equal to $(aa - ax + xx - a\dot{x} - 2x\dot{x})$, or to B - D. But (by Observation 2d.) the proportion of A + C to B - D is greater than the proportion of A to B. Therefore the proportion of the quinquinomial quantity $aa + 2ax + xx + 2a\dot{x} + 2x\dot{x}$ to the quinquinomial quantity $aa - ax + xx - a\dot{x} + 2x\dot{x}$ will be greater than the proportion of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$; and consequently, while x increases from 0 to $\frac{a}{2}$, the proportion of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$ will continually increase. Q. E. I.

Art. 6. When x is greater than $\frac{a}{2}$, $2x\dot{x}$ will be greater than $a\dot{x}$, and consequently the quinquinomial quantity $aa - ax + xx - a\dot{x} + 2x\dot{x}$ will be greater than the trinomial quantity $aa - ax + xx$. Therefore while x increases from $\frac{a}{2}$ *ad*

ad infinitum, the trinomial quantity $aa - ax + xx$ will continually increase, and it's increment will be $2xx - ax$. This increment we must now compare with the contemporary increment of the greater trinomial quantity $aa + 2ax + xx$, to wit, the increment $2ax + 2xx$, in order to discover whether the ratio of the greater trinomial quantity $aa + 2ax + xx$ to the lesser trinomial quantity $aa - ax + xx$ will increase at the same time, or will decrease, or will, first, increase till x is become equal to some quantity greater than $\frac{a}{2}$, and then decrease. This inquiry may be made in the manner following.

Art. 7. The quantity $2ax + 2xx$, (which is the increment of the trinomial quantity $aa + 2ax + xx$,) is to the quantity $2xx - ax$ (which is the contemporary increment of the trinomial quantity $aa - ax + xx$,) in the same proportion as $2a + 2x$ to $2x - a$. We must therefore inquire, whether this proportion is greater, or less, than the proportion of those two trinomial quantities themselves at the instant that x is equal to $\frac{a}{2}$, and begins to increase from that magnitude to a magnitude greater than $\frac{a}{2}$.

Now, when x is equal to $\frac{a}{2}$, $2ax$ will be $(= 2a \times \frac{a}{2}) = aa$, and xx will be $= \frac{aa}{4}$; and consequently $aa + 2ax + xx$ will be $(= aa + aa + \frac{aa}{4}) = 2aa + \frac{aa}{4} = \frac{8aa}{4} + \frac{aa}{4} = \frac{9aa}{4}$, and $aa - ax + xx$ will be $(= aa - \frac{aa}{2} + \frac{aa}{4}) = \frac{aa}{2} + \frac{aa}{4} = \frac{2aa}{4} + \frac{aa}{4} = \frac{3aa}{4}$. Therefore at this instant of time the proportion of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$ will be that of $\frac{9aa}{4}$ to $\frac{3aa}{4}$, or that of 3 to 1.

And at this time $2a + 2x$ will be $(= 2a + 2 \times \frac{a}{2} = 2a + a) = 3a$, and $2x - a$ will be $(= 2 \times \frac{a}{2} - a = a - a) = 0$. And consequently the proportion of the increment, $2ax + 2xx$, of the greater trinomial quantity $aa + 2ax + xx$, to the contemporary increment, $2xx - ax$, of the lesser trinomial quantity $aa - ax + xx$, at the instant that x is equal to $\frac{a}{2}$, will be that of $3a$ to 0, or an infinitely great ratio of majority, and therefore will be infinitely greater than the ratio of 3 to 1, or of $\frac{9aa}{4}$ to $\frac{3aa}{4}$, or of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$. Q. E. I.

Art. 8.

Art. 8. Since, when x is equal to $\frac{a}{2}$, the proportion of $2ax + 2xx$ to $2xx - ax$ is infinitely greater than the proportion of $aa + 2ax + xx$ to $aa - ax + xx$, it follows that, when x is a little greater than $\frac{a}{2}$, the former proportion will be very much greater than the latter. Therefore, if, when x is equal to, or a little greater than, $\frac{a}{2}$, we put A for $aa + 2ax + xx$, and C for it's increment $2ax + 2xx$, and B for $aa - ax + xx$, and D for it's increment $2xx - ax$, the proportion of C to D will be much greater than the proportion of A to B. Therefore (by Observation 3d,) the proportion of A + C to B + D will be greater than the proportion of A to B; that is, the proportion of the quinquinomial quantity $aa + 2ax + xx + 2ax + 2xx$ to the quinquinomial quantity $aa - ax + xx - ax + 2xx$ will be greater than the proportion of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$, or, in other words, the proportion of the former of these trinomial quantities to the latter will increase. Q. E. I.

Art. 9. And this increase of the proportion of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$ will continue as long as the proportion of the two increments, $2ax + 2xx$ and $2xx - ax$, of the said two trinomial quantities, continues to be greater than the proportion of the said two trinomial quantities themselves. And therefore, if the proportion of the said increments were to continue greater than the proportion of the said trinomial quantities themselves during the whole increase of x from $\frac{a}{2}$ *ad infinitum*, the proportion of these two trinomial quantities would continually increase from the ratio of 3 to 1, (to which it is equal, when x is equal to $\frac{a}{2}$;) to greater and greater ratios of majority during the whole increase of x from $\frac{a}{2}$ *ad infinitum*, without ever decreasing.

But it has been shewn above, in art. 3, that the proportion of these two trinomial quantities will not increase continually in that manner while x increases *ad infinitum*, but will increase only to some certain ratio of majority, (which it can never exceed,) and will then decrease from that ratio of majority till it approaches as near as we please to a ratio of equality. There will therefore be a point of time, during the increase of the variable line x from $\frac{a}{2}$ *ad infinitum*, at which the ratio of the two increments $2ax + 2xx$ and $2xx - ax$, or the ratio of $2a + 2x$ to $2x - a$, (which is equal to it,) will be equal to the ratio of $aa + 2ax + xx$ to $aa - ax + xx$. We will therefore suppose $2a + 2x$
to

to be to $2x - a$ as $aa + 2ax + xx$ is to $aa - ax + xx$, and will endeavour to find the value of x resulting from this supposition.

Art. 10. Now, if $2a + 2x$ is to $2x - a$ in the same proportion as $aa + 2ax + xx$ is to $aa - ax + xx$, we shall have $\frac{2a + 2x}{2x - a} \times \frac{aa - ax + xx}{aa + 2ax + xx} = \frac{2x - a}{2x - a}$. But $\frac{2a + 2x}{2x - a} \times \frac{aa - ax + xx}{aa + 2ax + xx}$ is $(= \frac{2a^3 - 2aax + 2axx + 2aax - 2axx + 2x^3}{2a^3 + 2x^3}) = \frac{2a^3 + 2x^3}{2a^3 + 2x^3}$; and $\frac{2x - a}{2x - a} \times \frac{aa + 2ax + xx}{aa + 2ax + xx}$ is $(= \frac{2aax + 4axx + 2x^3 - a^3 - 2aax - axx}{3axx + 2x^3 - a^3}) = \frac{3axx + 2x^3 - a^3}{3axx + 2x^3 - a^3}$. Therefore $2a^3 + 2x^3$ will be $= 3axx + 2x^3 - a^3$, and (adding a^3 to both sides,) $3a^3 + 2x^3$ will be $= 3axx + 2x^3$, and (subtracting $2x^3$ from both sides,) $3a^3$ will be $= 3axx$, and (dividing both sides by $3a$) aa will be $= xx$. Therefore x will be $= a$.

Therefore, when the ratio of the greater trinomial quantity $aa + 2ax + xx$ to the lesser trinomial quantity $aa - ax + xx$ is the greatest possible, the variable line x will be equal to the given line a .
Q. E. I.

Coroll. 1. The greatest possible proportion of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$ is that of 4 to 1.

For, when x is equal to a , the trinomial quantity $aa + 2ax + xx$ is $(= aa + 2aa + aa) = 4aa$, and the trinomial quantity $aa - ax + xx$ is $(= aa - aa + aa) = aa$; and $4aa$ is to aa as 4 is to 1.

Coroll. 2. When x increases from a to any greater magnitude, the proportion of the trinomial quantity $aa + 2ax + xx$ to the trinomial quantity $aa - ax + xx$ will decrease from the ratio of $4aa$ to aa , or of 4 to 1, to some lesser ratio of majority.

When x is of any magnitude greater than a , let the difference of it's said magnitude from a be denoted by the letter d , so that x shall be equal to $a + d$. Then we shall have the trinomial quantity $aa + 2ax + xx$ $(= aa + 2a \times a + d + a + d)^2 = aa + 2aa + 2ad + aa + 2ad + dd) = 4aa + 4ad + dd$, and the trinomial quantity $aa - ax + xx$ $(= aa - a \times a + d + a + d)^2 = aa - aa - ad + aa + 2ad + dd) = aa + ad + dd$. So that the increment received by the trinomial quantity $aa + 2ax + xx$, while x increases from a to $a + d$, will be $4ad + dd$; and the increment received by the trinomial quantity $aa - ax + xx$ in the same time will be $ad + dd$. But the ratio of the increment $4ad + dd$ to the increment $ad + dd$ is a less ratio of majority than the ratio of $4ad + 4dd$ to $ad + dd$, or than that of 4 to 1, or of $4aa$ to aa , or of the value of the trinomial quantity $aa + 2ax + xx$ to the value of the trinomial quantity $aa - ax + xx$, when x is equal to a . Therefore, (by Observation 4th,) the ratio of the whole quantity $4aa + 4ad + dd$ to the whole quantity $aa + ad + dd$ will be less than the ratio of $4aa$ to

to ax ; that is, the ratio of the value of the trinomial quantity $aa + 2ax + xx$ to the value of the trinomial quantity $aa - ax + xx$, when x is equal to $a + d$, will be less than the ratio of the value of the former trinomial quantity to the value of the latter, when x is equal to a ; or the ratio of the former trinomial quantity to the latter will have decreased from the ratio of $4aa$ to aa , or of 4 to 1, to the ratio of $4aa + 4ad + dd$ to $aa + ad + dd$, (which is less than the ratio of $4aa + 4ad + 4dd$ to $aa + ad + dd$, or of 4 to 1), while x has increased from a to $a + d$. Q. E. D.

Coroll. 3. It follows from this alternate increase of the ratio of $aa + 2ax + xx$ to $aa - ax + xx$ from a ratio of equality, or an infinitely small ratio of majority, to the ratio of 4 to 1, while x increases from 0 to a , and decrease of it from the ratio of 4 to 1 to a ratio of equality, while x increases further from a ad infinitum, that, if the ratio of $aa + 2ax + xx$ to $aa - ax + xx$ be of any magnitude less than the ratio of 4 to 1, or if the fraction $\frac{aa + 2ax + xx}{aa - ax + xx}$

be equal to any quantity less than $\frac{4}{1}$, or 4, there will be two values of x , the one less than a , and the other greater than a , that will make the fraction $\frac{aa + 2ax + xx}{aa - ax + xx}$ equal to the proposed quantity. Thus, for example, if it be

proposed to make the fraction $\frac{aa + 2ax + xx}{aa - ax + xx}$ equal to 2, such an equation will be possible, and will have two roots, of which the one will be less, and the other will be greater than a . These roots will be $a \times \frac{2 - \sqrt{3}}{2 + \sqrt{3}}$ and $a \times \frac{2 + \sqrt{3}}{2 - \sqrt{3}}$. For, if we suppose the fraction $\frac{aa + 2ax + xx}{aa - ax + xx}$ to be = 2, we shall

have $aa + 2ax + xx = 2 \times \frac{aa - ax + xx}{2 + \sqrt{3}}$ = $2aa - 2ax + 2xx$. Therefore (adding $2ax$ to both sides,) we shall have $aa + 4ax + xx = 2aa + 2xx$, and (subtracting $aa + xx$ from both sides,) we shall have $4ax = aa + xx$, and (subtracting xx from both sides,) $4ax - xx = aa$, and (subtracting both sides from $4aa$,) $4aa - 4ax + xx = 3aa$. Therefore $2a - x$ will be = $\sqrt{3aa} = a \times \sqrt{3}$, and $2a$ will be = $x + a \times \sqrt{3}$, and x will be = $2a - a \times \sqrt{3}$, or $a \times \frac{2 - \sqrt{3}}{2 + \sqrt{3}}$. And $x - 2a$ will likewise be = $\sqrt{3aa} = a \times \sqrt{3}$, and consequently x will be = $a \times \sqrt{3} + 2a$, or $a \times \frac{2 + \sqrt{3}}{2 - \sqrt{3}}$. Therefore either $a \times \frac{2 - \sqrt{3}}{2 + \sqrt{3}}$, or $a \times \frac{2 + \sqrt{3}}{2 - \sqrt{3}}$, being substituted instead of x in the fraction $\frac{aa + 2ax + xx}{aa - ax + xx}$, will make the said fraction equal to 2. And this will appear upon trial. For, if we suppose x to be = $a \times \frac{2 - \sqrt{3}}{2 + \sqrt{3}}$, we shall have ax (= $aa \times \frac{2 - \sqrt{3}}{2 + \sqrt{3}}$) = $2aa - aa \times \sqrt{3}$, and $2ax$ = $4aa - 2aa \times \sqrt{3}$, and xx (= $aa \times \frac{4 - 4 \times \sqrt{3} + 3}{2 + \sqrt{3}}$) = $aa \times \frac{7 - 4 \times \sqrt{3}}{2 + \sqrt{3}}$

$= 7aa - 4aa \times \sqrt{3}$. Therefore $aa + 2ax + xx$ will be $(= aa + 4aa - 2aa \times \sqrt{3} + 7aa - 4aa \times \sqrt{3}) = 12aa - 6aa \times \sqrt{3}$; and $aa - ax + xx$ will be $(= aa - 2aa - aa \times \sqrt{3} + 7aa - 4aa \times \sqrt{3} = aa - 2aa + aa \times \sqrt{3} + 7aa - 4aa \times \sqrt{3}) = 6aa - 3aa \times \sqrt{3}$, and consequently $\frac{aa + 2ax + xx}{aa - ax + xx}$ will be $= \frac{12aa - 6aa \times \sqrt{3}}{6aa - 3aa \times \sqrt{3}} = 2$. Q. E. D.

And, if we suppose x to be $= a \times 2 + \sqrt{3}$, we shall have $ax (= aa \times 2 + \sqrt{3}) = 2aa + aa \times \sqrt{3}$, and $2ax = 4aa + 2aa \times \sqrt{3}$, and $xx (= aa \times 4 + 4 \times \sqrt{3} + 3 = aa \times 7 + 4 \times \sqrt{3}) = 7aa + 4aa \times \sqrt{3}$. Therefore $aa + 2ax + xx$ will be $= aa + 4aa + 2aa \times \sqrt{3} + 7aa + 4aa \times \sqrt{3} = 12aa + 6aa \times \sqrt{3}$; and $aa - ax + xx$ will be $(= aa - 2aa - aa \times \sqrt{3} + 7aa + 4aa \times \sqrt{3}) = 6aa + 3aa \times \sqrt{3}$; and consequently $\frac{aa + 2ax + xx}{aa - ax + xx}$ will be $= \frac{12aa + 6aa \times \sqrt{3}}{6aa + 3aa \times \sqrt{3}} = 2$. Q. E. D.

Sept. 24, 1795.

N. B. The foregoing examination of the Variations of the Ratio, or Proportion, of $aa + 2ax + xx$ to $aa - ax + xx$ while x increases from 0 *ad infinitum*, was suggested to me by the Rev. Mr. John Hellins, in consequence of an answer given to a question that had been proposed by him in the Ladies' Diary for the year 1794, in the words following.

XIII. Question 981, by the Rev. Mr. JOHN HELLINS.

It is proposed to find the correct value of z from the equation $z = \frac{x}{1+x}$, (where x and z begin together,) in series, which shall converge when x is greater than 1; and to perform the whole by means of series.

Two answers were given to this question in the Diary for the year 1795, one by Mr. Hellins himself, and the other by another person under the signature of *Amicus*. It is in the latter answer that the variable quantity $\frac{aa + 2ax + xx}{aa - ax + xx}$, which gave rise to the examination contained in the preceeding pages, was

3 K

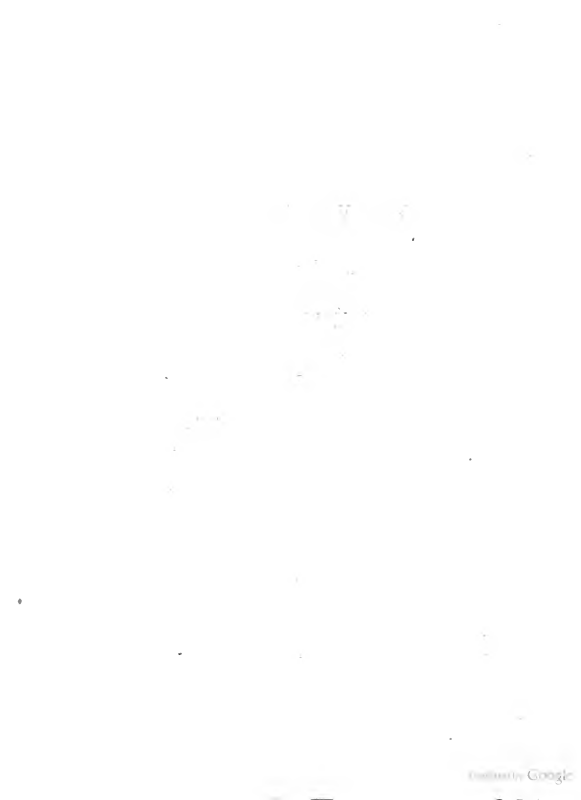
made

made use of. I, however, had not seen it there, and had no knowledge of it till it was mentioned to me by Mr. Hellins. The Question proposed by Mr. Hellins is not a mere idle Speculation to try the acuteness and sagacity of Mathematicians, but occurred, as he informed me, in the Solution of a curious and useful Problem in Physical Astronomy. And the foregoing examination of the Variations of the Proportion of $aa + 2ax + xx$ to $aa - ax + xx$ has also this further use in it, that it seems very well fitted to illustrate the principles of the curious, and often-times subtle and difficult, doctrine of finding the *maximums* of variable quantities.

OUVRAGES

OUVRAGES
DE
MATHÉMATIQUE
DE
BLAISE PASCAL.

Extraits de l'Édition des Œuvres de Monsieur PASCAL faite à La Haye
en l'Année 1779, chez *Detune*, Libraire.



E S S A I S
POUR
LES CONIQUES.

D É F I N I T I O N I.

QUAND plusieurs lignes droites concourent au même point, ou sont toutes parallèles entre elles : toutes ces lignes sont dites de *même ordre*, ou de *même ordonnance* ; et la multitude de ces lignes, est dite *ordre de lignes*, ou *ordonnance de lignes*.

D É F I N I T I O N II.

Par le mot de *Section* de cône, nous entendons la circonférence du cercle, l'ellipse, l'hyperbole, la parabole et l'angle * rectiligne : d'autant qu'un cône coupé parallèlement à sa base, ou par son sommet, ou des trois autres sens qui engendrent l'ellipse, l'hyperbole et la parabole, donne dans sa superficie, ou la circonférence d'un cercle, ou un angle †, ou l'ellipse, ou l'hyperbole, ou la parabole.

D É F I N I T I O N III.

Par le mot de *droite* mis seul, nous entendons la ligne droite.

* Peut-être devoit-on lire, *le triangle*.

† Peut être, *un triangle*.

L E M M E

MACHINE ARITHMÉTIQUE.

A MONSIEUR LE CHANCELIER*.

MONSIEUR,

SI le Public reçoit quelque utilité de l'invention que j'ai trouvée pour faire toutes sortes de Règles d'Arithmétique, par une manière aussi nouvelle que commode, il en aura plus d'obligation à votre Grandeur qu'à mes petits efforts, puisque je ne saurois me vanter que de l'avoir conçue, et qu'elle doit absolument sa naissance à l'honneur de vos commandements. Les longueurs et les difficultés des moyens ordinaires dont on se sert m'ayant fait penser à quelque secours plus prompt et plus facile pour me soulager dans les grands calculs où j'ai été occupé depuis quelques années en plusieurs affaires qui dépendent des emplois dont il vous a plu honorer mon père pour le service de Sa Majesté en la haute Normandie ; j'employai à cette recherche toute la connoissance que mon inclination et le travail de mes premières études m'ont fait acquérir dans les Mathématiques ; et, après une profonde méditation, je reconnus que ce secours n'étoit pas impossible à trouver. Les lumières de la Géométrie, de la Physique, et de la Mécanique m'en fournirent le dessein, et m'assurèrent que l'usage en seroit infailible, si quelque Ouvrier pouvoit former l'instrument dont j'avois imaginé le modèle. Mais ce fut en ce point que je rencontrai des obstacles aussi grands que ceux que je voulois éviter, et auxquels je cherchois un remède. N'ayant pas l'industrie de manier le métal et le marreau comme la plume et le compas ; et les Artisans ayant plus de connoissance de la pratique de leur Art que des sciences sur lesquelles il est fondé : je me vis réduit à quitter toute mon entreprise, dont il ne me revenoit que beaucoup de fatigues, sans aucun bon succès. Mais, Monsieur, votre Grandeur ayant soutenu mon courage, qui se laissoit aller, et m'ayant fait la grâce de parler du simple crayon, que mes amis vous avoient présenté, en des termes qui me le firent voir tout autre qu'il ne m'avoit paru auparavant : avec les nouvelles forces que vos louanges me donnèrent, je fis de nouveaux efforts ; et suspendant tout autre exercice, je ne songeai plus qu'à la construction de cette petite Machine, que j'ai osé, Monsieur, vous présenter, après l'avoir mise en état de faire, avec elle seule et sans aucun travail d'apprenti, les opérations de toutes les parties de l'Arithmétique, selon que je me l'étois proposé.

C'est donc à vous, Monsieur, que je devois ce petit essai, puisque c'est vous qui me l'avez fait faire ; et c'est de vous aussi que j'en attends une glorieuse protection. Les inventions qui ne sont pas connues, ont toujours plus de censeurs que d'approbateurs :

* Pierre Segurier.

on blâme ceux qui les ont trouvées, parce qu'on n'en a pas une parfaite intelligence ; et par un injuste préjugé, la difficulté que l'on s'imagine aux choses extraordinaires, fait qu'au lieu de les considérer pour les estimer, on les accuse d'impossibilité, afin de les rejeter ensuite comme impertinentes. D'ailleurs, Monseigneur, je m'attends bien que parmi tant de doctes qui ont pénétré jusques dans les derniers secrets des Mathématiques, il pourra s'en trouver qui d'abord estiment mon action téméraire, vu qu'en la jeunesse où je suis, et avec si peu de forces, j'ai osé tenter une route nouvelle dans un champ tout hérissé d'épines, et sans avoir de guide pour m'y frayer le chemin. Mais je veux bien qu'ils m'accusent, et même qu'ils me condamnent, s'ils peuvent justifier que je n'ai pas tenu exactement ce que j'avois promis ; et je ne leur demande que la faveur d'examiner ce que j'ai fait, et non pas celle de l'approuver sans le connoître. Aussi, Monseigneur, je puis dire à votre Grandeur, que j'ai déjà la satisfaction de voir mon petit Ouvrage, non-seulement autorisé de l'approbation de quelques-uns des principaux en cette véritable Science, qui, par une préférence toute particulière, a l'avantage de ne rien enseigner qu'elle ne démontre, mais encore honoré de leur estime et de leur recommandation : et que même celui d'entre eux, de qui la plupart des autres admirent tous les jours et recueillent les productions, ne l'a pas jugé indigne de se donner la peine, au milieu de ses grandes occupations, d'en enseigner, et la disposition, et l'usage à ceux qui auront quelque désir de s'en servir. Ce sont là véritablement, Monseigneur, de grandes récompenses du temps que j'ai employé et de la dépense que j'ai faite pour mettre la chose en l'état où je vous l'ai présentée. Mais permettez-moi de flatter ma vanité jusqu'au point de dire, qu'elles ne me satisferoient pas entièrement, si je n'en avois reçu une beaucoup plus importante et plus délicate de votre Grandeur. En effet, Monseigneur, quand je me représente que cette même bouche, qui prononce tous les jours des oracles sur le Trône de la Justice, a daigné donner des éloges au coup d'essai d'un homme de vingt ans ; que vous l'avez jugé digne d'être plus d'une fois le sujet de votre entretien, et de le voir placé dans votre Cabinet parmi tant d'autres choses rares et précieuses dont il est rempli : je suis comblé de gloire, et je ne trouve point de paroles pour faire paroître ma reconnaissance à votre Grandeur, et ma joie à tout le monde.

Dans cette impuissance, où l'excès de votre bonté m'a mis, je me contenterai de la révéler par mon silence : et toute la famille dont je porte le nom étant intéressée aussi-bien que moi, par ce bienfait et par plusieurs autres, à faire tous les jours des vœux pour votre prospérité, nous les ferons d'un cœur si ardent, et si continuel, que personne ne pourra se vanter d'être plus attaché que nous à votre service, ni de porter plus véritablement que moi la qualité, Monseigneur, de votre, &c.

PASCA L.

A V I S

A V I S

*Nécessaire à tous ceux qui auront curiosité de voir la Machine
Arithmétique, et de s'en servir.*

AMI Lecteur : cet avertissement servira pour te faire savoir que j'expose au Public une petite Machine de mon invention, par le moyen de laquelle seule tu pourras, sans peine quelconque, faire toutes les opérations de l'Arithmétique, et te soulager du travail qui t'a souventes fois fatigué l'esprit, lorsque tu as opéré par le jeton ou par la plume : je puis, sans présomption, espérer qu'elle ne te déplaira pas, après que M. le Chancelier l'a honorée de son estime, et que dans Paris, ceux qui sont le mieux versés aux Mathématiques, ne l'ont pas jugée indigne de leur approbation. Néanmoins, pour ne pas paroître négligent à lui faire acquiescer aussi la tienne, j'ai cru être obligé de t'éclaircir sur toutes les difficultés que j'ai estimées capables de choquer ton sens, lorsque tu prendras la peine de la considérer.

Je ne doute pas qu'après l'avoir vue, il ne tombe d'abord dans ta pensée que je devois avoir expliqué par écrit, et sa construction, et son usage ; et que, pour rendre ce discours intelligible, j'étois même obligé, suivant la méthode des Géomètres, de représenter par figures les dimensions, la disposition, et le rapport de toutes les pièces, et comment chacune doit être placée pour composer l'instrument, et mettre son mouvement en sa perfection. Mais tu ne dois pas croire qu'après n'avoir épargné, ni le temps, ni la peine, ni la dépense pour la mettre en état de t'être utile, j'eusse négligé d'employer ce qui étoit nécessaire pour te contenter sur ce point, qui sembloit manquer à son accomplissement, si je n'avois été empêché de le faire par une considération si puissante, que j'espéro même qu'elle te forcera de m'excuser. Oui, j'espère que tu approuveras que je me sois abstenu de ce discours, si tu prends la peine de faire réflexion d'une part sur la facilité qu'il y a d'expliquer de bouche, et d'entendre par une brève conférence, la construction et l'usage de cette Machine ; et d'autre part sur l'embarras et la difficulté qu'il y eût eu d'exprimer par écrit les mesures, les formes, les proportions, les situations, et le surplus des propriétés de tant de pièces différentes. Alors tu jugeras que cette doctrine est du nombre de celles qui ne peuvent être enseignées que de vive voix ; et qu'un discours par écrit en cette matière seroit autant et plus inutile et embarrassant, que celui qu'on emploieroit à la description de toutes les parties d'une montre, dont toutefois l'explication est si facile, quand elle est faite bouche à bouche ; et qu'apparemment un tel discours ne pourroit produire d'autre effet qu'un infailible dégoût en l'esprit de plusieurs, leur faisant concevoir mille difficultés où il n'y en a point du tout.

Maintenant, cher Lecteur, j'estime qu'il est nécessaire de t'avertir que je prévois deux choses capables de former quelques nuages en ton esprit. Je fais qu'il y a nombre de personnes qui font profession de trouver à redire par-tout, et qu'entre ceux-là il pourra s'en trouver qui te diront que cette Machine pouvoit être moins composée ; c'est là la première vapeur que j'estime nécessaire de dissiper. Cette proposition ne peut t'être faite que par certains esprits qui ont véritablement quelque connoissance de la Méchanique ou de la Géométrie, mais qui, pour ne les savoir joindre l'une à l'autre, et toutes deux ensemble à la Physique, se flattent ou se trompent dans leurs conceptions imaginaires, et se persuadent possibles beaucoup de choses qui ne le sont pas, pour ne posséder qu'une théorie imparfaite des choses en général, laquelle n'est pas suffisante de leur faire prévoir en particulier les inconvénients qui arrivent, ou de la part de la matière, ou des places que doivent occuper les pièces d'une machine dont les mouvements sont différents, afin qu'ils soient libres et qu'ils ne puissent s'empêcher, les uns les autres. Lors donc que ces Savants imparfaits te soutiendront que cette Machine pouvoit être moins composée, je te conjure de leur faire la réponse que je leur ferois moi-même, s'ils me faisoient une telle proposition, et de les assurer de ma part que je leur ferai voir, quand il leur plaira, plusieurs autres modèles, et même un instrument entier et parfait, beaucoup moins composé, dont je me suis publiquement servi pendant six mois entiers ; et ainsi que je n'ignore pas que la Machine ne peut être moins composée, et particulièrement si j'eusse voulu instituer le mouvement de l'opération par la face antérieure, ce qui ne pouvoit être qu'avec une incommodité ennuyeuse et insupportable, au lieu que maintenant il se fait par la face supérieure avec toute la commodité qu'on sauroit souhaiter, et même avec plaisir : tu leur diras aussi que, mon dessein n'ayant jamais visé qu'à réduire en mouvement réglé toutes les opérations de l'Arithmétique, je me suis en même-temps persuadé que mon dessein ne réussiroit qu'à ma propre confusion, si ce mouvement n'étoit simple, facile, commode et prompt à l'exécution, et que la Machine ne fût durable, solide, et même capable de souffrir sans altération la fatigue du transport ; et enfin que s'ils avoient autant médité que moi sur cette matière, et passé par tous les chemins que j'ai suivis pour venir à mon but, l'expérience leur auroit fait voir qu'un instrument moins composé ne pouvoit avoir toutes ces conditions que j'ai heureusement données à cette petite Machine.

Car pour la simplicité du mouvement des opérations, j'ai fait en sorte qu'en-cote que les opérations de l'Arithmétique soient en quelque façon opposées l'une à l'autre, comme l'addition à la soustraction, et la multiplication à la division, néanmoins elles se pratiquent toutes sur cette Machine par un seul et unique mouvement.

Pour la facilité de ce même mouvement des opérations, elle est toute apparente, en ce qu'il est aussi facile de faire mouvoir mille et dix mille roues toutes à la fois, si elles y étoient, quoique toutes achèvent leur mouvement très-parfait, que d'en faire mouvoir une seule, (je ne fais si après le principe sur lequel j'ai fondé cette facilité, il en reste un autre dans la Nature.) Que si tu

veux,

veux, outre la facilité du mouvement de l'opération, savoir quelle est la facilité de l'opération même, c'est-à-dire, la facilité qu'il y a en l'opération par cette Machine, tu le peux, si tu prends la peine de la comparer avec les méthodes d'opérer par le jeton et par la plume. Tu fais comme en opérant par le jeton, le Calculateur (sur-tout lorsqu'il manque d'habitude,) est souvent obligé, de peur de tomber en erreur, de faire une longue suite et extension de jetons, et comme la nécessité le contraint après d'abrégier et de relever ceux qui se trouvent inutilement étendus; en quoi tu vois deux peines inutiles, avec la perte de deux temps. Cette Machine facilite et retranche en ses opérations tout ce superflu; le plus ignorant y trouve autant d'avantage que le plus expérimenté; l'instrument supplée au défaut de l'ignorance ou du peu d'habitude; et par des mouvements nécessaires, il fait lui seul, sans même l'intention de celui qui s'en sert, tous les abrégés possibles à la Nature, toutes les fois que les nombres s'y trouvent disposés. Tu fais de même comme en opérant par la plume, on est à tout moment obligé de retenir, ou d'emprunter les nombres nécessaires, et combien d'erreurs se glissent dans ces retentions et emprunts, à moins d'une très-longue habitude, et en outre d'une attention profonde et qui fatigue l'esprit en peu de temps. Cette Machine délivre celui qui opère par elle, de cette vexation; il suffit qu'il ait le jugement; elle le relève du défaut de la mémoire, et sans rien retenir, ni emprunter, elle fait d'elle-même ce qu'il désire, sans même qu'il y pense. Il y a cent autres facilités que l'usage fait voir, dont le discours pourroit être ennuyeux.

Quant à la commodité de ce mouvement, il suffit de dire qu'il est insensible, allant de gauche à droite, et imitant notre méthode vulgaire d'écrire, fors qu'il procède circulairement.

Et enfin quant à sa promptitude, elle paroît de même, en la comparant avec celle des autres deux méthodes du jeton et de la plume; et si tu veux encore une plus parfaite explication de sa vitesse, je te dirai qu'elle est pareille à l'égalité de la main de celui qui opère: cette promptitude est fondée, non seulement sur la facilité des mouvements qui ne font aucune résistance, mais encore sur la petitesse des roues que l'on meut à la main, qui fait que le chemin étant plus court, le moteur peut le parcourir en moins de temps; d'où il arrive encore cette commodité, que par ce moyen, la Machine se trouvant réduite en plus petit volume, elle en est plus maniable et portative.

Et quant à la durée et solidité de l'instrument, la seule dureté du métal dont il est composé, pourroit en donner à quelque autre la certitude: mais d'y prendre une assurance entière, et la donner aux autres, je n'ai pu le faire qu'après en avoir fait l'expérience, par le transport de l'instrument durant plus de deux cents cinquante lieues de chemin, sans aucune altération.

Ainsi, cher Lecteur, je te conjure encore une fois de ne point prendre pour imperfection que cette Machine soit composée de tant de pièces, puisque sans cette composition, je ne pouvois lui donner toutes les conditions ci-devant déduites, qui toutefois lui étoient toutes nécessaires; en quoi tu pourras remarquer

quer une espèce de paradoxe, que, pour rendre le mouvement de l'opération plus simple, il a fallu que la Machine ait été construite d'un mouvement plus composé.

La seconde cause que je prévois capable de te donner de l'ombrage, ce sont, cher Lecteur, les mauvaises copies de cette Machine qui pourroient être produites par la présomption des Artisans : en ces occasions, je te conjure d'y porter soigneusement l'esprit de distinction, te garder de la surprise, distinguer entre la copie et la copie, et ne pas juger des véritables originaux, par les productions imparfaites de l'ignorance et de la témérité des Ouvriers : plus ils sont excellents en leur Art, plus il est à craindre que la vanité ne les enlève par la persuasion, qu'ils se donnent trop légèrement, d'être capables d'entreprendre et d'exécuter d'eux-mêmes des ouvrages nouveaux, desquels ils ignorent, et les principes, et les règles ; puis, enivrés de cette fausse persuasion, ils travaillent en tâtonnant, c'est-à-dire, sans mesures certaines et sans proportions réglées par art : d'où il arrive qu'après beaucoup de temps et de travail, ou ils ne produisent rien qui revienne à ce qu'ils ont entrepris ; ou, au plus, ils font paroître un petit monstre auquel manquent les principaux membres, les autres étant informes et sans aucune proportion : ces imperfections le rendant ridicule, ne manquent jamais d'attirer le mépris de tous ceux qui le voient, desquels la plupart rejettent, sans raison, la faute sur celui qui, le premier, a eu la pensée d'une telle invention ; au lieu de s'en éclaircir avec lui, et puis blâmer la présomption de ces Artisans, qui, par une fausse hardiesse d'oser entreprendre plus que leurs semblables, produisent ces inutiles avortons. Il importe au Public de leur faire connoître leur foiblesse, et leur apprendre que, pour les nouvelles inventions, il faut nécessairement que l'Art soit aidé par la théorie, jusqu'à ce que l'usage ait rendu les règles de la théorie si communes, qu'il les ait enfin réduites en art, et que le continuel exercice ait donné aux Artisans l'habitude de suivre et pratiquer ces règles avec assurance. Et tout ainsi qu'il n'étoit pas en mon pouvoir, avec toute la théorie imaginable, d'exécuter moi seul mon propre dessein, sans l'aide d'un Ouvrier qui possédât parfaitement la pratique du tour, de la lime, et du marteau, pour réduire les pièces de la Machine dans les mesures et proportions que par les règles de la théorie je lui prescrivois : il est de même absolument impossible à tous les simples Artisans, si habiles qu'ils soient en leur Art, de mettre en perfection une pièce nouvelle qui consiste, comme celle-ci, en mouvements compliqués, sans l'aide d'une personne qui, par les règles de la théorie, lui donne les mesures et les proportions de toutes les pièces dont elle doit être composée.

Cher Lecteur, j'ai sujet particulier de te donner ce dernier avis, après avoir vu de mes yeux une fausse exécution de ma pensée, faite par un Ouvrier de la ville de Rouen, Horloger de profession, lequel, sur le simple récit qui lui fut fait de mon premier modèle, que j'avois fait quelques mois auparavant, eut assez de hardiesse pour en entreprendre un autre, et, qui plus est, par une autre espèce de mouvement ; mais, comme le bon homme n'a autre talent que celui de manier adroitement ses outils, et qu'il ne fait pas seulement si la Géométrie

et

et la Méchanique sont au monde : aussi (quoiqu'il soit très-habile en son Art, et même très-industrieux en plusieurs choses qui n'en sont point) ne fit-il qu'une pièce inutile, propre véritablement, polie et très-bien limée par le dehors, mais tellement imparfaite au-dedans, qu'elle n'est d'aucun usage. Toutefois à cause seulement de sa nouveauté, elle ne fut pas sans estime parmi ceux qui n'y connoissent rien, et, nonobstant tous les défauts essentiels que l'épreuve y fit reconnoître, ne laissa pas de trouver place dans le Cabinet d'un Curieux de la même Ville, rempli de plusieurs autres pièces rares et ingénieuses. L'aspect de ce petit avorton me déplut au dernier point, et refroidit tellement l'ardeur avec laquelle je faisois alors travailler à l'accomplissement de mon modèle, qu'à l'instant même je donnai congé à tous mes Ouvriers, résolu de quitter entièrement mon entreprise, par la juste appréhension que je conçus qu'une pareille hardiesse ne pût à plusieurs autres, et que les fausses copies qu'ils pouvoient produire de cette nouvelle pensée, n'en ruinaissent l'estime dès sa naissance, avec l'utilité que le Public pouvoit en recevoir. Mais quelque temps après, M. le Chancelier ayant daigné honorer de sa vue mon premier modèle, et donner le témoignage de l'estime qu'il faisoit de cette invention, me fit commandement de la mettre en sa perfection ; et, pour dissiper la crainte qui m'avoit retenu quelque temps, il lui plut de retrancher le mal dès sa racine, et d'empêcher le cours qu'il pouvoit prendre au préjudice de ma réputation et au désavantage du Public, par la grace qu'il me fit de m'accorder un Privilège, qui n'est pas ordinaire, et qui étouffe avant leur naissance tous ces avortons illégitimes qui pourroient être engendrés d'ailleurs que de la légitime et nécessaire alliance de la Théorie avec l'Art.

Au reste, si quelquefois tu as exercé ton esprit à l'invention des Machines, je n'aurai pas grande peine à te persuader que la forme de l'instrument, en l'état où il est à présent, n'est pas le premier effet de l'imagination que j'ai eue sur ce sujet : j'avois commencé l'exécution de mon projet par une marche très-différente de celle-ci, et en sa matière, et en sa forme, laquelle (bien qu'en état de satisfaire à plusieurs) ne me donna pas pourtant la satisfaction entière, ce qui fit qu'en la corrigeant peu à peu, j'en fis insensiblement une seconde, en laquelle ; rencontrant encore des inconvénients que je ne pus souffrir, pour y apporter le remède j'en composai une troisième, qui va par ressorts, et qui est très-simple en sa construction. C'est celle de laquelle, comme j'ai déjà dit, je me suis servi plusieurs fois, au vû et sù d'une infinité de personnes, et qui est encore en état de servir autant que jamais. Cependant en la perfectionnant toujours, je trouvai des raisons de la changer ; et enfin reconnoissant dans toutes, ou de la difficulté d'agir, ou de la rudesse aux mouvements, ou de la disposition à se corrompre trop facilement par le temps ou par le transport, j'ai pris la patience de faire jusqu'à plus de cinquante modèles, tous différents, les uns de bois, les autres d'ivoire et d'ébène, et les autres de cuivre, avant que d'être venu à l'accomplissement de la Machine que maintenant je fais paroître, laquelle, bien que composée de tant de petites pièces différentes, comme tu pourras voir, est toutefois tellement solide, qu'après l'expérience dont j'ai parlé ci-devant, j'ose te donner assurance que tous les efforts qu'elle pourroit recevoir en la transportant si loin
que

que tu voudras, ne sauroient la corrompre, ni lui faire souffrir la moindre altération.

Enfin, cher Lecteur, maintenant que j'estime l'avoir mise en état d'être vue, et que même tu peux, si tu en as la curiosité, la voir et t'en servir, je te prie d'agréer la liberté que je prends d'espérer que la seule pensée à trouver une troisième méthode pour faire toutes les opérations arithmétiques, totalement nouvelle, et qui n'a rien de commun avec les deux méthodes vulgaires de la plume et du jeton, recevra de toi quelque estime; et qu'en approuvant le dessein que j'ai eu de te plaire en te soulageant, tu me feras gré du soin que j'ai pris pour faire que toutes les opérations qui, par les précédentes méthodes, sont pénibles, composées, longues et peu certaines, deviennent faciles, simples, promptes et assurées.

LETTRE DE PASCAL

A

LA REINE CHRISTINE,

En lui envoyant la Machine Arithmétique.

MADAME,

SI j'avois autant de fanté que de zèle, j'irois moi-même présenter à Votre Majesté un Ouvrage de plusieurs années, que j'ose lui offrir de si loin; et je ne souffrirais pas que d'autres mains que les miennes eussent l'honneur de le porter aux pieds de la plus grande Princesse du monde. Cet Ouvrage, MADAME, est une Machine pour faire les règles d'Arithmétique sans plume et sans jetons. Votre Majesté n'ignore pas la peine et le temps que coûtent les productions nouvelles, sur-tout lorsque les inventeurs veulent les porter eux-mêmes à la dernière perfection: c'est pourquoi il seroit inutile de dire combien il y a que je travaille à celle-ci; et je ne pourrois mieux l'exprimer qu'en disant, que je m'y suis attaché avec autant d'ardeur, que si j'eusse prévu qu'elle devoit paroître un jour devant une personne si auguste. Mais, MADAME, si cet honneur n'a pas été le véritable motif de mon travail, il en sera du moins la récompense; et je m'estimerai trop heureux, si, à la suite de tant de veilles, il peut donner à Votre Majesté une satisfaction de quelques moments. Je n'importunerai pas non plus Votre Majesté du particulier de ce qui compose cette Machine: si

si elle en a quelque curiosité, elle pourra se contenter dans un Discours * que j'ai adressé à M. de Bourdelot † ; j'y ai touché en peu de mots toute l'histoire de cet Ouvrage, l'objet de son invention, l'occasion de sa recherche, l'utilité de ses efforts, les difficultés de son exécution, les degrés de son progrès, le succès de son accomplissement et les règles de son usage. Je dirai donc seulement ici le sujet qui me porte à l'offrir à Votre Majesté, ce que je considère comme le couronnement et le dernier bonheur de son aventure. Je fais, MADAME, que je pourrai être suspect d'avoir recherché de la gloire, en le présentant à Votre Majesté, puisqu'il ne sauroit passer que pour extraordinaire, quand on verra qu'il s'adresse à elle ; et qu'au lieu qu'il ne devoit lui être offert que par la considération de son excellence, on jugera qu'il est excellent, par cette seule raison qu'il lui est offert. Ce n'est pas néanmoins cette espérance qui m'a inspiré un tel dessein. Il est trop grand, MADAME, pour avoir d'autre objet que Votre Majesté même. Ce qui m'y a véritablement porté, est l'union qui se trouve en sa Personne sacrée, de deux choses qui me comblent également d'admiration et de respect, qui sont l'autorité souveraine et la science solide. Car j'ai une vénération toute particulière pour ceux qui sont élevés au suprême degré, ou de puissance, ou de connoissances. Les derniers peuvent, si je ne me trompe, aussi-bien que les premiers, passer pour des Souverains. Les mêmes degrés se rencontrent entre les Génies qu'entre les conditions : et le pouvoir des Rois sur leurs Sujets n'est, ce me semble, qu'une image du pouvoir des esprits sur les esprits qui leur sont inférieurs, sur lesquels ils exercent le droit de persuader ; ce qui est parmi eux ce que le droit de commander est dans le gouvernement politique. Ce second empire me paroît même d'un ordre d'autant plus élevé, que les esprits sont d'un ordre plus élevé que les corps ; et d'autant plus équitable, qu'il ne peut être départi et conservé que par le mérite, au lieu que l'autre peut l'être par la naissance ou par la fortune. Il faut donc avouer que chacun de ces empires est grand en soi ; mais, MADAME, que Votre Majesté me permette de le dire, elle n'y est pas blessée ; l'un sans l'autre me paroît défectueux. Quelque puissant que soit un Monarque, il manque quelque chose à sa gloire, s'il n'a la prééminence de l'esprit ; et quelque éclairé que soit un Sujet, sa condition est toujours rabaisée par sa dépendance. Les hommes qui désirent naturellement ce qui est le plus parfait, avoient jusqu'ici continuellement aspiré à rencontrer ce Souverain par excellence. Tous les Rois et tous les Savants en étoient autant d'ébauches, qui ne remplissoient qu'à demi leur attente ; ce chef-d'œuvre étoit réservé à notre siècle. Et afin que cette grande merveille parût accompagnée de tous les sujets possibles d'étonnement, le degré où les hommes n'avoient pu atteindre, est rempli par une jeune Reine, dans laquelle se rencontrent ensemble l'avantage de l'expérience avec la tendresse de l'âge ; le loisir de l'étude avec l'occupation d'une royale naissance ; et l'éminence de la science avec la faiblesse du sexe. C'est Votre Majesté, MADAME, qui

* Ce Discours paroît être celui de la page 12 ci-dessus, avec quelques additions qu'on n'a pu retrouver.

† Médecin de la Reine Christine.

fournit à l'Univers cet unique exemple qui lui manquoit ; c'est elle en qui la puissance est dispensée par les lumières de la science, et la science relevée par l'éclat de l'autorité. C'est cette union si merveilleuse, qui fait que comme Votre Majesté ne voit rien qui soit au-dessus de sa puissance, elle ne voit rien aussi qui soit au-dessus de son esprit ; et qu'elle fera l'admiration de tous les siècles. Regnez donc, incomparable Princesse, d'une manière toute nouvelle ; que votre génie vous assujettisse tout ce qui n'est pas soumis à vos armes : regnez par le droit de la naissance, par une longue suite d'années, sur tant de triomphantes Provinces ; mais regnez toujours par la force de votre mérite sur toute l'étendue de la terre. Pour moi, n'étant pas né sous le premier de vos Empires, je veux que tout le monde sache que je fais gloire de vivre sous le second ; et c'est pour le témoigner, que j'ose lever les yeux jusqu'à ma Reine, en lui donnant cette première preuve de ma dépendance. Voilà, MADAME, ce qui me porte à faire à Votre Majesté ce présent, quoiqu'indigne d'elle. Ma faiblesse n'a pas arrêté mon ambition. Je me suis figuré, qu'encore que le seul nom de Votre Majesté semble éloigner d'elle tout ce qui lui est disproportionné, elle ne rejette pas néanmoins tout ce qui lui est inférieur ; autrement sa grandeur seroit sans hommages, et sa gloire sans éloges. Elle se contente de recevoir un grand effort d'esprit, sans exiger qu'il soit l'effort d'un esprit grand comme le sien. C'est par cette condescendance qu'elle daigne entrer en communication avec le reste des hommes : et toutes ces considérations jointes, me font lui proposer avec toute la soumission dont l'un des plus grands admirateurs de ses héroïques qualités est capable, que je ne souhaite rien avec tant d'ardeur que de pouvoir être adopté, MADAME, de Votre Majesté, pour son très humble, très-obéissant et très-fidèle serviteur,

BLAISE PASCAL.

PRIVILEGE

PRIVILEGE DU ROI,

POUR

LA MACHINE ARITHMÉTIQUE.

LOUIS, par la grace de Dieu, Roi de France et de Navarre, &c. ; Salut.

Notre très-cher et bien-ami le Sieur PASCAL nous a fait remontrer, qu'à l'imitation du Sieur Pascal, son pere, notre Conseiller en nos Conseils, et Préfident en notre Cour des Aides d'Auvergne, il auroit eu, dès ses plus jeunes années, une inclination particuliere aux Sciences Mathématiques, dans lesquelles, par ses études et ses observations, il a inventé plusieurs choses, et particulièrement une Machine par le moyen de laquelle on peut faire toutes sortes de supputations, additions, soustractions, multiplications, divisions, et toutes les autres règles arithmétiques, tant en nombres entiers que rompus, sans se servir de plume, ni jetons, par une méthode beaucoup plus simple, plus facile à apprendre, plus prompte à l'exécution, et moins pénible à l'esprit que les autres façons de calculer qui ont été en usage jusqu'à présent ; et qui, outre ces avantages, a celui d'être hors de tout danger d'erreur, qui est la condition la plus importante de toutes dans les calculs. De laquelle Machine il auroit fait plus de cinquante modèles, tous différens, les uns composés de verges ou lames droites, d'autres de courbes, d'autres avec des chaînes, les uns avec des rouages concentriques, d'autres avec des excentriques, les uns mouvans en ligne droite, d'autres circulairement, les uns en cônes, d'autres en cylindres, et d'autres tous différens de ceux-là, soit pour la matière, soit pour la figure, soit pour le mouvement : de toutes lesquelles manières différentes, l'invention principale et le mouvement essentiel consistent en ce que chaque roue ou verge d'un ordre faisant un mouvement de dix figures arithmétiques, fait mouvoir la prochaine d'une figure seulement. Après tous lesquels essais, auxquels il a employé beaucoup de temps et de frais, il seroit enfin arrivé à la construction d'un modèle achevé qui a été reconnu infaillible, par les plus doctes Mathématiciens de ce temps, qui l'ont universellement honoré de leur approbation, et estimé très-utile au Public. Mais d'autant que ledit instrument peut être aisément contrefait par des Ouvriers, et qu'il est néanmoins impossible qu'ils parviennent à l'exécuter dans la justesse et perfection nécessaire pour s'en servir utilement, s'ils n'y sont conduits expressément par ledit Pascal, ou par une personne qui ait une entière

intelligence de l'artifice de son mouvement, il seroit à craindre que s'il étoit permis à toutes sortes de personnes de tenter d'en construire de semblables, les défauts qui s'y rencontreroient infailliblement par la faute des Ouvriers, ne rendissent cette invention aussi inutile qu'elle doit être profitable étant bien exécutée. C'est pourquoi il desireroit qu'il nous plût faire défenses à tous Artisans et autres personnes, de faire ou faire faire ledit instrument sans son consentement, nous suppliant à cette fin de lui accorder nos Lettres sur ce nécessaires ; et parce que ledit instrument est à présent à un prix excessif, qui le rend, par sa cherté, comme inutile au Public, et qu'il espère le réduire à moindre prix et tel qu'il puisse avoir cours, ce qu'il prétend faire par l'invention d'un mouvement plus simple et qui opère néanmoins le même effet, à la recherche duquel il travaille continuellement, et en y s'ylant peu à peu les Ouvriers encore peu habitués, lesquelles choses dépendent d'un temps qui ne peut être limité.

A ces causes, desirant gratifier et favorablement traiter ledit Pascal, fils, en considération de sa capacité en plusieurs Sciences, et sur-tout aux Mathématiques, et pour l'exciter d'en communiquer de plus en plus les fruits à nos Sujets, et ayant égard au notable soulagement que cette Machine doit apporter à ceux qui ont de grands calculs à faire, et à raison de l'excellence de cette invention, nous avons permis et permettons par ces présentes signées de notre main, audit Sieur Pascal, fils, et à ceux qui auront droit de lui, dès-à-présent et à toujours, de faire construire ou fabriquer par tels Ouvriers, de telle matière et en telle forme qu'il avisera bon être, en tous les lieux de notre obéissance, ledit instrument par lui inventé, pour compter, calculer, faire toutes additions, soustractions, multiplications, divisions et autres règles d'arithmétique, sans plume, ni jetons ; et faisons très-expresses défenses à toutes personnes, Artisans et autres, de quelque qualité et condition qu'ils soient, d'en faire, ni faire faire, vendre, ni débiter dans aucun lieu de notre obéissance, sans le consentement dudit Sieur Pascal, fils, ou de ceux qui auront droit de lui, sous prétexte d'augmentation, changement de matière, forme ou figure, ou diverses manières de s'en servir, soit qu'ils fussent composés de roues excentriques ou concentriques, ou parallèles, de verges ou bâtons et autres choses, ou que les roues se meuvent seulement d'une part ou de toutes deux, ni pour quelque déguisement que ce puisse être, même à tous étrangers, tant Marchands, que d'autres professions, d'en exposer, ni vendre en ce Royaume, quoiqu'ils eussent été faits hors d'ice-lui : le tout à peine de trois mille livres d'amende, payables sans déport par chacun des contrevenants, et applicables, un tiers à nous, un tiers à l'Hôtel-Dieu de Paris, et l'autre tiers audit Sieur Pascal, ou à ceux qui auront son droit ; de confiscation des instruments contrefaits, et de tous dépens, dommages et intérêts. Enjoignons à cet effet à tous Ouvriers qui construiront ou fabriqueront lesdits instruments en vertu des présentes, d'y faire apposer par ledit Sieur Pascal, ou par ceux qui auront son droit, telle contre-marque qu'ils auront choisie, pour témoigner qu'ils auront visité lesdits instruments, et qu'ils les auront reconnus sans défaut. Voulons que tous ceux où ces formalités ne seront pas gardées, soient confiscués, et que ceux qui les auront faits ou qui en seront trouvés

trouvés faisis, soient sujets aux peines et amendes susdites ; à quoi ils seront contraints en vertu des présentes, ou de copies d'icelles duement collationnées par l'un de nos amés et féaux Conseillers-Secrétaires, auxquelles foi sera ajoutée comme à l'original : du contenu duquel nous vous mandons que vous le fassiez jouir et user pleinement et paisiblement, et ceux auxquels il pourra transporter son droit, sans souffrir qu'il leur soit donné aucun empêchement. Mandons au premier notre Huissier ou Sergent sur ce requis, de faire, pour l'exécution des présentes, tous les Exploits nécessaires, sans demander autre permission. Car tel est notre plaisir : nonobstant tous Edits, Ordonnances, Déclarations, Arrêts, Réglemens, Privilèges, Statuts et confirmation d'iceux, Clameur de Haro, Charte Normande, et autres Lettres à ce contraires, auxquelles et aux dérogratives y contenues, nous dérogeons par ces présentes. Donné à Compiègne, le vingt-deuxième jour de Mai, l'an de grace mil, six cent, quarante-neuf, et de notre regne le septième.

Signé, L O U I S.

Et plus bas, la REINE REGENTE, sa Mere, présente.

Par le Roi, PHELYPEAUX, *gratis*.

L'original en parchemin scellé du grand Sceau de cire jaune.

DESCRIP-

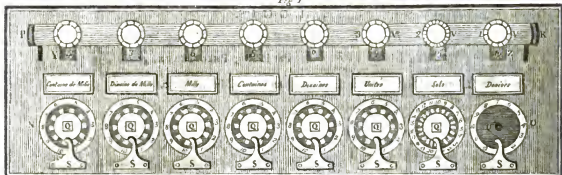
DESCRIPTION

DE

LA MACHINE ARITHMÉTIQUE DE PASCAL,

PAR M. DIDEROT*.

Fig. 1



DANS la *Figure 1*, NOPR est une plaque de cuivre qui forme la surface supérieure de la *Machine*. On voit à la partie inférieure de cette plaque une rangée NO de cercles Q, Q, Q, &c., tous mobiles, autour de leurs centres Q; le premier à la droite a douze dents; le second en allant de droite à gauche en a vingt; et tous les autres en ont dix. Les pièces qu'on aperçoit en S, S, S, &c. et qui s'avancent sur les disques des cercles mobiles Q, Q, Q, &c., sont des étochios ou arrêts, qu'on appelle *potences*. Ces étochios sont fixes et immobiles; ils ne posent point sur les cercles qui peuvent se mouvoir librement sous leurs pointes; ils ne servent qu'à arrêter un stylet, qu'on appelle *dirigreur*, qu'on tient à la main, et dont on place la pointe entre les dents des cercles mobiles Q, Q, Q, &c., pour les faire tourner dans la direction 6, 5, 4, 3, &c. quand on se sert de la *Machine*.

* Cette excellente Description est tirée du premier Volume de l'*Encyclopédie*. La *Machine* dont il s'agit étant aujourd'hui peu connue, et nullement en usage, le seul moyen d'en donner une idée suffisante au Lecteur, étoit de la décrire; les raisons que Pascal a alléguées ci-dessus pour se dispenser lui-même de ce travail, n'ont plus lieu.

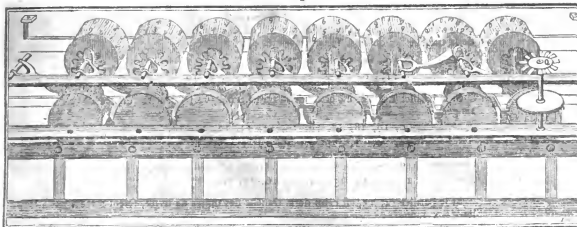
Il est évident, par le nombre des dents des cercles mobiles Q, Q, Q, &c, que le premier à droite marque les deniers; le second en allant de droite à gauche, les sous; le troisième, les unités de livres; le quatrième, les dizaines; le cinquième, les centaines; le sixième, les mille; le septième, les dizaines de mille; le huitième, les centaines de mille: et quoiqu'il n'y en ait que huit, on auroit pu, en agrandissant la Machine, pousser plus loin le nombre de ses cercles.

La ligne YZ est une rangée de trous, à travers lesquels on aperçoit des chiffres. Les chiffres aperçus ici sont 436,809 l. 15 s. 10 d.; mais on verra par la suite qu'on peut en faire paroître d'autres à discrétion par les mêmes ouvertures.

La bande P, R, est mobile de bas en haut: on peut, en la prenant par ses extrémités P, R, la faire descendre sur la rangée des ouvertures 436,809 l. 15 s. 10 d. qu'elle couvrirait: mais alors on apercevrait une autre rangée parallèle de chiffres à travers des trous placés directement au-dessus des premiers.

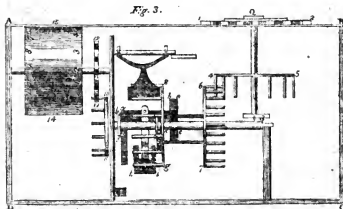
La même bande P, R porte de petites roues gravées de plusieurs chiffres, toutes avec une aiguille au centre, à laquelle la petite roue sert de cadran: chacune de ces roues porte autant de chiffres que les cercles mobiles Q, Q, Q, &c, auxquels elles correspondent perpendiculairement. Ainsi V₁ porte douze chiffres, ou plutôt a douze divisions; V₂ en a vingt; V₃ en a dix; V₄ dix, et ainsi de suite.

Fig. 2



On voit (Fig. 2.) la Machine entière. On a découvert la roue des deniers, pour faire voir l'effet de cette roue sur les autres. Il en est de même pour l'effet de toute autre roue.

Fig. 3.



ABCD (Fig. 3.) est une coupe verticale de la Machine. 1^{re} Q₂ représente un des cercles mobiles Q de la Fig. 1 ; ce cercle entraîne par son axe Q₃ la roue à chevilles 4, 5. Les chevilles de la roue 4, 5 font mouvoir la roue 6, 7, la roue 8, 9, et la roue 10, 11, qui sont toutes fixées sur un même axe. Les chevilles de la roue 10, 11, engrenent dans la roue 12, 13, et la font mouvoir, et avec elle le barillet 14, 15.

Sur le barillet 14, 15 sont tracées l'une au-dessus de l'autre, deux rangées de chiffres de la manière qu'on va dire. Si l'on suppose que ce barillet soit celui de la tranche des deniers, soient tracés les deux rangées :

0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1,
11, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Si le barillet 14, 15, est celui de la tranche des sous, soient tracées les deux rangées :

0, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10,
19, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
9, 8, 7, 6, 5, 4, 3, 2, 1.
10, 11, 12, 13, 14, 15, 16, 17, 18.

Si le barillet 14, 15, est celui de la tranche des unités de livres, soient tracées les deux rangées :

0, 9, 8, 7, 6, 5, 4, 3, 2, 1.
9, 0, 1, 2, 3, 4, 5, 6, 7, 8.

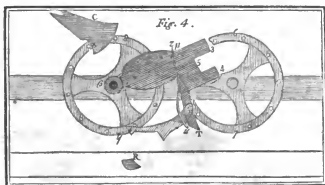
Il est évident, 1^{er}, que c'est de la rangée inférieure des chiffres tracés sur les barillets, que quelques-uns paroissent à travers les ouvertures de la ligne YZ (Fig. 1.)

(Fig. 1.) et que ceux qui paroissent à travers les ouvertures couvertes de la bande mobile P, R, sont de la rangée supérieure.

2°. Qu'en tournant (Fig. 1.) le cercle mobile Q, on arrêtera sous une des ouvertures de la ligne Y Z, tel chiffre que l'on voudra ; et, que le chiffre retranché de 11 sur le barillet des deniers, donnera celui qui lui correspond dans la rangée supérieure des deniers ; retranché de 19 sur le barillet des sous, il donnera celui qui lui correspond dans la rangée supérieure des sous ; retranché de 9 sur le barillet des unités de livres, il donnera celui qui lui correspond dans la rangée supérieure des unités de livres, et ainsi de suite.

3°. Que pareillement celui de la bande supérieure du barillet des deniers retranché de 11, donnera celui qui lui correspond dans la rangée inférieure, &c.

La piece *abcdefgbikl*, qu'on entrevoit (même Figure 3.), est celle qu'on appelle *fautoir*. Il est important de bien en considérer la figure, la position et le jeu ; car sans une connoissance très-exacte de ces trois choses, il ne faut pas espérer d'avoir une idée précise de la Machine. Aussi avons-nous répété cette piece en quatre figures différentes. *abcdefgbikl* (Fig. 3.) est le *fautoir*, comme nous venons d'en avertir : 1 2 3 4 5 6 7 8 *xy* T *zu* l'est aussi (Fig. 4.) ; et 1 2 3 4 5 6 7 8 9 l'est encore (Fig. 6.) Voyez également la Fig. 5.

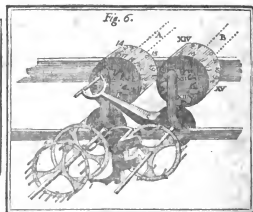
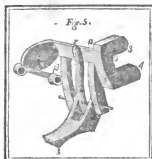


Le *fautoir* (Fig. 3.) a deux anneaux ou portions de douilles, dans lesquelles passe la portion *fk* et *gl* de l'axe de la roue à chevilles 8, 9 ; il est mobile sur cette partie d'axe. Le *fautoir* (Fig. 4.), a une concavité ou partie échancrée 3, 4, 5 ; un coude *μ*, pratiqué pour laisser passer les chevilles attachées à la roue 8, 9 ; deux anneaux dont on voit un en *ε*, l'autre est couvert par une portion de la roue 6, 7 ; en 2, une espèce de coulisse dans laquelle le cliquet 1, 2, est suspendu par le tenon 2, et pressé par un ressort entre les chevilles de la roue 8, 9. Ce ressort est représenté par *zu* ; en appuyant sur le talon du cliquet, il pousse son extrémité 1 entre les chevilles de la roue 8, 9.

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Fig. 5.



On a représenté (Fig. 5.) le sautoir avec tous ses développements, pour en faire mieux sentir la figure et le jeu. Comparez cette figure, lettre à lettre, avec la figure 4.

Ce qui précède bien entendu, nous pouvons passer au jeu de la Machine. Soit (Fig. 3.) le cercle mobile 1Q2, mu dans la direction 1Q2 : la roue à chevilles 4, 5, sera mue, et la roue à chevilles 6, 7 ; et (Fig. 6.) la roue VIII, IX, car c'est la même que la roue 8, 9 de la Fig. 3. Cette roue VIII, IX sera mue dans la direction VIII, VIII, IX, IX. La première de ses deux chevilles *r*, *s*, entrera dans l'échancrure du sautoir ; le sautoir continuera d'être élevé, à l'aide de la seconde cheville *s*. Dans ce mouvement, l'extrémité 1 du cliquet sera entraînée ; et se trouvant à la hauteur de l'entre-deux de deux chevilles immédiatement supérieur à celui où elle étoit, elle y sera poussée par le ressort. Mais la Machine est construite de manière que ce premier échappement n'est pas plutôt fait, qu'il s'en fait un autre, celui de la seconde cheville *s*, de dessous la partie 3, 4 du sautoir : ce second échappement laisse le sautoir abandonné à lui-même : le poids de la partie 4 5 6 7 fait agir l'extrémité 1 du cliquet contre la cheville de la roue 6, 7, sur laquelle elle vient de s'appuyer par le premier échappement ; fait tourner la roue 8, 9 dans le sens 8, 8, 9, 9, et par conséquent aussi dans le même sens la roue 10, 11, 11, et la roue 12, 13, 13, en sens contraire, ou dans la direction 13, 13, 12 ; et dans le même sens que la roue 12, 13, 13, le barillet 14, 15. Mais telle est encore la construction de la Machine, que quand par le second échappement, celui de la cheville *s* de dessous la partie 3, 4 du sautoir, ce sautoir se trouve abandonné à lui-même, il ne peut descendre et entraîner la roue 8, 9 que d'une certaine quantité déterminée. Quand il est descendu de cette quantité, la partie T (Fig. 4.) de la coulisse rencontre l'étochio R, qui l'arrête.

Maintenant

Maintenant (I.) si l'on suppose, 1°. que la roue VIII, IX a douze chevilles, la roue X, XI autant, et la roue XII, XIII autant encore; 2°. que la roue 8, 9 a vingt chevilles, la roue 10, 11 vingt, et la roue 12, 13 autant; 3°. que l'extrémité T du sautoir (Fig. 4.) rencontre l'étochio R précisément quand la roue 8, 9 (Fig. 6.) a tourné d'une vingtième partie, il s'ensuivra évidemment que le barillet XIV, XV fera un tour sur lui-même, tandis que le barillet 14, 15 ne tournera sur lui-même que de sa vingtième partie.

(II.) Si l'on suppose, 1°. que la roue VIII, IX a vingt chevilles, la roue X, XI autant, et la roue XII, XIII autant; 2°. que la roue 8, 9 ait dix chevilles, la roue 10, 11 autant, et la roue 12, 13 autant; 3°. que l'extrémité T du sautoir ne soit arrêtée (Fig. 3.) par l'étochio R, que quand la roue 8, 9 (Fig. 6.) a tourné d'une dixième partie, il s'ensuivra évidemment que le barillet XIV, XV fera un tour entier sur lui-même, tandis que le barillet 14, 15 ne tournera sur lui-même, que de sa dixième partie.

(III.) Si l'on suppose, 1°. que la roue VIII, IX ait dix chevilles, la roue X, XI autant, et la roue XII, XIII autant; 2°. que la roue 8, 9 ait pareillement dix chevilles, la roue 10, 11 autant, et la roue 12, 13 autant aussi; 3°. que l'extrémité T du sautoir (Fig. 4.) ne soit arrêtée par l'étochio R, que quand la roue 8, 9 (Fig. 6.) aura tourné d'un dixième, il s'ensuivra évidemment que le barillet XIV, XV fera un tour entier sur lui-même, tandis que le barillet 14, 15 ne tournera sur lui-même que d'un dixième.

On peut donc, en général, établir tel rapport qu'on voudra entre un tour entier du barillet XIV, XV, et la partie dont le barillet 14, 15 tournera dans le même temps.

Donc si l'on écrit sur le barillet XIV, XV les deux rangées de nombre suivantes, l'une au-dessus de l'autre, comme on le voit :

0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1,
11, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

et sur le barillet 14, 15, les deux rangées suivantes, comme on les voit,

0, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10,
19, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

9, 8, 7, 6, 5, 4, 3, 2, 1,
10, 11, 12, 13, 14, 15, 16, 17, 18.

et que les zéros des deux rangées inférieures des barillets correspondent exactement aux intervalles A, B, il est clair qu'au bout d'une révolution du barillet XIV, XV, le zéro correspondra encore à l'intervalle B; mais que ce sera le chiffre 1 du barillet 14, 15, qui correspondra dans le même temps à l'intervalle A.

Donc si l'on écrit sur le barillet XIV, XV, les deux rangées suivantes, comme on les voit,

0, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10,
19, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

9, 8, 7, 6, 5, 4, 3, 2, 1,
10, 11, 12, 13, 14, 15, 16, 17, 18.

et sur le barillet 14, 15, les deux rangées suivantes, comme on les voit,

0, 9, 8, 7, 6, 5, 4, 3, 2, 1,
9, 0, 1, 2, 3, 4, 5, 6, 7, 8.

et que les zéros des deux rangées inférieures des barillets correspondent en même-temps aux intervalles A, B, il est clair que dans ce cas, de même que dans le premier, lorsque le zéro du barillet XIV, XV correspondra, après avoir fait un tour, à l'intervalle B, le barillet 14, 15 présentera à l'ouverture ou espace A le chiffre 1.

Il en fera toujours ainsi, quelles que soient les rangées de chiffres que l'on trace sur le barillet XIV, XV, et sur le barillet 14, 15. Dans le premier cas, le barillet XIV, XV tournera sur lui-même, et présentera les douze caractères à l'intervalle B, quand le barillet 14, 15 n'ayant tourné que d'un vingtième, présentera à l'intervalle A le chiffre 1. Dans le second cas, le barillet 14, 15 tournera sur lui-même, et présentera ses vingt caractères à l'ouverture ou intervalle B, pendant que le barillet 14, 15 n'ayant tourné que d'un dixième, présentera à l'ouverture ou intervalle A le chiffre 1. Dans le troisième cas, le barillet XIV, XV tournera sur lui-même, et aura présenté ses dix caractères à l'ouverture B, quand le barillet 14, 15 n'ayant tourné que d'un dixième, présentera à l'ouverture ou intervalle A le chiffre 1.

Mais au lieu de faire toutes ces suppositions sur deux barillets, je peux les faire sur un grand nombre de barillets, tous assemblés les uns avec les autres, comme on voit ceux de la Fig. 6. Rien n'empêche de supposer à côté du barillet 14, 15 un autre barillet placé par rapport à lui, comme il est placé par rapport au barillet XIV, XV, avec les mêmes roues, un sautoir et tout le reste de l'assemblage : rien n'empêche que je ne puisse supposer douze chevilles à la roue VIII, IX, et les deux rangées 0, 11, 10, 9, &c.

11, 0, 1, 2, &c.

tracées sur le barillet XIV, XV ; vingt chevilles à la roue 8, 9, et les deux rangées 0, 19, 18, 17, 16, &c.

19, 0, 1, 2, 3, &c.

tracées sur le barillet 14, 15 ; dix chevilles à la première, pareille à la roue 8, 9, et les deux rangées 0, 9, 8, 7, 6, &c. sur le troisième barillet ; dix

9, 0, 1, 2, 3, &c.

chevilles

chevilles à la seconde, pareille de 8, 9, et les deux rangées 0, 9, 8, 7, 6, &c. sur le quatrième barillet ; dix 9, 0, 1, 2, 3, &c.

chevilles à la troisième, pareille de 8, 9, et les deux rangées 0, 9, 8, 7, 6, &c. sur le cinquième barillet, 9, 0, 1, 2, 3, &c. et ainsi de suite.

Rien n'empêche non plus de supposer que tandis que le premier barillet présentera ses douze chiffres à son ouverture, le second ne présentera plus que le chiffre 1 à la sienne ; que tandis que le second barillet présentera ses vingt chiffres à son ouverture ou intervalle, le troisième ne présentera que le chiffre 1 ; que tandis que le troisième barillet présentera ses dix caractères à son ouverture, le quatrième n'y présentera que le chiffre 1 ; que tandis que le quatrième barillet présentera ses dix caractères à son ouverture, le cinquième ne présentera à la sienne que le chiffre 1, et ainsi de suite.

D'où il s'ensuivra, 1°. qu'il n'y aura aucun nombre qu'on ne puisse écrire avec ces barillets ; car après les deux échappements, chaque équipage de barillet demeure isolé, est indépendant de celui qui le précède du côté de la droite, peut tourner sur lui-même tant qu'on voudra dans la direction VIII, VIII, IX, IX, et par conséquent offrir à son ouverture celui des chiffres de sa rangée inférieure qu'on jugera à propos : mais les intervalles A, B sont aux cylindres nus XIV, XV, 14, 15, ce que leur font les ouvertures de la ligne Y, Z, (Fig. 1.) quand ils sont couverts de la plaque NORP.

2°. Que le premier barillet marquera des deniers, le second des sous, le troisième des unités de livres, le quatrième des dizaines, le cinquième des centaines, &c.

3°. Qu'il faut un tour du premier barillet pour un vingtième du second ; un tour du second pour un dixième du troisième ; un tour du troisième pour un dixième du quatrième ; et que par conséquent les barillets suivent entre leurs mouvements la proportion qui règne entre les chiffres de l'Arithmétique, quand ils expriment des nombres ; que la proportion des chiffres est toujours gardée dans les mouvements des barillets, quelle que soit la quantité de tours qu'on fasse faire au premier, ou au second, ou au troisième, et que par conséquent de même qu'on fait les opérations de l'Arithmétique avec des chiffres, on peut les faire avec les barillets et les rangées de chiffres qu'ils ont.

4°. Que pour cet effet, il faut commencer par mettre tous les barillets de manière que les zéros de leur rangée inférieure correspondent en même-temps aux ouvertures de la bande Y, Z et de la plaque NORP ; car si tandis que le premier barillet, par exemple, présente 0 à son ouverture, le second présente 4 à la sienne, il est à présumer que le premier barillet a fait déjà quatre tours ; ce qui n'est pas vrai.

5°. Qu'il est assez indifférent de faire tourner les barillets dans la direction VIII, VIII, IX ; que ce mouvement ne dérange rien à l'effet de la Machine ;
mais

mais qu'il ne faut pas qu'ils aient la liberté de rétrograder ; et c'est aussi la fonction du cliquet supérieur C de la leur ôier.

Il permet, comme on voit, aux roues de tourner dans le sens VIII, VIII, IX ; mais il les empêche de tourner dans le sens contraire.

6°. Que les roues ne pouvant tourner que dans la direction VIII, VIII, IX, c'est de la ligne ou rangée de chiffres inférieure des barillets, qu'il faut se servir pour écrire un nombre ; par conséquent pour faire l'addition ; par conséquent encore pour faire la multiplication ; et que comme les chiffres des rangées sont dans un ordre renversé, la soustraction doit se faire sous la rangée supérieure, et par conséquent aussi la division.

Tous ces corollaires s'éclairciront davantage par l'usage de la Machine, et la manière de faire les opérations.

Mais avant que de passer aux opérations, nous ferons observer encore une fois que chaque roue 6, 7 (Fig. 6.) a sa correspondante 4, 5 (Fig. 2.) et chaque roue 4, 5 son cercle mobile Q ; que chaque roue 8, 9, a son cliquet supérieur et son cliquet inférieur ; que ces deux cliquets ont une de leurs fonctions commune ; c'est d'empêcher les roues VIII, IX, 8, 9, &c., de rétrograder ; enfin que le talon 1, pratriqué au cliquet inférieur, lui est essentiel.

USAGE DE LA MACHINE ARITHMÉTIQUE POUR L'ADDITION.

COMMENCEZ par couvrir de la bande P, R la rangée supérieure d'ouvertures, en sorte que cette bande soit dans l'état où vous la voyez (Fig. 1.) ; mettez ensuite toutes les roues de la bande inférieure ou rangée à zéro ; et soient les sommes à ajouter,

69	7	8
584	15	6
342	12	9

Prenez le conducteur ; portez sa pointe dans la huitième denture du cercle Q, le plus à la droite ; faites tourner ce cercle jusqu'à ce que l'arrêt ou la potence S vous empêche d'avancer.

Passiez à la roue des sous ou au cercle Q, qui suit immédiatement celui sur lequel vous avez opéré, en allant de la droite à la gauche ; portez la pointe du conducteur dans la septième denture, à compter depuis la potence ; faites tourner ce cercle jusqu'à ce que la potence S vous arrête ; passez aux livres, aux dizaines, et faites la même opération sur leurs cercles Q.

En vous y prenant ainsi, votre première somme sera évidemment écrite : opérez sur la seconde précisément comme vous avez fait sur la première, sans

VOUS

vous embarrasser des chiffres qui se présentent aux ouvertures ; puis sur la troisième. Après votre troisième opération, remarquez les chiffres qui paroîtront aux ouvertures de la ligne YZ : ils marqueront la somme totale de vos trois sommes partiels.

DÉMONSTRATION.

Il est évident que si vous faites tourner le cercle Q des deniers de huit parties, vous aurez 8 à l'ouverture correspondante à ce cercle : il est encore évident que si vous faites tourner le même cercle de six autres parties, comme il est divisé en douze, c'est la même chose que si vous l'aviez fait tourner de douze parties, plus 2 : mais en le faisant tourner de douze, vous auriez remis à zéro le barillet des deniers correspondant à ce cercle de deniers, puisqu'il eût fait un tour exact sur lui-même : il n'a pu faire un tour sur lui-même, que le second barillet, ou celui des sous, n'ait tourné d'un vingtième ; et par conséquent mis le chiffre 1 à l'ouverture des sous. Le chiffre des deniers n'a pu rester à 0 ; car ce n'est pas seulement de douze parties que vous l'avez fait tourner, mais de douze parties, plus deux. Vous avez donc fait en sus comme si le barillet des deniers étant à zéro, et celui des sous à 1, vous eussiez fait tourner le cercle Q des deniers de deux dentures ; mais en faisant tourner le cercle Q des deniers de deux dentures, on met le barillet des deniers à 2, où ce barillet présente 2 à son ouverture. Donc le barillet des deniers offrira 2 à son ouverture, et celui des sous 1 : mais 8 deniers et 6 deniers font 14 deniers, ou un sou, plus 2 deniers ; ce qu'il falloit en effet ajouter, et ce que la Machine a donné. La démonstration fera la même pour tout le reste de l'opération.

EXEMPLE DE SOUSTRACTION.

Commencez par baisser la bande P, R sur la ligne YZ d'ouvertures inférieures ; écrivez la plus grande somme sur les ouvertures de la ligne supérieure, comme nous l'avons prescrit pour l'addition, par le moyen du conducteur ; faites l'addition de la somme à soustraire, ou de la plus petite avec la plus grande, comme nous l'avons prescrit à l'exemple de l'addition ; cette addition faite, la soustraction le sera aussi. Les chiffres qui paroîtront aux ouvertures, marqueront la différence des deux sommes, ou l'excès de la grande sur la petite ; ce que l'on cherchoit.

	£.	s.	d.
Soit	9	12	1
Dont il faut soustraire	8	9	8
	1	3	3

Si vous exécutez ce que nous avons prescrit, }
vous trouverez aux ouvertures } 13¹ 12 3

DÉMON-

D É M O N S T R A T I O N .

Quand j'écris le nombre 9121 l. 9 s. 2 d. : pour faire paroître 2 à l'ouverture des deniers, je suis obligé de faire passer avec le directeur onze dentures du cercle Q des deniers ; car il y a à la rangée supérieure du barillet des deniers onze termes depuis 0 jusqu'à 2 : si à ce 2 j'ajoute encore 11, je tomberai sur 33, car il faut encore que je fasse faire onze dentures au cercle Q ; or comptant 11, depuis 2, on tombe sur 3. La démonstration est la même pour le reste. Mais remarquez que le barillet des deniers n'a pu tourner de 22, sans que le barillet des sous n'ait tourné d'un vingtième ou de douze deniers. Et comme à la rangée d'en-haut les chiffres vont en rétrogradant dans le sens que les barillets tournent, à chaque tour du barillet des deniers, les chiffres du barillet des sous diminuent d'une unité : c'est-à-dire, que l'emprunt que l'on fait pour un barillet est acquitté sur l'autre, ou que la soustraction s'exécute comme à l'ordinaire.

E X E M P L E D E M U L T I P L I C A T I O N .

Revenez aux ouvertures inférieures ; faites remonter la bande P, R sur les ouvertures supérieures ; mettez toutes les roues à zéro, par le moyen du conducteur, comme nous avons dit plus haut. Ou le multiplicateur n'a qu'un caractère, ou il en a plusieurs ; s'il n'a qu'un caractère, on écrit, comme pour l'addition, autant de fois le multiplicande qu'il y a d'unités dans ce chiffre du multiplicateur : ainsi la somme de 1245 l. étant à multiplier par 3, j'écris ou pose trois fois cette somme à l'aide de mes roues et des cercles Q ; après la dernière fois, il paroît aux ouvertures 3735 l., qui est en effet le produit de 1245 l. par 3.

Si le multiplicateur a plusieurs caractères, il faut multiplier tous les chiffres du multiplicande par chacun de ceux du multiplicateur, les écrire de la même manière que pour l'addition ; mais il faut observer au second multiplicateur de prendre pour première roue celle des dixaines.

La multiplication n'étant qu'une espèce d'addition, et cette règle se faisant évidemment ici par voie d'addition, l'opération n'a pas besoin de démonstration.

E X E M P L E

EXEMPLE DE DIVISION.

POUR faire la division, il faut se servir des ouvertures supérieures : faites donc descendre la bande P, R sur les inférieures ; mettez à zéro toutes les roues fixées sur cette bande, et qu'on appelle *roues de quotient* ; faites paroître aux ouvertures votre nombre à diviser, et opérez comme nous allons dire. Soit la somme de 65 à diviser par cinq ; vous dites, en fix, cinq y est, et vous ferez tourner votre roue comme si vous vouliez additionner 5 et 6 ; cela fait, les chiffres des roues supérieures allant toujours en rétrogradant, il est évident qu'il ne paroîtra plus que 1 à l'ouverture où il paroîssoit 6 ; car dans 0, 9, 8, 7, 6, 5, 4, 3, 2, 1 ; 1 est le cinquième terme après 6.

Mais le diviseur 5 n'est plus dans 1 ; marquez donc 1 sur la roue des quotiens, qui répond à l'ouverture des dizaines ; passez ensuite à l'ouverture des unités, ôtez-en 5 autant de fois qu'il sera possible, en ajoutant 5 au caractère qui paroît à travers cette ouverture, jusqu'à ce qu'il vienne à cette ouverture, ou zéro, ou un nombre plus petit que 5, et qu'il n'y ait que des zéros aux ouvertures qui précèdent : à chaque addition, faites passer l'aiguille de la roue des quotiens qui est au-dessous de l'ouverture des unités, du chiffre 1 sur le chiffre 2, sur le chiffre 3, en un mot sur un chiffre qui ait autant d'unités que vous ferez de soustractions ; ici après avoir ôté trois fois 5 du chiffre qui paroîssoit à l'ouverture des unités, il est venu zéro ; donc 5 est treize fois en 65.

Il faut observer qu'en ôtant ici une fois 5 du chiffre qui paroît aux unités, il vient tout de suite 0 à cette ouverture ; mais que pour cela l'opération n'est pas achevée, parce qu'il reste une unité à l'ouverture des dizaines, qui fait avec le zéro qui suit, 10, qu'il faut épuiser ; or il est évident que 5 ôté deux fois de 10, il ne restera plus rien ; c'est-à-dire, que pour exhaustion totale, ou que pour avoir zéro à toutes les ouvertures, il faut encore soustraire 5 deux fois.

Il ne faut pas oublier que la soustraction se fait exactement comme l'addition, et que la seule différence qu'il y ait, c'est que l'une se fait sur les nombres d'en-bas, et l'autre sur les nombres d'en-haut.

Mais, si le diviseur a plusieurs caractères, voici comment on opérera. Soit 9989 à diviser par 124, on ôtera 1 de 9, chiffre qui paroît à l'ouverture des mille ; 2 du chiffre qui paroît à l'ouverture des centaines ; 4 du chiffre qui paroît à l'ouverture des dizaines ; et l'on mettra l'aiguille des cercles de quotient, qui répond à l'ouverture des dizaines, sur le chiffre 1. Si le diviseur 124 peut s'ôter encore une fois de ce qui paroît, après la première soustraction, aux ouvertures des mille, des centaines et des dizaines, on l'ôtera, et on tournera l'aiguille du même cercle de quotient sur 2, et on continuera jusqu'à l'exhaustion la plus complète qu'il sera possible : pour cet effet, il faudra réitérer ici la soustraction huit fois sur les trois mêmes ouvertures ; l'aiguille du cercle du quotient qui répond aux dizaines, fera donc sur 8, et il ne se trouvera plus aux ouvertures que 69, qui ne peut plus se diviser par 124 ; on mettra donc l'aiguille du cercle de quotient, qui répond à l'ouverture des unités, sur 9 ; ce qui marquera que 124 ôté 80 fois de 9989, il reste ensuite 69.

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MANIERE

MANIERE DE REDUIRE LES LIVRES EN SOUS, ET LES SOUS EN DENIERS.

REDUIRE les livres en sous, c'est multiplier par 20 les livres données ; et réduire les sous en deniers, c'est multiplier par 12. Voyez MULTIPLICATION.

Convertir les sous en livres et les deniers en sous, c'est diviser dans le premier cas par 20, et dans le second par 12. Voyez DIVISION.

Convertir les deniers en livres, c'est diviser par 240. Voyez DIVISION.

IL parut en 1725 une autre Machine *Arithmétique* d'une composition plus simple que celle de M. Pascal, et que celles qu'on avoit déjà faites à l'imitation ; elle est de M. de l'Epine ; et l'Académie des Sciences a jugé qu'elle contenoit plusieurs choses nouvelles et ingénieusement pensées. On la trouvera dans le Recueil des Machines approuvées par cette Académie ; on y en verra encore une autre de M. de Boitiffendeau, dont l'Académie fait aussi l'éloge. Le principe de ces Machines une fois connu, il y a peu de mérite à les varier : mais il falloit trouver ce principe ; il falloit s'appercevoir que, si l'on fait tourner verticalement de droite à gauche un barillet chargé de deux suites de nombres placées l'une au-dessus de l'autre en cette sorte, 0, 9, 8, 7, 6, &c.
9, 0, 1, 2, 3, &c.

l'addition se faisoit sur la rangée supérieure, et la soustraction sur l'inférieure, précisément de la même manière.

Fin de la Description de la Machine Arithmétique de Monsieur Blaise Pascal.

LETTRE

L E T T R E

*De MM. PASCAL et ROBERVAL à M. FERMAT, sur un Principe de Géostatique, mis en avant par ce dernier *.*

MONSIEUR,

LE principe que vous demandez pour la Géostatique est, que si deux poids égaux sont joints par une ligne droite ferme et de soi sans poids, et qu'étant ainsi disposés, ils puissent descendre librement, ils ne reposeront jamais, jusqu'à ce que le milieu de la ligne (qui est le centre de pesanteur des anciens) s'unisse au centre commun des choses pesantes. Ce principe, (lequel nous avons considéré il y a long-temps, ainsi qu'il vous a été mandé,) paroît d'abord fort plausible : mais, quand il est question de principe, vous savez quelles conditions lui sont requises pour être reçu ; desquelles conditions, au principe dont il s'agit, la principale manque ; savoir, que nous ignorons quelle est la cause radicale qui fait que les corps pesants descendent, et quelle est l'origine de leur pesanteur. Ce qui n'étant point en notre connoissance (comme il faut librement avouer, et en ceci, et quasi en toutes les autres choses physiques) il est évident qu'il nous est impossible de déterminer ce qui arriveroit au centre, où les choses pesantes aspirent, ni aux autres lieux hors la surface de la terre ; sur laquelle, parce que nous y habitons, nous avons quelques expériences assez constantes, desquelles nous tirons ces principes en vertu desquels nous raisonnons en la Mécanique.

La diversité des opinions touchant l'origine de la pesanteur des corps, desquelles aucune n'a été jusqu'ici, ni démontrée, ni convaincue de fausseté par démonstration, est un ample témoignage de l'ignorance humaine en ce point.

Diverses opinions sur l'origine de la pesanteur des corps.

La commune opinion est, que la pesanteur est une qualité qui réside dans le corps même qui tombe. D'autres sont d'avis que la descente des corps procède de l'attraction d'un autre corps qui attire celui qui descend, comme le globe de la terre paroît attirer une pierre qui tombe. Il y a une troisième opinion qui n'est pas hors de vraisemblance ; que c'est une attraction mutuelle entre les corps, causée par un desir naturel que ces corps ont de s'unir ensemble, comme il est évident au fer et à l'aimant, lesquels sont tels, que, si l'aimant est arrêté, le fer, ne l'étant pas, ira le trouver ; et, si le fer est arrêté, l'aimant ira vers lui ;

* Tirée du Recueil des Œuvres de Fermat.

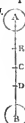
et, si tous deux sont libres, ils s'approcheront réciproquement l'un de l'autre ; en sorte toutefois que le plus fort des deux fera le moins de chemin *.

Or de ces trois causes possibles de la pesanteur ou des centres des corps, les conséquences sont fort différentes, particulièrement de la première et des deux autres ; comme nous ferons voir en les examinant.

Conséquences
de la première
opinion.

Car, si la première est vraie, le sens commun nous dicte qu'en quelque lieu que soit un corps pesant, près ou loin du centre de la terre, il pèsera toujours également, ayant toujours en soi la même qualité qui le fait peser, et en même degré. Le sens commun nous dicte aussi (posée cette même opinion première) qu'alors un corps reposera au centre commun des choses pesantes, quand les parties du corps qui seront de part et d'autre du même centre, seront d'égale pesanteur, pour contrepeser l'une à l'autre, sans considérer si elles sont peu ou beaucoup, également ou inégalement, éloignées du centre commun.

Fig. 1.



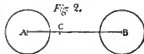
Si cette première est véritable, nous ne voyons point que le principe que vous demandez pour la Géostatique puisse subsister. Car soient (Fig. 1.) deux poids égaux A, B joints ensemble par la ligne droite ferme et de soi sans poids AB ; soit C le point du milieu de la même ligne AB ; et soient D, E, deux autres points tels quels dans ladite ligne entre les poids A et B. Vous demandez qu'on vous accorde que les poids A, B tombant librement avec leur ligne, ne reposeront point jusqu'à ce que le point du milieu C s'unisse au centre commun des choses pesantes. Suivant cette première opinion, nous accordons que, si le point C est uni au centre des choses pesantes, le composé des poids A, B demeurera immobile véritablement. Mais il nous semble aussi que, si le point D ou E convient avec le même centre commun des choses pesantes, combien que l'un des poids en soit plus proche que l'autre, ils contrepeseront encore et demeureront en équilibre : puisque (pour nous servir de vos propres termes) ces deux poids sont égaux, et ont tous deux même inclination de s'unir au même centre commun des choses pesantes, et l'un n'a aucun avantage sur l'autre pour le déplacer de son lieu. Et il ne sert de rien d'alléguer le centre de pesanteur du corps AB, lequel centre, selon les Anciens, est au milieu C ; car il n'a pas été démontré que le point C soit le centre de pesanteur du composé AB, sinon lorsque la descente des corps se fait naturellement par des lignes parallèles, ce qui est contre vos suppositions et les nôtres, et contre la vérité : et même nous ne voyons pas qu'aucun corps, hormis la sphère, ait un centre de pesanteur, posée la définition de ce centre selon Pappus et les autres Auteurs ; et quand il y en auroit un en chaque corps, il ne paroît pas (et n'a jamais été démontré) que ce seroit ce point-là par lequel le corps s'uniroit au centre des choses pesantes : même cela, pour les raisons précédentes, répugne à notre commune connoissance en plusieurs figures, comme en la seconde des deux figures suivantes. En tout cas, nous ne voyons point que ce centre de pesanteur des Anciens doive être considéré autre part qu'aux poids qui sont pendus ou soutenus hors du lieu auquel ils aspirent.

Du centre de
pesanteur
d'un corps.

* Cette troisième opinion est celle du Chevalier Isaac Newton et de la plupart des Philosophes du temps présent, 1297. F. M.

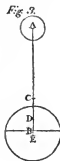
Quant

Quant à la comparaison qui vous a été faite d'un levier horizontal, lequel étant pressé horizontalement aux deux bouts par deux forces ou puissances égales, demeure en l'état qu'il est : elle vous semble entièrement pareille au levier précédent A B (puisque vous voulez l'appeler ainsi) d'autant que ces poids ne pressent le levier que par la force ou puissance qu'ils ont de se porter vers leur centre commun. Comme si le levier horizontal est A B (Fig. 2.) et les forces ou puissances égales A et B pressant horizontalement le levier pour le porter à un certain point commun C, auquel elles aspirent, et lequel est posé également ou inégalement entre les mêmes puissances dans la ligne A B : ces forces pressant également le levier, se résisteront l'une à l'autre, selon notre sens ; encore même que l'une comme A, fût plus proche que l'autre du point commun auquel toutes deux aspirent. Et quand le levier ne seroit pas horizontal, mais en telle autre position que l'on voudra, étant considéré de foi sans poids, et toutes les autres choses comme auparavant, le même effet s'ensuivra, selon notre jugement.



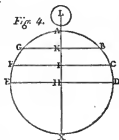
Nous ajouterons ici ce que nous pensons, suivant cette première opinion, de deux poids qui seroient inégaux, joints comme dessus à une ligne droite ferme et de foi sans poids.

Soient donc (Fig. 3.) deux poids inégaux A et B, desquels A soit le moindre ; et soit A B la ligne ferme qui les joint, dans laquelle le point C soit le centre de pesanté du composé des corps A, B, selon les Anciens : ce point C ne sera pas au milieu de la ligne A B. Si donc on met le composé des poids A, B, de sorte que le point C convienne au centre commun des choses pesantes, nous ne pouvons croire que ce composé demeurera en cet état, le poids A étant entièrement d'une part du centre des choses pesantes, et le poids B entièrement de l'autre part. Mais il nous semble que le plus grand poids B doit s'approcher du même centre des choses pesantes, jusqu'à ce qu'une partie dudit poids B soit au-delà dudit centre vers A comme la partie D, en sorte que cette partie D avec tout le poids A étant d'une même part, soit de même pesanté que la partie E restant de l'autre part.



Si la seconde opinion touchant la cause de la descente des poids est véritable, voici les conséquences qu'on peut en tirer, selon notre jugement.

Soit (Fig. 4.) le corps attirant A D X E sphérique duquel le centre soit H ; et que la vertu d'attraction soit également épandue par toutes les parties du même corps, en sorte que chacune selon sa puissance, tire à foi le corps attiré, ainsi que supposent les Auteurs de cette opinion.



Conséquences de la seconde opinion.

Sur cette position, le sens commun nous dicte que
les

les distances et autres conditions étant pareilles, les parties égales du corps attirant attireront également, et les inégales, inégalement.

Soit donc le corps attiré *L* considéré, premièrement, hors le corps attirant en *A* ; soit menée la ligne droite *AH*, à laquelle soit un plan perpendiculaire *EHD*, coupant le corps *ADXE* en deux parties égales, et partant d'égale vertu. Soient aussi dans la ligne *AH* pris tant de points que l'on voudra, comme *K*, *I*, par lesquels soient menés des plans *FIC*, *GKB* parallèles au plan *EHD*, coupant le corps attirant *ADXE* en parties inégales, et partant d'inégale vertu. Alors le corps *L* étant en *A*, sera attiré vers *H* par la vertu entière de tout le corps *ADXE* ; et le chemin étant libre, il viendra en *K*, où étant, il sera attiré vers *H* par la plus grande et forte partie *BDXEG*, et contre-attiré vers *A* par la plus petite et plus foible partie *BAG*. Il en fera de même quand il sera parvenu en *I*, où il sera moins attiré que quand il étoit en *K* ou en *A* ; toutefois il sera toujours contraint de s'approcher du centre *H*, tant qu'il y soit venu : mais la partie qui attire diminuant toujours, et celle qui contre-attire s'augmentant toujours, il sera continuellement attiré avec moins de vertu, jusqu'à ce qu'étant arrivé en *H*, il sera également attiré de toutes parts, et demeurera en cet état.

Si cette proposition est vraie, il est facile de voir que le corps *L* pesera d'autant moins, qu'il sera plus proche du centre *H* ; mais cette diminution ne sera pas en la raison des lignes *HA*, *HK*, *HI*, ce que vous connoîtrez en le considérant sans autre explication.

Si la troisième opinion de la descente des corps est véritable, les conclusions que l'on peut en tirer sont les mêmes, ou fort approchant de celles que nous avons tirées de la seconde opinion.

Conséquences
de la troisième
opinion.

Puis donc que de ces trois causes possibles de la pesanteur nous ne savons quelle est la vraie, et que même nous ne sommes pas assurés que ce soit l'une d'elles, pouvant se faire que la vraie cause soit composée des deux autres, ou que c'en soit une toute autre, de laquelle on tireroit des conséquences toutes différentes, il nous semble que nous ne pouvons poser d'autres principes pour raisonner en cette matière, que ceux desquels l'expérience, assistée d'un bon jugement, nous a rendus certains.

Pour ces considérations, dans nos conférences de Mécanique, nous appelons des *poids égaux* ou *inégaux*, ceux qui ont égale ou inégale puissance de se porter vers le centre commun des choses pesantes ; et nous entendons un même corps avoir un même poids, quand il a toujours cette même puissance : que si cette puissance augmente ou diminue, alors, quoique ce soit le même corps, nous ne le considérons plus comme le même poids. Or que cela arrive ou non aux corps qui s'éloignent ou s'approchent du centre commun des choses pesantes, c'est chose que nous désirerions bien de savoir : mais ne trouvant rien qui nous satisfasse sur ce sujet, nous laissons cette question indécidée, raisonnant seulement sur ce que les Anciens et nous avons pu découvrir de vrai jusqu'à maintenant.

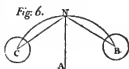
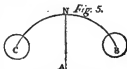
Ce qu'il faut
entendre par
poids égaux,
ou inégaux.

Voilà

Voilà ce que nous avons à vous dire pour le présent touchant votre principe de la Géostatique : laissant à part beaucoup d'autres doutes, pour éviter la proximité du discours.

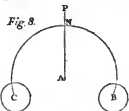
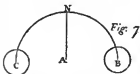
Quant à la nouvelle proportion des angles que vous mettez en avant ; afin de la démontrer, vous supposez deux principes, desquels le premier est vrai : mais le second est si éloigné d'être vrai, qu'il y a des cas où il arrive tout le contraire de ce que vous demandez qu'on vous accorde pour vrai.

Le premier est tel. Soit (Fig. 5.) A le centre commun des choses pesantes ; l'appui du levier, N ; du centre A intervalle AN, soit décrite une portion de circonférence telle quelle CNB, pourvu que l'arc CN soit égal à l'arc NB ; et soit considérée la circonférence CNB, comme une balance ou un levier de soi sans poids, qui se remue librement à l'entour de l'appui N ; soient aussi des poids égaux posés en C et B. Vous supposez que ces poids seront équilibre étant balancés sur le point N. Et il semble que tacitement vous supposez encore l'équilibre quand les bras du levier NC et NB seroient des lignes droites (Fig. 6.), pourvu que les extrémités C et B soient également éloignées du centre A, et les lignes NC et NB, soutendantes, ou cordes, en effet ou en puissance, d'arcs égaux NC, NB.



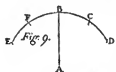
Toutes ces choses sont vraies en général ; mais nous ne les croyons telles que pour les avoir démontrées par des principes qui nous sont plus clairs et plus connus.

Toutefois en particulier il y a une distinction à faire, laquelle est de grande considération ; savoir, que quand les arcs NC et NB sont chacun moindres qu'un quart de circonférence, le levier CNB, chargé des poids C et B, pèse sur l'appui N, poussant vers le centre A pour s'en approcher. Mais quand les arcs CN, NB sont chacun un quart de circonférence (Fig. 7.), le levier CNB, chargé des poids C, B, ne pèse nullement sur l'appui N, d'autant que les poids sont diamétralement opposés ; et partant le levier demeurera de même sans appui qu'avec un appui. Finalement quand les arcs égaux NC, NB sont chacun plus grands qu'un quart de circonférence (Fig. 8.), le levier CNB, chargé des poids égaux C, B, pèse sur l'appui N poussant vers P, pour s'éloigner du centre A.



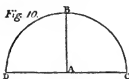
Cette distinction étant vraie comme elle est, votre second principe ne peut subsister ; ce qui paraîtra assez par l'examen d'icelui.
Votre

Votre second principe est tel. Soient A le centre commun des choses pesantes ; la balance ou le levier, EFBCD (Fig. 9.), dont l'appui est B. Soit posé un poids comme B, tout entier au point B pesant de toute sa puissance sur l'appui B. Ou bien soit divisé le poids B en parties égales E, F, B, C, D, lesquelles soient posées sur le levier aux points E, F, B, C, D, étant les arcs EF, FB, BC, CD égaux, et tout l'arc EFBCD décrit alentour du centre A. Vous supposez que le poids B mis tout entier au point B, pesera de même sur l'appui B, qu'étant posé, par parties égales, aux points E, F, B, C, D. Cela est tellement éloigné du vrai, que quelquefois le poids B, ainsi posé par parties sur le levier, ne pesera plus du tout sur l'appui B, quelquefois au lieu de peser sur l'appui B pour tirer le levier vers A, il pesera tout au contraire sur le même appui B, pour éloigner le levier de A. Et toutesfois étant ramassé tout entier au point B, il pesera toujours de toute sa force sur l'appui B, pour emporter le levier vers A. Et généralement étant divisé et étendu, il pesera toujours moins sur l'appui, qu'étant ramassé au point B. Et vous supposez qu'entier et divisé, il pèse toujours de même.



Toutes ces choses sont démontrées ensuite de nos principes, et nous vous en expliquerons les principaux cas, que vous connaîtrez véritables sans aucune démonstration.

Soit derechef A (Fig. 10.) le centre commun des choses pesantes, alentour duquel soit décrit le levier CBD qui soit de foi sans poids, prolongé tant que de besoin : et soit B le point de l'appui, auquel si un poids est posé, nous demeurons d'accord avec vous qu'il pesera de toute sa puissance sur l'appui B, lequel appui, s'il n'est assez fort, rompra, et le poids s'en ira avec son levier jusqu'au centre A. Maintenant soit divisé le poids, premièrement, en deux parties égales : et

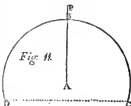


ayant pris les arcs BC et CD chacun d'un quart de circonférence, soit posée une moitié du poids en D, l'autre en C ; alors ces deux poids C et D pesant vers A, ne feront point d'autre effet sur le levier CBD, sinon qu'ils le presseront également par les deux extrémités C et D pour le courber. Supposant donc qu'il est assez roide pour ne pas plier, ils demeureront sur le levier de même que s'ils étoient attachés aux bouts du diamètre DAC, sans qu'il soit besoin de l'appui B, sur lequel le levier chargé de ces deux poids ne fait aucun effort : et quand cet appui sera ôté, le tout demeurera de même qu'avec l'appui : ce qui est assez clair.

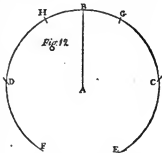
Que si le poids est divisé en plus de deux parties égales, et qu'étant étendu sur des portions égales du levier, deux d'icelles parties se rencontrent aux points C, D, et les autres dans l'espace CBD, alors celles qui seront en C et D ne chargeront point l'appui B. Quant aux autres, elles le chargeront, mais d'autant moins, que plus elles approcheront des points C, D, auxquels finit la charge.

charge. Ainsi il s'en faudra beaucoup que toutes ensemble étendues, chargent autant l'appui que lorsqu'elles sont ramassées en B : elles ne pèsent donc pas de même.

Davantage soient pris les arcs égaux BC et BD (Fig. 11.) chacun plus grand qu'un quart de circonférence, et soit imaginée la ligne droite CD ; puis étant divisé le poids en deux parties égales seulement, soient attachées l'une en C, et l'autre en D : alors il est clair que le levier chargé des poids C, D, pesera sur l'appui B ; mais ce sera tout au contraire, que si les deux poids étoient ramassés en B : car si l'appui n'est pas assez fort, il rompra, et les poids emportant le levier, que nous supposons être de soi sans poids, ne cesseront de se mouvoir tant que la ligne droite CD soit venue au point A, le levier étant monté en partie au-dessus de B vers P, au lieu de s'abaisser vers A, comme il arriveroit si les poids, étant ramassés en B, avoient rompu l'appui. Voyez quelle différence !

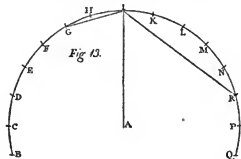


Enfin soit le levier comme auparavant, auquel soient des quarts de circonférence BC, BD, (Fig. 12.) ; et de part et d'autre du point C, soient pris des arcs égaux CG, CE, chacun moindre qu'un quart. De même de part et d'autre du point D soient pris les arcs égaux entre eux et aux précédents DH, DF, tous commensurables au quart. Soit aussi divisé tout l'arc EBF en tant de parties égales que l'on voudra, en sorte que les points E, C, G, B, H, D, F soient du nombre de ceux qui font la division ; et soit divisé le poids en autant de parties égales que l'arc EBF, lesquelles parties de poids soient posées sur les parties de la division du levier. Alors les poids qui se trouveront posés sur les arcs EC et FD, déchargeront autant l'appui B, qu'il étoit chargé par ceux des arcs CG, DH : partant tous ceux qui seront sur les arcs EG et FH ne chargeront point l'appui B, lequel, par ce moyen, ne sera chargé que par ceux qui seront sur l'arc GBH ; et si entre BG et BH il n'y a aucun poids (ce qui arrivera quand les arcs BG et BH ne seront chacun qu'une partie de la susdite division du levier) alors l'appui B sera entièrement déchargé. Voyez donc combien il y a de différence entre les poids ramassés en B, et étendus par parties sur le levier EBF ; voyez aussi qu'un même poids, divisé par parties et étendu sur le levier, pèse d'autant moins sur l'appui B, que plus grande est la portion qu'il occupe de la circonférence décrite alentour du point A, centre commun des choses pesantes.



Cette dernière considération pourroit bien être cause qu'un même corps pèse-
 roit moins, plus proche que plus éloigné du centre commun des choses pe-
 Vol. IV. 3 P santes :

fantes : mais la proportion de ces pesanteurs ne seroit nullement pareille à celle des distances, et seroit peut-être très-difficile à examiner.



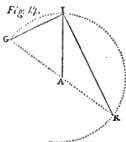
Maintenant, pour venir à votre démonstration : soit le levier $G I R$ (Fig. 13.) duquel l'appui soit I , et que les extrémités G , R et l'appui I soient également éloignés de A , centre commun des choses pesantes, alentour duquel soit imaginée la portion de circonférence $G I R$; soit fait que comme l'arc $G I$ est à l'arc $I R$, ainsi le poids R soit au poids G . Vous dites que le levier chargé des poids G , R , demeurera en équilibre sur son appui I . Quant à la démonstration, vous supposez qu'elle est facile en conséquence de vos deux principes précédents. Et de fait, si ces principes étoient vrais, il ne resteroit aucune difficulté, et la chose pourroit se conclure ainsi. Soit faite la préparation suivant la méthode d'Archimède, en sorte que les arcs $R Q$, $R M$ soient égaux, tant entre eux qu'à l'arc $I G$; et les arcs $G B$, $G M$ égaux, tant entre eux qu'à l'arc $I R$. Et soit étendu le poids R également depuis Q jusqu'en M , et le poids G aussi également depuis M jusqu'en B ; ainsi les deux poids G , R seront également étendus sur tout l'arc $B G I M R Q$, lequel arc sera quelquefois moindre que la circonférence entière, quelquefois égal à icelle, et quelquefois plus grand. Et d'autant que les portions $I B$, $I Q$ sont égales, le levier $B G I R Q$ demeurera en équilibre, par le premier principe, sur l'appui I . Mais le poids G étendu depuis B jusqu'en M , pèse de même qu'étant ramassé au point G , par le second principe : et par le même principe, le poids R pèse de même étant étendu depuis M jusqu'en Q , qu'étant ramassé au point R . Partant puisque ces deux poids étant ramassés en G et en R , pèsent de même sur le levier qu'étant étendus, et qu'étant étendus ils sont équilibre sur le levier ; ils seront encore équilibre étant ramassés en G et en R .

En cette démonstration, tout ce qui est fondé sur le second principe, reçoit les mêmes difficultés que le principe même : et partant la conclusion ne s'enfuit point que les poids G , R fassent équilibre sur le levier $G I R$.

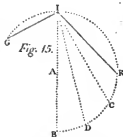
Nous pourrions nous contenter de ce que dessus, croyant que vous serez satisfait : mais nous vous prions de considérer encore deux instances, dont la première est telle.

Au

Au levier GIR (Fig. 14) soit l'angle GIR droit, et partant l'arc GIR une demi-circonférence décrite autour de A , centre commun des choses pesantes. Si l'on pèse l'arc GI , moindre que l'arc IR , par exemple, que GI soit le tiers de IR , et le poids R de 20 livres; il faudroit donc en G 60 livres, selon vous, pour faire équilibre sur le levier GIR appuyé au point I ; et toutefois si vous mettez des poids égaux en G et en R , ils seront diamétralement oppoés, et partant par le principe de la Géostatique au cas dudit principe, accordé par vous et par nous, lesdits poids égaux feront encore équilibre, comme s'ils pesoient sur les extrémités du diamètre GR vers le centre A : et quand il y a une fois équilibre, pour peu que l'on augmente ou diminue l'un des poids, l'équilibre se perd. Voyez comme cela peut s'accorder avec votre position.



La seconde instance est telle. Soit A (Fig. 15.) le centre commun des choses pesantes, alentour duquel soit la circonférence GIR , l'appui du levier I , et les bras IG , IR , desquels GI soit le moindre; et soit prolongée la ligne droite IA tant qu'elle rencontre la circonférence en B . Partant, selon vous, il faudra en G un plus grand poids qu'en R . Et, si on prend l'arc IC plus grand que IR , mettant en C le même poids qui étoit en R , il faudra en G un plus grand poids qu'auparavant, pour faire l'équilibre. De même prenant l'arc ID encore plus grand que IC , et faisant ID être le bras du levier, et mettant en D le même poids qui étoit en C , il faudra encore augmenter le poids G . Ainsi plus le bras du levier qui est en la circonférence IRB aboutira près du point B , étant chargé du même poids, plus il faudra en G un grand poids pour contrepefer. Et selon le sens commun par le raisonnement ordinaire, le bras du levier étant la ligne droite IB chargée comme dessus, il faudroit en G le plus grand poids. Et toutefois alors le poids qui seroit en B , pesant vers A , seroit tout son effort sur la roideur du bras BI , et le moindre poids qui seroit en G seroit balancer le bras IB vers D : et pour peu que le poids qui sera en G fasse balancer le bras IB avec son poids vers D (ce qui est facile à démontrer) alors, encore que tant G que B sortent hors la circonférence, on conclura quelque chose de choquant de votre position.



Enfin, Monsieur, parce que l'expérience de ce que dessus ne peut se faire par les hommes, des poids à l'égard de leur centre naturel; si vous voulez prendre la peine de la faire alentour d'un centre artificiel, supposons pour levier un petit cercle artificiel, au lieu du grand cercle naturel, et des puillances qui agissent

ou aspirent vers le centre du petit cercle, au lieu des poids qui tendent vers le centre du grand : vous trouverez que l'expérience est du tout conforme à ce raisonnement.

Si vous avez agréable de continuer nos communications sur ce sujet ou sur celui de la Géométrie, en laquelle nous savons que vous excellez entre tous ceux de ce temps, nous tâcherons à vous donner contentement : et ce que nous vous proposerons ne sera point par forme de questions ; car nous en enverrons les démonstrations en même-temps, pour en avoir votre jugement. Vous nous obligerez aussi de nous faire part de vos pensées. Nous sommes, &c.

A Paris, le 16 Août 1636.

Voyez la Réponse à cette Lettre dans les Œuvres de Fermat, avec le reste de la même discussion entre Roberval et Fermat. Pages 121, 122, &c - - 157.

CELE-

C E L E B E R R I M Æ
M A T H E S E O S
A C A D E M I Æ P A R I S I E N S I *.

HÆC vobis, doctissimi et celeberrimi viri, aut dono, aut reddo : vestra enim esse fateor quæ non, nisi inter vos educatus, mea fecissem ; propria autem agnosco quæ adeo præcellentibus Geometris indigna video. Vobis enim nonnisi magna ac egregia demonstrata placent. Paucis verò genium audax inventionis, paucioribus (ut reor) genium elegans demonstrationis, paucissimis utrumque. Silerem itaque, nihil vobis congruum habens, nisi ea benignitas quæ me à junioribus annis in erudito Lyceo sustinuit, hæc oblata qualiacunque sint, exciperet.

Horum Opusculorum primum, magnâ ex parte agit de ambitibus, seu peripheriis numerorum quadratorum, cuborum, quadrato-quadratorum, et in quocunque gradu constitutorum ; et ideò *de numericarum potestatum ambitibus* inscribitur.

Secundum circa numeros aliorum multiplices versatur, et ut ex solâ additione characterum numericorum agnoscantur, methodum tradit.

Deinceps autem, si juvat Deus, prodibunt et alii tractatus quos omninò paratos habemus, et quorum sequuntur tituli :

De numeris magico-magicis ; seu methodus ordinandi numeros omnes in quadrato numero contentos, ita ut non solum quadratus totus sit magicus ; sed, quod difficilius sanè est, ut ablati singulis ambitibus reliquum semper magicum remaneat, idque omnibus modis possibilibus, nullo omisso.

Promotus Apollonius Gallus ; id est tactiones circulares, non solum quales veteribus notæ, et à Vietâ repertæ, sed et adeò ulterius promotæ ut vix eundem patiantur titulum.

* On doit entendre par le mot *Académie*, la société des Savants qui s'assembloient dans ce temps-là librement les uns chez les autres, et non pas l'Académie des Sciences, qui ne fut fondée qu'en 1666.

Tacticones sphaericae, pari amplitudine dilatæ, quippe eâdem methodo tractatæ. Utrarumque autem methodus, singula earum problemata per plana resolvens, ex singulari conicarum sectionum proprietate oritur, quæ aliis multis difficillimis problematibus succurrit; et vix unicam adimplet paginam.

Tacticones etiã conicae: ubi ex quinque punctis et quinque rectis datis, quinque quibuscumque, &c.

Loci solidi, cum omnibus casibus et omni ex parte absolutissimi.

Loci plani: non solum illi quos à veteribus tempus abripuit, nec solum illi quos his restitutus perillustris hujus ævi Geometra subjunxit, sed et alii huc usque non noti, utrosque complectentes, et multò latius exuberantes, methodo, ut conjicere est, omninò novâ, quippe nova præstante, viâ tamen longè breviori.

Conicorum opus completum, et conica Apollonii et alia innumera unicâ ferè propositione amplectens; quod quidem nondum sex-decimum ætatis annum affectus excogitavi, et deindè in ordinem congesti.

Perspectivæ methodus, quâ nec inter inventas, nec inter inventu possibiles, ulla compendiosior esse videtur; quippe quæ puncta ichnographiæ per duarum solummodò rectarum intersectionem præbet, quo sanè nihil brevius esse potest.

Novissima autem ac penitus intentatæ materiæ tractatio, scilicet de *compositione alexæ in ludis ipsi suis*, quod gallico nostro idiomate dicitur (*faire les partis des jeux*): ubi anceps fortuna æquitate rationis ita reprimitur ut utrique lusorum quod jure competit exactè semper assignetur. Quod quidem eo fortius ratiocinando quarendum, quo minùs tentando investigari possit: ambigui enim sortis eventus fortuitæ contingentiæ potius quàm naturali necessitati meritò tribuuntur. Ideò res hæc hucusque erravit incerta; nunc autem quæ experimento rebellis fuerat, rationis dominium effugere non potuit: eam quippe tantâ securitate in artem per Geometriam reduximus, ut certitudinis ejus particeps facta, jam audaciè prodeat; et sic Matheseos demonstrationes cum alexæ incertitudine jungendo, et quæ contraria videntur conciliando, ab utràque nominationem suam accipiens stupendum hunc titulum jure sibi arrogat: *alexæ Geometria*.

Non de *Gnomoni* loquor, nec de innumeris *miscellaneis*, quæ satis in promptu habeo; verùm nec parata, nec parari digna.

De vacuo quoque subiceo, quippè brevi typis mandandum, et non solum vobis (ut ista) sed et cunctis proditurum: non tamen sine nutu vestro, quem si mereatur, nihil metuendum: quod equidem aliquandò aliàs expertus sum, maximè in instrumento illo arithmetico quod timidus inveneram, et vobis hortantibus exponens, agnovi approbationis vestræ pondus.

Illi sunt Geometriæ nostræ maturi fructus: felices et immane lucrum facturi, si hos impertiendo quosdam ex vestris repottemus.

B. P A S C A L.

Datum Parisiis, 1654.

PRE-

PREMIÈRE LETTRE DE PASCAL

À F E R M A T*.

MONSIEUR,

L'IMPATIENCE me prend aussi-bien qu'à vous ; et quoique je sois encore au lit, je ne puis m'empêcher de vous dire que je reçus hier au soir, de la part de M. de Carcavi, votre Lettre sur les partis, que j'admire si fort, que je ne puis vous le dire. Je n'ai pas le loisir de m'étendre ; mais en un mot vous avez trouvé les deux partis des dés et des parties dans la parfaite justice : j'en suis tout satisfait ; car je ne doute plus maintenant que je ne sois dans la vérité, après la rencontre admirable où je me trouve avec vous. J'admire bien davantage la méthode des parties que celle des dés : j'avois vu plusieurs personnes trouver celle des dés, comme M. le Chevalier de Meré, (qui est celui qui m'a proposé ces questions,) et aussi M. de Roberval ; mais M. de Meré n'avoit jamais pu trouver la juste valeur des parties, ni de biais pour y arriver : de sorte que je me trouvois seul qui eusse connu cette proportion. Votre méthode est très-sûre, et c'est la première qui m'est venue à la pensée dans cette recherche. Mais, parce que la peine des combinaisons est excessive, j'en ai trouvé un abrégé, et proprement une autre méthode bien plus courte et plus nette, que je voudrois pouvoir vous dire ici en peu de mots : car je voudrois désormais vous ouvrir mon cœur, s'il se pouvoit ; tant j'ai de joie de voir notre rencontre. Je vois bien que la vérité est la même à Toulouse et à Paris. Voici à peu près comme je fais pour savoir la valeur de chacune des parties, quand deux joueurs jouent, par exemple, en trois parties, et chacun a mis 32 pistoles au jeu.

Pofons que le premier en ait deux et l'autre une : ils jouent maintenant une partie dont le sort est tel, que si le premier la gagne, il gagne tout l'argent qui est au jeu, savoir, 64 pistoles : si l'autre la gagne, ils font deux parties à deux parties ; et par conséquent s'ils veulent se séparer, il faut qu'ils retirent chacun leur mise, savoir, chacun 32 pistoles. Confidérez donc, Monsieur, que, si le premier gagne, il lui appartient 64 ; s'il perd, il lui appartient 32. Donc s'ils ne veulent point hasarder cette partie, et se séparer sans la jouer, le premier doit dire : " je suis sûr d'avoir 32 pistoles ; car la perte même me les donne : mais

* Tirée du Recueil des Œuvres de Fermat. Il paroît que cette Lettre avoit été précédée par d'autres sur la même matière ; mais je n'ai pu les recouvrer.

pour

pour les 32 autres, peut-être je les aurai, peut-être vous les aurez; le hafard est égal; partageons donc ces 32 pistoles par la moitié, et donnez-moi outre cela mes 32 qui me font sûres." Il aura donc 48 pistoles, et l'autre 16.

Pofons maintenant que le premier ait deux parties, l'autre point, et qu'ils commencent à jouer une partie: le fort de cette partie est tel, que si le premier la gagne, il tire tout l'argent, 64 pistoles; si l'autre la gagne, les voilà reveus au cas précédent, auquel le premier aura deux parties et l'autre une. Or nous avons déjà montré qu'en ce cas il appartient à celui qui a les deux parties, 48 pistoles; donc s'ils veulent ne point jouer cette partie, il doit dire ainsi: "si je la gagne, je gagnerai tout, qui est 64; si je la perds, il m'appartiendra légitimement 48. Donc donnez-moi les 48 qui me font certaines, au cas même que je perde, et partageons les 16 autres par la moitié, puisqu'il y a autant de hafard que vous les gagniez comme moi." Ainsi il aura 48 et 8, qui font 56 pistoles.

Pofons enfin que le premier n'ait qu'une partie et l'autre point. Vous voyez, Monsieur, que s'ils commencent une partie nouvelle, le fort en est tel, que si le premier la gagne, il aura deux parties à point, et partant, par le cas précédent, il lui appartient 56; s'il la perd, ils sont partie à partie, donc il lui appartient 32 pistoles. Donc il doit dire: "si vous voulez ne pas la jouer, donnez-moi 32 pistoles qui me font sûres, et partageons le reste de 56 par la moitié; de 56 ôtez 32, reste 24; partagez donc 24 par la moitié, prenez en 12 et moi 12, qui, avec 32, font 44."

Or par ce moyen vous voyez par les simples soustractions, que pour la première partie il appartient sur l'argent de l'autre 12 pistoles, pour la seconde autres 12, et pour la dernière 8.

Or pour ne plus faire de mystère, puisque vous voyez aussi-bien tout à découvert, et que je n'en faisois que pour voir si je ne me trompois pas, la valeur (j'entends la valeur sur l'argent de l'autre seulement) de la dernière partie de 2 est double de la partie de 3; et quadruple de la dernière partie de 4; et octuple de la dernière partie de 5, &c.

Mais la proportion des premières parties n'est pas si aisée à trouver: elle est donc ainsi, car je ne veux rien déguiser; et voici le problème dont je faisois tant de cas, comme en effet il me plaît fort.

Etant donné tel nombre de parties qu'on voudra, trouver la valeur de la première?

Soit le nombre des parties donné, par exemple, 8: prenez les huit premiers nombres pairs et les huit premiers nombres impairs, savoir:

2, 4, 6, 8, 10, 12, 14, 16.

et 1, 3, 5, 7, 9, 11, 13, 15.

Multipliez les nombres pairs en cette sorte: le premier par le second, le produit par le troisième, le produit par le quatrième, le produit par le cinquième, &c. Multipliez les nombres impairs de la même sorte: le premier par le second,

le

le produit par le troisième, &c. : le dernier produit des pairs est le dénominateur, et le dernier produit des impairs est le numérateur de la fraction qui exprime la valeur de la première partie de 8 ; c'est-à-dire, que si on joue chacun le nombre des pistoles exprimé par le produit des pairs, il en appartiendra sur l'argent de l'autre le nombre exprimé par le produit des impairs.

Ce qui se démontre, mais avec beaucoup de peine, par les combinaisons, telles que vous les avez imaginées : je n'ai pu le démontrer par cette autre voie que je viens de vous dire, mais seulement par celle des combinaisons ; et voici les propositions qui y mènent, qui sont proprement des propositions arithmétiques touchant les combinaisons, dont j'ai d'assez belles propriétés.

Si d'un nombre quelconque de lettres, par exemple, de huit, A, B, C, D, E, F, G, H, vous prenez toutes les combinaisons possibles de quatre lettres, et ensuite toutes les combinaisons possibles de cinq lettres, et puis de six, de sept et de huit, &c. ; et qu'ainsi vous preniez toutes les combinaisons possibles depuis la multitude, qui est la moitié de la toute, jusqu'au tout : je dis que si vous joignez ensemble la moitié de la combinaison de quatre avec chacune des combinaisons supérieures, la somme sera le nombre tantième de la progression quaternaire, à commencer par le binaire, qui est la moitié de la multitude. Par exemple, et je vous le dirai en latin, car le françois n'y vaut rien.

Si quolibet litterarum, verbi gratia, octo, A, B, C, D, E, F, G, H, sumantur omnes combinationes quaternarii, quinquenarii, senarii, &c. usque ad octonarium : dico, si jungas dimidium combinationis quaternarii, nempe 35 (dimidium 70) cum omnibus combinationibus quinquenarii, nempe 56, plus omnibus combinationibus senarii, nempe 28, plus omnibus combinationibus septenarii, nempe 8, plus omnibus combinationibus octonarii, nempe 1, factum esse quartum numerum progressionis quaternarii cujus origo est 2 : dico quartum numerum, quia 4 octonarii dimidium est.*

Sunt enim numeri progressionis quaternarii quibus origo est 2, isti : 2, 8, 32, 128, 512, &c. quorum 2 primus est, 8 secundus, 32 tertius, et 128 quartus, cui 128 æquantur + 35, dimidium combinationis 4 litterarum, + 56 combinationis 5 litterarum, + 28 combinationis 6 litterarum, + 8 combinationis 7 litterarum, + 1 combinationis 8 litterarum.

Voilà la première proposition, qui est purement arithmétique.

L'autre regarde la doctrine des partis, et est telle. Il faut dire auparavant : Si on a une partie de 5, par exemple, et qu'ainsi il en manque quatre, le jeu sera infailliblement décidé en 8, qui est double de 4 : la valeur de la première partie de 5 sur l'argent de l'autre, est la fraction qui a pour numérateur la moitié de la combinaison de 4 sur 8 (je prends 4, parce qu'il est égal au nombre des parties qui manquent, et 8, parce qu'il est double de 4,) et pour dénominateur ce même numérateur, plus toutes les combinaisons supérieures.

Ainsi si j'ai une partie de 5, il m'appartient sur l'argent de mon joueur $\frac{35}{128}$, c'est-à-dire, que, s'il a mis 128 pistoles, j'en prends 35, et lui laisse le reste 93.

* Potius quinarum.

Or cette fraction $\frac{35}{128}$, est la même que celle-là $\frac{105}{384}$, laquelle est faite par la multiplication des pairs pour le dénominateur, et de la multiplication des impairs pour le numérateur.

Vous verrez bien sans doute tout cela, si vous vous en donnez tant soit peu la peine. C'est pourquoi je trouve inutile de vous en entretenir davantage : je vous envoie néanmoins une de mes vieilles Tables. Je n'ai pas le loisir de la copier, je la referai ; vous y verrez comme toujours la valeur de la première partie est égale à celle de la seconde, ce qui se trouve aisément par les combinaisons.

Vous verrez de même que les nombres de la première ligne augmentent toujours.

Ceux de la seconde, de même.

Ceux de la troisième, de même.

Mais ensuite ceux de la quatrième diminuent.

Ceux de la cinquième, &c.

Ce qui est étrange.

Je n'ai pas le temps de vous envoyer la démonstration d'une difficulté qui étonnoit fort M. de Meré : car il a très-bon esprit, mais il n'est pas Géomètre ; c'est, comme vous savez, un grand défaut ; et même il ne comprend pas qu'une ligne mathématique soit divisible à l'infini, et étoit fort bien entendre qu'elle est composée de points en nombre fini ; et jamais je n'ai pu l'en tirer : si vous pouviez le faire, on le rendroit parfait. Il me disoit donc qu'il avoit trouvé fausseté dans les nombres par cette raison.

Si on entreprend de faire un 6 avec un dé, il y a avantage de l'entreprendre en 4, comme de 671 à 625.

Si on entreprend de faire sonner avec deux dés, il y a désavantage de l'entreprendre en 24.

Et néanmoins 24 est à 36, qui est le nombre des faces de deux dés, comme 4 à 6, qui est le nombre des faces d'un dé.

Voilà quel étoit son grand scandale, qui lui faisoit dire hautement que les propositions n'étoient pas constantes, et que l'Arithmétique se démentoit. Mais vous en verrez bien aisément la raison, par les principes où vous êtes.

Je mettrai par ordre tout ce que j'en ai fait, quand j'aurai achevé des Traités Géométriques où je travaille il y a déjà quelque temps.

J'en ai fait aussi d'arithmétiques, sur le sujet desquels je vous supplie de me mander votre avis sur cette démonstration.

Je pose le Lemme que tout le monde sçait, que la somme de tant de nombres qu'on voudra de la progression continuée depuis l'unité, comme 1, 2, 3, 4, étant prise deux fois, est égale au dernier 4, multiplié par le prochainement plus grand 5, c'est-à-dire, que la somme des nombres contenus dans A, étant prise deux fois, est égale au produit de A par A + 1.

Maintenant

Maintenant je viens à ma proposition.

Duorum quorumlibet euborum proximorum differentia, unitate demptâ, sextupla est omnium numerorum in minoris radice contentorum.

Sint due radices R, S, unitate differentes : dico $R^3 - S^3 - 1$ æquari summæ numerorum in S contentorum, sexies sumptæ. Etenim S vocetur A ; ergò R est A + 1. Igitur cubus radicis R, seu A + 1, est $A^3 + 3A^2 + 3A + 1$; cubus verò S, seu A, est A^3 ; et horum differentia est $3A^2 + 3A + 1$, id est, $R^3 - S^3$. Igitur si auferatur unitas, $3A^2 + 3A$ æquatur $R^3 - S^3 - 1$. Sed duplum summæ numerorum in A, seu S, contentorum æquatur, ex Lemmate, A in A + 1, hoc est $A^2 + A$. Igitur sextuplum summæ numerorum in A contentorum æquatur $3A^2 + 3A$. Sed $3A^2 + 3A$ æquatur $R^3 - S^3 - 1$. Igitur $R^3 - S^3 - 1$ æquatur sextuplo summæ numerorum in A seu S contentorum. Quod erat demonstrandum.

On ne m'a pas fait de difficulté là-dessus ; mais on m'a dit qu'on ne m'en faisoit pas, par cette raison que tout le monde est accoutumé aujourd'hui à cette méthode : et moi je prétends que sans me faire grace, on doit admettre cette démonstration comme d'un genre excellent. J'en attends néanmoins votre avis avec toute soumission : tout ce que j'ai démontré en arithmétique, est de cette nature. Voici encore deux difficultés.

J'ai démontré une proposition plane, en me servant du cube d'une ligne, comparé au cube d'une autre. Je prétends que cela est purement géométrique, et dans la sévérité la plus grande.

De même j'ai résolu ce problème : *De quatre plans, quatre points et quatre sphères, quatre quelconques étant données, trouver une sphère qui, touchant les sphères données, passe par les points donnés ; et laisse sur les plans des portions de sphères capables d'angles donnés : et celui-ci : De trois cercles, trois points, trois lignes, trois quelconques étant donnés, trouver un cercle qui, touchant les cercles et les points, laisse sur la ligne un arc capable d'angle donné.*

J'ai résolu * ces problèmes pleinement, n'employant dans la construction que des cercles et des lignes droites. Mais dans la démonstration, je me fers des lieux solides, de paraboles ou hyperboles. Je prétends néanmoins qu'attendu que la construction est plane, ma solution est plane, et doit passer pour telle.

C'est bien mal reconnoître l'honneur que vous me faites de souffrir mes entretiens, que de vous importuner si long-temps : je ne pense jamais vous dire que deux mots, et si je ne vous dis pas ce que j'ai le plus sur le cœur, qui est, que, plus je vous connois, plus je vous admire et vous honore ; et que si vous voyiez à quel point cela est, vous donneriez une place dans votre amitié à celui qui est, Monsieur, votre, &c.

P A S C A L.

Le 29 Juillet 1654.

* Il y a apparence que toutes ces recherches sont perdues.

T A B L E

Dont il est fait mention dans la Lettre précédente.

Si on joue chacun 256, en

	6 Parties.	5 Parties.	4 Parties.	3 Parties.	2 Parties.	1 Partie.
1 Partie.	63	70	80	96	128	256
2 Partie.	63	70	80	96	128	
3 Partie.	56	60	64	64		
4 Partie.	42	40	32			
5 Partie.	24	16				
6 Partie.	8					

Il m'appartient
sur les 256
pistoles de mon
Joueur,
pour la

Si on joue 256, chacun en

	6 Parties.	5 Parties.	4 Parties.	3 Parties.	2 Parties.	1 Partie.
La 1ere Partie.	63	70	80	96	128	256
2 1eres Parties.	126	140	160	192	256	
3 1eres Parties.	182	200	224	256		
4 1eres Parties.	224	240	256			
5 1eres Parties.	248	256				
6 1eres Parties.	256					

Il m'appartient
sur les 256
de mon
Joueur,
pour les

SECONDE

SECONDE LETTRE DE PASCAL

A F E R M A T*.

MONSIEUR,

JE ne pus vous ouvrir ma pensée entière touchant les partis de plusieurs joueurs, par l'ordinaire passé; et même j'ai quelque répugnance à le faire, de peur qu'en ceci cette admirable convenance qui étoit entre nous, et qui m'étoit si chère, ne commence à se démentir; car je crains que nous ne soyons de différens avis sur ce sujet. Je veux vous ouvrir toutes mes raisons; et vous me ferez la grace de me redresser, si j'erre, ou de m'affermir, si j'ai bien rencontré. Je vous le demande tout de bon et sincèrement; car je ne me tiendrai pour certain que quand vous serez de mon côté.

Quand il n'y a que deux joueurs, votre méthode, qui procède par les combinaisons, est très-lure. Mais quand il y en a trois, je crois avoir démonstration qu'elle est mal juste, si ce n'est que vous y procédiez de quelqu'autre manière que je n'entends pas. Mais la méthode que je vous ai ouverte, et dont je me fers par-tout, est commune à toutes les conditions imaginables de routes sortes de partis, au lieu que celle des combinaisons (dont je ne me fers qu'aux rencontres particulières où elle est plus courte que la générale) n'est bonne qu'en ces seules occasions, et non pas aux autres.

Je suis sûr que je me donnerai à entendre; mais il me faudra un peu de discours, et à vous un peu de patience.

Voici comment vous procédez, quand il y a deux joueurs. Si deux joueurs jouant en plusieurs parties, se trouvent en cet état qu'il manque deux parties au premier et trois au second, pour trouver le parti il faut (dites-vous) voir en combien de parties le jeu sera décidé absolument.

Il est aisé de supputer que ce sera en quatre parties; d'où vous concluez qu'il faut voir combien quatre parties se combinent entre deux joueurs, et voir combien il y a de combinaisons pour faire gagner le premier, et combien pour le second, et partager l'argent suivant cette proportion. J'eusse eu peine à entendre ce discours-là, si je ne l'eusse su de moi-même auparavant; aussi vous

* Tirée du Recueil des Œuvres de Fermat.

l'aviez écrit dans cette pensée. Donc pour voir combien quatre parties se combinent entre deux joueurs, il faut imaginer qu'ils jouent avec un dé à deux faces (puisque'ils ne sont que deux joueurs) comme à croix et pile, et qu'ils jettent quatre de ces dés (parce qu'ils jouent en quatre parties,) et maintenant il faut voir combien ces dés peuvent avoir d'affiètes différentes : cela est aisé à supputer ; ils peuvent en avoir 16, qui est le second degré de 4, c'est-à-dire, le carré ; car figurons-nous qu'une des faces est marquée A, favorable au premier joueur, et l'autre B, favorable au second ; donc ces quatre dés peuvent s'affieter sur une de ces 16 affiètes, a a a a b b b b.

a a a a	1
a a a b	1
a a b a	1
a a b b	1
a b a a	1
a b a b	1
a b b a	1
a b b b	2
b a a a	1
b a a b	1
b a b a	1
b a b b	2
b b a a	1
b b a b	2
b b b a	2
b b b b	2

Et parce qu'il manque deux parties au premier joueur, toutes les faces qui ont deux A le font gagner ; donc il en a 11 pour lui ; et parce qu'il manque trois parties au second, toutes les faces où il y a trois B peuvent le faire gagner ; donc il y en a 5 ; donc il faut qu'ils partagent la somme, comme 11 à 5.

Voilà votre méthode quand il y a deux joueurs. Sur quoi vous dites que, s'il y en a davantage, il ne sera pas difficile de faire les partis par la même méthode.

Sur cela, Monsieur, j'ai à vous dire que ce parti pour deux joueurs, fondé sur les combinaisons, est très-juste et très-bon ; mais que, s'il y a plus de deux joueurs, il ne sera pas toujours juste : et je vous dirai la raison de cette différence.

Je communiquai votre méthode à nos Messieurs ; sur quoi M. de Roberval me fit cette objection.

Que c'est à tort que l'on prend l'art de faire le parti, sur la supposition qu'on joue en quatre parties : vû que, quand il manque deux parties à l'un et trois à l'autre, il n'est pas de nécessité que l'on joue quatre parties, pouvant arriver qu'on n'en jouera que deux ou trois, ou, à la vérité, peut-être quatre ; et ainsi qu'il ne voyoit pas pourquoi on prétendoit de faire le parti juste sur une condition feinte, qu'on jouera quatre parties : vû que la condition naturelle du jeu est qu'on ne jouera plus dès que l'un des joueurs aura gagné ; et qu'au moins, si cela n'étoit faux, cela n'étoit pas démontré. De sorte qu'il avoit quelque soupçon que nous avions fait un paralogisme.

Je lui répondis que je ne me fondois pas tant sur cette méthode des combinaisons, laquelle véritablement n'est pas en son lieu en cette occasion, comme sur mon autre méthode universelle, à qui rien n'échappe, et qui porte sa démonstration avec soi, qui trouve le même parti précisément que celle des combinaisons ; et de plus je lui démontrai la vérité du parti entre deux joueurs par les combinaisons en cette sorte.

N'est-il pas vrai que si deux joueurs, se trouvant en cet état de l'hypothèse qu'il manque 2 parties à l'un et 3 à l'autre, conviennent maintenant de gré à gré qu'on joue quatre parties complètes, c'est-à-dire, qu'on jette les quatre dés à deux

à deux faces tout à la fois : n'est-il pas vrai, dis-je, que, s'ils ont délibéré de jouer les quatre parties, le parti doit être tel que nous avons dit, suivant la multitude des assiettes favorables à chacun ?

Il en demeura d'accord, et cela, en effet, est démonstratif ; mais il nioit que la même chose subsistât, en ne s'astreignant pas à jouer les quatre parties. Je lui dis donc ainsi :

N'est-il pas clair, que les mêmes joueurs, n'étant pas astreints à jouer quatre parties, mais voulant quitter le jeu dès que l'un auroit atteint son nombre, peuvent, sans dommage, ni avantage, s'astreindre à jouer les quatre parties entières, et que cette convention ne change en aucune manière leur condition ? Car si le premier gagne les 2 premières parties de 4, et qu'ainsi il ait gagné, refusera-t-il de jouer encore deux parties, vu que, s'il les gagne, il n'a pas mieux gagné ; et s'il les perd, il n'a pas moins gagné ; car ces 2 que l'autre a gagnées, ne lui suffisent pas, puisqu'il lui en faut 3 ; et ainsi il n'y a pas assez de quatre parties pour faire qu'ils puissent tous deux atteindre le nombre qui leur manque ?

Certainement il est aisé de considérer qu'il est absolument égal et indifférent à l'un et à l'autre de jouer en la condition naturelle à leur jeu, qui est de finir dès qu'on aura son compte, ou de jouer les quatre parties entières ; donc puisque ces deux conditions sont égales et indifférentes, le parti doit être tout pareil en l'une et en l'autre. Or il est juste quand ils sont obligés de jouer quatre parties comme je l'ai montré ; donc il est juste aussi en l'autre cas.

Voilà comment je le démontrai : et, si vous y prenez garde, cette démonstration est fondée sur l'égalité des deux conditions, vraie et feinte, à l'égard de deux joueurs, et qu'en l'une et en l'autre un même gagnera toujours ; et si l'un gagne ou perd en l'une, il gagnera ou perdra en l'autre, et jamais deux n'auront leur compte.

Suivons la même pointe pour trois joueurs, et posons qu'il manque une partie au premier, qu'il en manque deux au second et deux au troisième. Pour faire le parti suivant la même méthode des combinaisons *, il faut chercher d'abord en combien de parties le jeu sera décidé, comme nous avons fait quand il y avoit deux joueurs : ce sera en 3. Car ils ne sauroient jouer trois parties sans que la décision soit arrivée nécessairement.

Il faut voir maintenant combien trois parties se combinent en trois joueurs ; combien il y en a de favorables à l'un, combien à l'autre, et combien au dernier ; et, suivant cette proportion, distribuer l'argent, de même qu'on a fait en l'hypothèse de deux joueurs.

Pour voir combien il y a de combinaisons en tout, cela est aisé ; c'est la troisième puissance de trois ; c'est-à-dire, son cube 27.

* Pascal emploie d'une manière défectueuse la méthode des combinaisons pour trois Joueurs, comme on le verra par la Réponse de Fermat.

aaa	1		
aab	1		
aac	1		
aba	1		
abb	1	2	
abc	1		
aca	1		
acb	1		
acc	1		3
baa	1		
bab	1	2	
bac	1		
bba	1	2	
bbb		2	
bbc		2	
bca	1		
bcb		2	
bcc			3
caa	1		
cab	1		
cac	1		3
cba	1		
cbb		2	
cbc			3
cca	1		3
ccb			3
ccc			3

Car si on jette trois dés à la fois (puisque'il faut jouer trois parties) qui aient chacun trois faces, puisqu'il y a trois joueurs, l'une marquée A favorable au premier, l'autre B pour le second, l'autre C pour le troisième; il est incontestable que ces trois dés jetés ensemble, peuvent s'affecier sur 27 affiettes différentes, savoir :

Or il ne manque qu'une partie au premier; donc toutes les affiettes où il y a un A sont pour lui; donc il y en a 19.

Il manque deux parties au second; donc toutes les affiettes où il y a deux B sont pour lui; donc il y en a 7.

Il manque deux parties au troisième; donc toutes les affiettes où il y a deux C sont pour lui; donc il y en a 7.

Si de-là on concluoit qu'il faudroit donner à chacun suivant la proportion de 19, 7, 7, on se tromperoit trop grossièrement, et je n'ai garde de croire que vous le fassiez ainsi: car il y a quelques faces favorables au premier et au second tout ensemble, comme ABB; car le premier y trouve un A, qu'il lui faut, et le second deux BB, qui lui manquent; ainsi ACC est pour le premier et le troisième.

Donc il ne faut pas compter ces faces qui sont communes à deux, comme valant la somme entiere à chacun, mais seulement la moitié.

Car s'il arrivoit l'affiette ACC, le premier et le troisième auroient même droit à la somme, ayant chacun leur compte; donc ils partageroient l'argent par la moitié: mais s'il arrive l'affiette ABB, le premier gagne seul; il faut donc faire la supputation ainsi.

Il y a treize affiettes qui donnent l'entier au premier, et six qui lui donnent la moitié, et huit qui ne lui valent rien.

Donc si la somme entiere est une pistole:

Il y a treize faces qui lui valent chacune 1 pistole.

Il y a six faces qui lui valent chacune $\frac{1}{2}$ pistole.

Et huit qui ne valent rien.

Donc, en cas de parti, il faut multiplier,

13	par une pistole, qui font	-	13
6	par une demi, qui font	-	3
8	par zéro, qui font	-	0

somme 27

somme 16.

Et

Et diviser la somme des valeurs 16, par la somme des affiettes 27, qui fait la fraction $\frac{16}{27}$, qui est ce qui appartient au premier en cas de parti; savoir, 16 pistoles de 27.

Le parti du second et du troisième joueur se trouvera de même.

Il y a 4 affiettes, qui lui valent 1 pistole :	multipliez	-	-	4
Il y a 3 affiettes, qui lui valent $\frac{1}{2}$ pistole :	multipliez	-	-	$1\frac{1}{2}$.
Et 20 affiettes, qui ne lui valent rien	-	-	-	0

somme 27

somme $5\frac{1}{2}$.

Donc il appartient au second joueur 5 pistoles et $\frac{1}{2}$ sur 27 et autant au troisième, et ces trois sommes $5\frac{1}{2}$, $5\frac{1}{2}$ et 16 étant jointes, font les 27.

Voilà, ce me semble, de quelle manière il faudroit faire les partis par les combinaisons suivant votre méthode, si ce n'est que vous ayez quelque autre chose sur ce sujet que je ne puis savoir. Mais, si je ne me trompe, ce parti est mal juste.

La raison en est qu'on suppose une chose fautive, qui est qu'on joue en trois parties infailliblement, au lieu que la condition naturelle de ce jeu-là est qu'on ne joue que jusqu'à ce qu'un des joueurs ait atteint le nombre de parties qui lui manque, auquel cas le jeu cesse.

Ce n'est pas qu'il ne puisse arriver qu'on joue trois parties; mais il peut arriver aussi qu'on n'en jouera qu'une ou deux, et rien de nécessité.

Mais d'où vient, dira-t-on, qu'il n'est pas permis de faire en cette rencontre, la même supposition feinte que quand il y avoit deux joueurs? En voici la raison.

Dans la condition véritable de ces trois joueurs, il n'y en a qu'un qui peut gagner : car la condition est que dès qu'on a gagné, le jeu cesse; mais, en la condition feinte, deux peuvent atteindre le nombre de leurs parties; savoir, si le premier en gagne une qui lui manque, et un des autres, deux qui lui manquent; car ils n'auront joué que trois parties : au lieu que quand il n'y avoit que deux joueurs, la condition feinte et la véritable convenoient pour l'avantage des joueurs en tour, et c'est ce qui met l'extrême différence entre la condition feinte et la véritable.

Que si les joueurs se trouvant en l'état de l'hypothèse, c'est-à-dire, s'il manque une partie au premier, deux au second, et deux au troisième, veulent maintenant, de gré à gré, et conviennent de cette condition, qu'on jouera trois parties complètes, et que ceux qui auront atteint le nombre qui leur manque, prendront la somme entière (s'ils se trouvent seuls qui l'aient atteint) ou s'il se trouve que deux l'aient atteint, qu'ils la partageront également : en ce cas le parti doit se faire comme je viens de le donner, que le premier ait 16, le second $5\frac{1}{2}$, le troisième $5\frac{1}{2}$ de 27 pistoles; et cela porte sa démonstration de soi-même, en supposant cette condition ainsi.

Mais s'ils jouent simplement à condition, non pas qu'on joue nécessairement trois parties, mais seulement jusqu'à ce que l'un d'entre eux ait atteint ses parties, et qu'alors le jeu cesse, sans donner moyen à un autre d'y arriver, alors il appartient au premier 17 pistoles, au second 5, au troisième 5, de 27.

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3 R

Et

Et cela se trouve par ma méthode générale, qui détermine aussi qu'en la condition précédente il en faut 16 au premier, $5\frac{1}{2}$ au second, et $5\frac{1}{4}$ au troisième, sans se servir des combinaisons; car elle va par-tout seule et sans obstacle.

Voilà, Monsieur, mes pensées sur ce sujet, sur lequel je n'ai d'autre avantage sur vous que celui d'y avoir beaucoup plus médité. Mais c'est peu de chose à votre égard, puisque vos premières vues sont plus pénétrantes que la longueur de mes efforts.

Je ne laisse pas de vous ouvrir mes raisons pour en attendre le jugement de vous. Je crois vous avoir fait connoître par-là que la méthode des combinaisons est bonne entre deux joueurs par accident, comme elle l'est aussi quelquefois entre trois joueurs, comme quand il manque une partie à l'un, une à l'autre et deux à l'autre; parce qu'en ce cas le nombre des parties dans lesquelles le jeu sera achevé, ne suffit pas pour en faire gagner deux; mais elle n'est pas générale, et n'est généralement bonne qu'en cas seulement qu'on soit astreint à jouer un certain nombre de parties exactement. De sorte que comme vous n'aviez pas ma méthode, quand vous m'avez proposé le parti de plusieurs joueurs, mais seulement celle des combinaisons, je crains que nous soyons de sentiments différents sur ce sujet. Je vous supplie de me mander de quelle sorte vous procédez à la recherche de ce parti. Je recevrai votre réponse avec respect et avec joie, quand même votre sentiment me seroit contraire.

Je suis, &c.

P A S C A L.

Du 24 Août 1654.

PREMIERE LETTRE DE FERMAT A PASCAL *.

MONSIEUR,

NOS coups fourrés continuent toujours; et je suis aussi-bien que vous dans l'admiration de quoi nos pensées s'ajustent si exactement, qu'il semble qu'elles aient pris une même route et fait un même chemin: vos derniers *Traité du Triangle Arithmétique* et de *son application*, en sont une preuve authentique; et si mon calcul ne me trompe, votre onzième conséquence courroit la poste de Paris à Toulouse, pendant que ma proposition des nombres figurés, qui en effet est la même, alloit de Toulouse à Paris. Je n'ai garde de faillir, tandis que je rencontrerai de cette sorte; et je suis persuadé que le vrai moyen pour s'empêcher de faillir, est celui de concourir avec vous. Mais si j'en disois davantage, la chose tiendrait du compliment, et nous avons banni cet ennemi des conversations douces et aisées.

* Cette Lettre, qui n'avoit pas encore été imprimée, paroît répondre à une Lettre de Pascal, que nous n'avons point. Nous donnons, suivant l'ordre chronologique, ce qui nous reste de cette correspondance.

Ce seroit maintenant à mon tour à vous débiter quelque une de mes inventions numériques ; mais la fin du Parlement augmente mes occupations, et j'use espérer de votre bonté que vous m'accorderez un répit juste et quasi nécessaire. Cependant je répondrai à votre question des trois joueurs qui jouent en deux parties. Lorsque le premier en a une, et que les autres n'en ont pas une, votre première solution est la vraie, et la division de l'argent doit se faire en dix-sept, cinq et cinq, de quoi la raison est manifeste et se prend toujours du même principe, les combinaisons faisant voir d'abord que le premier a pour lui dix-sept hasards égaux, lorsque chacun des autres n'en a que cinq.

Au reste, il n'est rien à l'avenir que je ne vous communique avec toute franchise. Songez cependant, si vous le trouvez à propos, à cette proposition.

Les puissances quarrées de 2, augmentées de l'unité, sont toujours des nombres premiers*.

Le quarré de 2, augmenté de l'unité, fait 5, qui est nombre premier.

Le quarré du quarré fait 16, qui augmenté de l'unité, fait 17, nombre premier.

Le quarré de 16 fait 256, qui augmenté de l'unité, fait 257, nombre premier.

Le quarré de 256 fait 65536, qui augmenté de l'unité, fait 65537, nombre premier ; et ainsi à l'infini.

C'est une propriété de la vérité de laquelle je vous répons. La démonstration en est très-mal-aisée, et je vous avoue que je n'ai pu encore la trouver pleinement ; je ne vous la proposerois pas pour la chercher, si j'en étois venu à bout.

Cette proposition sert à l'invention des nombres qui sont à leurs parties aliquotes en raison donnée, sur quoi j'ai fait des découvertes considérables. Nous en parlerons une autre fois.

Je suis, Monsieur, votre, &c.

F E R M A T.

A Toulouse, le 29 Août 1654.

SECONDE LETTRE DE FERMAT A PASCAL,

En réponse à celle de la page 485 +.

MONSIEUR,

N'APPREHENDEZ pas que notre convenance se démente ; vous l'avez confirmée vous-même en pensant la détruire, et il me semble qu'en répondant à M. de Roberval pour vous, vous avez aussi répondu pour moi. Je

* Cette proposition n'est pas vraie généralement. M. Euler a remarqué (*Ancient Mém. de l'Acad. de Pétersbourg, Tom. IV. ann. 1732 et 1733, pag. 104.*) que la trente-deuxième puissance de 2, augmentée de l'unité, c'est-à-dire, $4,294,967,197$, est divisible par 641.

† Imprimée pour la seconde fois.

prends l'exemple des trois joueurs, au premier desquels il manque une partie, et à chacun des deux autres deux, qui est le cas que vous m'opposez. Je n'y trouve que dix-sept combinaisons pour le premier, et cinq pour chacun des deux autres; car quand vous dites que la combinaison ACC est bonne pour le premier et pour le troisième, il semble que vous ne vous souveniez plus que tout ce qui se fait après que l'un des joueurs a gagné, ne sert plus de rien. Or cette combinaison ayant fait gagner le premier dès la première partie, qu'importe que le troisième en gagne deux ensuite, puisque, quand il en gagneroit trente, tout cela seroit superflu? Ce qui vient de ce que, comme vous avez très-bien remarqué, cette fiction d'étendre le jeu à un certain nombre de parties, ne sert qu'à faciliter la règle, et (suivant mon sentiment) à rendre tous les hasards égaux, ou bien, plus intelligiblement, à réduire toutes les fractions à une même dénomination. Et afin que vous n'en doutiez plus, si, au lieu de trois parties, vous étendez, au cas proposé, la feinte jusqu'à quatre, il y aura non-seulement vingt-sept combinaisons, mais quatre-vingt-une, et il faudra voir combien de combinaisons seront gagnées au premier une partie, plutôt que deux à chacun des autres, et combien seront gagnées à chacun des deux autres deux parties plutôt qu'une au premier. Vous trouverez que les combinaisons pour le gain du premier, seront 51, et celles de chacun des autres deux 15; ce qui revient à la même raison. Que si vous prenez cinq parties, ou tel autre nombre qu'il vous plaira, vous trouverez toujours trois nombres en proportion de 17, 5, 5. Et ainsi j'ai droit de dire que la combinaison ACC n'est que pour le premier et non pour le troisième, et que CCA n'est que pour le troisième et non pour le premier, et que partant ma règle des combinaisons est la même en trois joueurs qu'en deux, et généralement en tous nombres.

Vous aviez déjà pu voir par ma précédente, que je n'hésitois point à la solution véritable de la question des trois joueurs dont je vous avois envoyé les trois nombres décisifs 17, 5, 5. Mais parce que M. Roberval sera peut-être bien aise de voir une solution sans rien feindre, et qu'elle peut quelquefois produire des abrégés en beaucoup de cas, la voici en l'exemple proposé.

Le premier peut gagner, ou en une seule partie, ou en deux, ou en trois.

S'il gagne en une seule partie, il faut qu'avec un dé qui a trois faces, il rencontre la favorable du premier coup. Un seul dé produit trois hasards; ce joueur a donc pour lui $\frac{1}{3}$ des hasards, lorsqu'on ne joue qu'une partie.

Si on en joue deux, il peut gagner de deux façons, ou lorsque le second joueur gagne la première et lui la seconde, ou lorsque le troisième gagne la première et lui la seconde. Or deux dés produisent neuf hasards: ce joueur a donc pour lui $\frac{2}{9}$ des hasards lorsqu'on joue deux parties.

Si on en joue trois, il ne peut gagner que de deux façons, ou lorsque le second gagne la première, le troisième la seconde et lui la troisième; ou lorsque le troisième gagne la première, le second la seconde, et lui la troisième. Car si le second ou le troisième joueur gagnaient les deux premières, il gagneroit le jeu, et non pas le premier joueur. Or trois dés ont 27 hasards; donc ce premier joueur a $\frac{2}{27}$ de hasards lorsqu'on joue trois parties.

La somme des hasards, qui font gagner ce premier joueur, est par conséquent $\frac{1}{3}$, $\frac{2}{9}$, et $\frac{2}{27}$; ce qui fait en tout $\frac{11}{27}$.

Et

Et la règle est bonne et générale en tous les cas ; de sorte que sans recourir à la feinte, les combinaisons véritables en chaque nombre des parties portent leur solution, et font voir ce que j'ai dit au commencement, que l'extension à un certain nombre de parties, n'est autre chose que la réduction de diverses fractions à une même dénomination. Voilà en peu de mots tout le mystère, qui nous remettra sans doute en bonne intelligence, puisque nous ne cherchons l'un et l'autre que la raison et la vérité.

J'espère vous envoyer à la S. Martin un Abrégé de tout ce que j'ai inventé de considérable aux nombres. Vous me permettrez d'être concis, et de me faire entendre seulement à un homme qui comprend tout à demi-mot.

Ce que vous y trouverez de plus important, regarde la proposition que tout nombre est composé d'un, de deux, ou de trois triangles ; d'un, de deux, de trois ou de quatre carrés ; d'un, de deux, de trois, de quatre ou de cinq pentagones ; d'un, de deux, de trois, de quatre, de cinq ou de six hexagones, et à l'infini. Pour y parvenir, il faut démontrer que tout nombre premier qui surpasse de l'unité un multiple de quatre, est composé de deux carrés, comme 5, 13, 17, 29, 37, &c.

Etant donné un nombre premier de cette nature, comme 53, trouver par règle générale les deux carrés qui le composent.

Tout nombre premier qui surpasse de l'unité un multiple de 3, est composé d'un carré et du triple d'un autre carré, comme 7, 13, 19, 31, 37, &c.

Tout nombre premier qui surpasse d'un ou de trois un multiple de huit, est composé d'un carré et du double d'un autre carré, comme 11, 17, 19, 41, 43, &c.

Il n'y a aucun triangle en nombres duquel l'aire soit égale à un nombre carré.

Cela sera suivi de l'invention de beaucoup de propositions que Bachet avoue avoir ignorées, et qui manquent dans le Diophante.

Je suis persuadé que dès que vous aurez connu ma façon de démontrer en cette nature de propositions, elle vous paraîtra belle, et vous donnera lieu de faire beaucoup de nouvelles découvertes ; car il faut, comme vous savez, que *multi pertransit ut augeatur scientia*.

S'il me reste du temps, nous parlerons ensuite des nombres magiques, et je rappellerai mes vieilles espèces sur ce sujet.

Je suis, de tout mon cœur, Monsieur, votre, &c.

F E R M A T.

Ce 25 Septembre.

Je souhaite la santé de M. de Carcavi comme la mienne, et suis tout à lui.

Je vous écris de la campagne, et c'est ce qui retardera par aventure mes réponses pendant ces vacances.

TROI-

TROISIEME LETTRE DE FERMAT A PASCAL *.

MONSIEUR,

Si j'entreprends de faire un point avec un seul dé en huit coups ; si nous convenons, après que l'argent est dans le jeu, que je ne jouerai pas le premier coup : il faut, par mon principe, que je tire du jeu un sixième du total pour être défintéressé, à raison dudit premier coup. Que si encore nous convenons après cela que je ne jouerai pas le second coup, je dois, pour mon indemnité, tirer le sixième du restant, qui est $\frac{5}{36}$ du total. Et si après cela nous convenons que je ne jouerai pas le troisième coup, je dois, pour mon indemnité, tirer le sixième du restant, qui est $\frac{25}{216}$ du total. Et si après cela nous convenons encore que je ne jouerai pas le quatrième coup, je dois tirer le sixième du restant, qui est $\frac{125}{1296}$ du total. Et je conviens avec vous, que c'est la valeur du quatrième coup, supposé qu'on ait déjà traité des précédents. Mais vous me proposez dans l'exemple dernier de votre Lettre (je mets vos propres termes,) que si j'entreprends de trouver le six en huit coups, et que j'en aie joué trois sans le rencontrer ; si mon joueur me propose de ne point jouer mon quatrième coup, et qu'il veuille me défintéresser à cause que je pourrois le rencontrer ; il m'appartiendra $\frac{125}{1296}$ de la somme entiere de nos mises ; ce qui pourtant n'est pas vrai, suivant mon principe. Car, en ce cas, les trois premiers coups n'ayant rien acquis à celui qui tient le dé, la somme totale restant dans le jeu, celui qui tient le dé et qui convient de ne pas jouer son quatrième coup, doit prendre pour son indemnité un sixième du total ; et s'il avoit joué quatre coups sans trouver le point cherché, et qu'on convint qu'il ne joueroit pas le cinquième, il auroit de même pour son indemnité un sixième du total ; car la somme entiere restant dans le jeu, il ne fuit pas seulement du principe, mais il est même du sens naturel, que chaque coup doit donner un égal avantage. Je vous prie donc que je sache si nous sommes conformes au principe, ainsi que je crois, ou si nous différons seulement en l'application.

Je suis, de tout mon cœur, &c.

F E R M A T.

* Imprimée pour la première fois. Cette Lettre est sans date dans la copie que j'en ai : elle paroît répondre à une Lettre de Pascal, que je n'ai pu recouvrer.

TROI-

TROISIEME LETTRE DE PASCAL A FERMAT.

En réponse à celle de la page 491.

MONSIEUR,

VOTRE dernière Lettre m'a parfaitement satisfait ; j'admire votre méthode pour les partis, d'autant mieux que je l'entends fort bien ; elle est entièrement vôtre, et n'a rien de commun avec la mienne, et arrive au même but facilement. Voilà notre intelligence rétablie. Mais, Monsieur, si j'ai concouru avec vous en cela, cherchez ailleurs qui vous suive dans vos inventions numériques, dont vous m'avez fait la grace de m'envoyer les énonciations : pour moi je vous confesse que cela me passe de bien loin : je ne suis capable que de les admirer, et vous supplie très-humblement d'occuper votre prochain loisir à les achever. Tous nos Messieurs les virent Samedi dernier, et les estimèrent de tout leur cœur : on ne peut pas aisément supporter l'attente de choses si belles et si souhaitables ; pensez-y donc, s'il vous plaît, et assurez-vous que je suis, &c.

P A S C A L.

Paris, 27 Octobre 1654.

LETTRE DE M. FERMAT A M. DE CARCAVI.

MONSIEUR,

J'AI été ravi d'avoir eu des sentiments conformes à ceux de M. Pascal ; car j'estime infiniment son génie, et je le crois très-capable de venir à bout de tout ce qu'il entreprendra. L'amitié qu'il m'offre m'est si chère et si considérable, que je crois ne point devoir faire difficulté d'en faire quelque usage en l'impression de mes Traités. Si cela ne vous choquoit point, vous pourriez tous deux procurer cette impression, de laquelle je consens que vous soyez les maîtres ; vous pourriez éclaircir, ou augmenter ce qui semble trop concis, et me décharger d'un soin que mes occupations m'empêchent de prendre : je desiré même que cet Ouvrage paroisse sans mon nom, vous remettant, à cela près, le choix de toutes les désignations qui pourront marquer le nom de l'Auteur que vous qualifierez votre ami. Voici le biais que j'ai imaginé pour la seconde Partie, qui contiendra mes inventions pour les nombres : c'est un travail qui n'est encore qu'une idée, et que je n'aurois pas le loisir de coucher au long sur le papier ; mais j'enverrai succinctement à M. Pascal tous mes principes et mes premières démonstrations, de quoi je vous réponds à l'avance qu'il tirera des choses non-seulement nouvelles et jusqu'ici inconnues, mais encore surprenantes. Si vous joignez votre travail avec le sien, tout pourra succéder et s'achever dans peu de temps,

temps, et cependant on pourra mettre au jour la première Partie que vous avez en votre pouvoir. Si M. Pascal goûte mon ouverture, (qui est principalement fondée sur la grande estime que je fais de son génie, de son savoir et de son esprit;) je commencerai d'abord à vous faire part de mes inventions numériques.

Adieu, je suis, Monsieur, &c.

F E R M A T.

A Toulouse, ce 9 Août 1659.

QUATRIEME LETTRE DE FERMAT A PASCAL.

MONSIEUR,

DES que j'ai su que nous sommes plus proches l'un de l'autre que nous n'étions auparavant, je n'ai pu résister à un dessein d'amitié dont j'ai prié M. de Carcavi d'être le médiateur : en un mot je prétends vous embrasser, et converser quelques jours avec vous ; mais parce que ma santé n'est guères plus forte que la vôtre, j'ose espérer qu'en cette considération vous me ferez la grace de la moitié du chemin, et que vous m'obligerez de me marquer un lieu entre Clermont et Toulouse, où je ne manquerai pas de me rendre vers la fin de Septembre ou le commencement d'Octobre. Si vous ne prenez pas ce parti, vous courrez hasard de me voir chez vous, et d'y avoir deux malades en même-temps. J'attends de vos nouvelles avec impatience, et suis de tout mon cœur, tout à vous,

F E R M A T.

A Toulouse, le 25 Juillet 1660.

LETTRE DE PASCAL A FERMAT,

En réponse à la précédente.

MONSIEUR,

VOUS êtes le plus galant homme du monde, et je suis assurément un de ceux qui fais le mieux reconnoître ces qualités-là et les admirer infiniment, sur-tout quand elles sont jointes aux talents qui se trouvent singulièrement en vous : tout cela m'oblige à vous témoigner de ma main ma reconnaissance pour l'offre que vous me faites, quelque peine que j'aie encore d'écrire et de lire moi-même : mais l'honneur que vous me faites m'est si cher, que je ne puis trop me hâter d'y répondre. Je vous dirai donc, Monsieur, que si j'étois en sante, je serois volé à Toulouse, et que je n'aurois pas souffert qu'un homme comme vous eût fait un pas pour un homme comme moi. Je vous dirai aussi que, quoique

quoique vous soyez celui de toute l'Europe que je tiens pour le plus grand Géometre, ce ne seroit pas cette qualité là qui m'auroit attiré ; mais que je me figure tant d'esprit et d'honnêteté en votre conversation, que c'est pour cela que je vous rechercherois. Car pour vous parler franchement de la Géométrie, je la trouve le plus haut exercice de l'esprit ; mais en même-temps je la connois pour si inutile, que je fais peu de différence entre un homme qui n'est que Géometre et un habile Artisan. Aussi je l'appelle le plus beau métier du monde ; mais enfin ce n'est qu'un métier ; et j'ai dit souvent qu'elle est bonne pour faire l'essai, mais non pas l'emploi, de notre force : de sorte que je ne ferois pas deux pas pour la Géométrie, et je m'assure que vous êtes fort de mon humeur. Mais il y a maintenant ceci de plus en moi, que je suis dans des études si éloignées de cet esprit-là, qu'à peine me souviens-je qu'il y en ait. Je m'y étois mis il y a un an ou deux, par une raison tout-à-fait singulière, à laquelle ayant faicisfait, je suis au hasard de ne jamais plus y penser : outre que ma santé n'est pas encore assez forte ; car je suis si foible, que je ne puis marcher sans bâton, ni me tenir à cheval. Je ne puis même faire que 3 ou 4 lieues au plus en carrosse ; c'est ainsi que je suis venu de Paris ici en vingt-deux jours. Les Médecins m'ordonnent les eaux de Bourbon pour le mois de Septembre, et je suis engagé, autant que je puis l'être depuis deux mois, d'aller de-là en Poitou par eau jusqu'à Saumur, pour demeurer jusqu'à Noël avec M. le Duc de Rohanne, Gouverneur de Poitou, qui a pour moi des sentimens que je ne vauz pas. Mais comme je passerai par Orléans en allant à Saumur par la riviere, si ma santé ne me permet pas de passer outre, j'irai de-là à Paris. Voilà, Monsieur, tout l'état de ma vie présente, dont je suis obligé de vous rendre compte, pour vous assurer de l'impossibilité où je suis de recevoir l'honneur que vous daignez m'offrir, et que je souhaite de tout mon cœur de pouvoir un jour reconnoître, ou en vous, ou en Messieurs vos enfans, auxquels je suis tout dévoué, ayant une vénération particuliere pour ceux qui portent le nom du premier homme du monde.

Je suis, &c.

De Bienassis, le 10 Août 1660.

P A S C A L.

LETTRE DE M. FERMAT A M. * * *.

MONSIEUR MON CHER MAITRE,

JE suis embarrassé en affaires non-géométriques ; je vous envoie pourtant un petit Ecrit que le P. Lalouvere m'a fait porter ce matin. J'ai reçu le Traité de M. Pascal depuis deux jours, et n'ai pu m'y appliquer encore sérieusement à le lire ; j'en ai pourtant conçu une grande opinion, aussi-bien que de tout ce qui part de cet illustre.

Je suis tout à vous,

A Toulouse, ce 16 Février 1659.

F E R M A T.

VOL. IV.

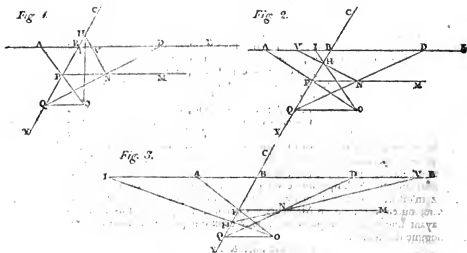
3 S

P O R I S.

PORISMATA DUO:

AUTORE PETRO FERMAT*.

PORISMA PRIMUM.



DATIS positione duabus rectis ABE, YBC (Fig. 1, 2, 3.) sese in puncto B secantibus: datis etiam punctis A et D in recta ABE: quaeruntur duo puncta, exempli gratia, O et N, à quibus si ad quodlibet rectae YBC punctum, ut H, recta OHN inflectatur, rectam ABD in punctis I et V secans, rectangulum sub AI in DV aequetur spatio dato; videlicet, rectangulo sub AB in BD?

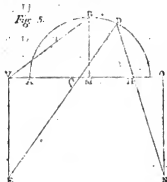
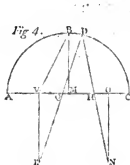
Ita procedit Porismatica EUCLIDIS constructio, et generalissimam problematis solutionem repræsentabit.

Sumatur punctum quodvis O; jungatur recta AO secans rectam YBC in puncto P; à puncto O ducatur recta OQ ipsi ABD parallela, et rectae YBC occurrens in Q; ducatur etiam infinita PNM eidem ABD parallela; et juncta QD secet rectam PNM in puncto N. Aio duo puncta P et N ad-

* On a trouvé, parmi les papiers de Pascal, ces deux Porismes et le Problème suivant, écrits de la main de Fermat: on croit que le Lecteur les verra ici avec d'autant plus de plaisir, qu'ils n'avoient pas encore été imprimés.

implere propositum; sumpto quippe ubilibet in rectâ YBC puncto H, et ductis rectis OH, NH, rectæ ABD occurrentibus in punctis I et V, rectangulum sub AI in DV, in quibuslibet omnino casibus (tres tantum triplex figura repræsentat) rectangulo AB in BD æquale erit.

P O R I S M A S E C U N D U M .



Dato circulo ABDC (Fig. 4, 5,) cujus diameter AC, centrum M; quærentur duo puncta, ut E et N, à quibus si ad quodvis circumferentiæ punctum, ut D, inflectatur recta EDN diametrum in punctis Q et H secans, summa quadratorum QD et DH, ad triangulum QDH, habeat rationem datam, idemque in quolibet inflexione generalitèr et perpetuè contingat?

A centro M excitetur ad diametrum perpendicularis MB; fiat ratio data eadem quæ quadrupla rectæ BU ad rectam UM; à puncto U excitetur UE ad diametrum perpendicularis, et ipsi UB æqualis; sumptâ rectâ MO ipsi MU æquali, fiat ON æqualis et parallela rectæ UE: dico puncta quæsitâ esse puncta E et N. Sumpto quippe quovis in circumferentiâ puncto ut D, et junctis ED, UD, rectis, diametrum in punctis Q et H secantibus, summa quadratorum QD et DH ad triangulum QDH erit, in quocumque casu, in ratione datâ, hoc est in ratione quadruplæ BU ad rectam UM.

Non solum proponitur inquirenda istius Porismatis demonstratio, sed videant etiâ subtiliores Mathematici an duo alia puncta præter E et N possint problemati proposito satisfacere, et utrum solutiones quæstionis, sicut in primo Porismate, suppetant infinitæ. Si nihil respondeant, Geometriæ in hac parte laboranti non dedignabimur opitulari.

SOLUTIO PROBLEMATIS

A DOMINO PASCAL PROPOSITI.

EODEM AUTORE FERMAT.

PROPOSUIT Dominus PASCAL hoc Problema: *Dato trianguli angulo ad verticem, et ratione quam habet perpendicularum ad differentiam laterum: invenire speciem trianguli?*

Fig. 6.

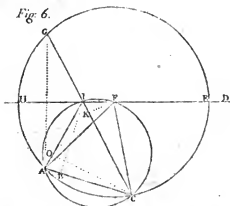
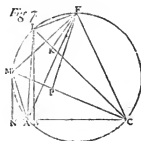


Fig. 7.



Exponatur recta quævis data AC (Fig. 6, 7,) super quam portio circuli AIFC capax anguli dati describatur. Eò quæstionem deduximus ut, datâ basi AC, angulo verticis AIC, et ratione quam habet perpendicularum ad differentiam laterum, quætur triangulum.

Ponatur jam factum esse, et triangulum quæsitum esse AIC; demittatur perpendicularum IB; et diviso arcu AIFC bifariam in F, jungantur AF, FC; et junctâ IF, demittantur in rectas AI, IC, perpendiculares CO, FK; deindè centro F, intervallo AF, describatur circulus AHGEC, cui rectæ CI, IF continuatæ occurrant in punctis G, H, E; denique jungatur GA. Angulus AFC ad centrum duplex est anguli AGC ad circumferentiam; sed angulus AIC æquatur angulo AFC in eadem portione; igitur angulus AIC duplex est anguli AGC. Sed angulus AIC æquatur duobus angulis AGC, IAG; igitur anguli IGA, IAG sunt æquales, ideòque rectæ IA, IG. Sed, cum à centro F in rectam GC cadat perpendicularis FK, æquales sunt GK, KC; ideòque KI est dimidia differentia inter rectas CI, IG, hoc est, inter rectas CI, IA.

IA. Data est autem ratio perpendicularis IB ad differentiam laterum CI, IA; ergo datur ratio BI ad IK; et singulis in rectam AC ductis, data est ratio rectanguli sub AC in BI ad rectangulum sub AC in IK. Sed rectangulum sub AC in BI æquatur rectangulo sub AI in CO; est enim utrumque duplum trianguli AIC: ergo ratio rectanguli sub AI in CO ad rectangulum sub AC in IK data est. Datur autem ex hypothesi angulus AIC, et rectus est COI ex constructione; ergo datur specie triangulum COI. Ratio igitur CO ad CI data est, ideoque rectanguli sub AI in CO ad rectangulum sub AI in IC ratio datur. Sed probavimus rationem rectanguli sub AI in CO ad rectangulum sub AC in IK dari; ergo datur ratio rectanguli AIC ad rectangulum sub AC in IK. Jam in triangulo AFC, datur angulus AFC ex hypothesi; ergo angulus FAC datur; cui æqualis CIF idcirco dabitur; est autem rectus angulus FKI; ergo triangulum FIK datur specie; ideoque rectæ KI ad IF ratio data est; ideoque rectanguli AC in IK ad rectangulum sub AC in IF datur ratio. Probatum est autem dari rationem rectanguli AI in IC ad rectangulum AC in IK; ergo datur ratio rectanguli AI in IC ad rectangulum AC in IF. Est autem rectangulum CIG æquale rectangulo CIA, quia rectæ IG, IA sunt æquales, et rectangulo CIG æquatur rectangulum HIE: ergo ratio rectanguli HIE ad rectangulum sub AC in IF data est. Sit data ratio ED ad AC: cum igitur AC sit data, dabitur ED, quæ ponatur rectæ HE in directum, ut in figurâ 6; rectangulum igitur HIE ad rectangulum AC in IF est in ratione datâ ED ad AC. Sed ut DE ad AC ita DE in IF ad AC in IF; igitur ut rectangulum HIE est ad rectangulum ad AC in IF, ita rectangulum DE in IF ad rectangulum AC in IF; rectangulum igitur DE in IF æquatur rectangulo HIE. Probatum est triangulum AFC dari specie; sed datur basis AC magnitudine; ergo datur AF, ideoque dupla ipsius EH datur. Æqualibus rectangulis DE in IF et HIE addatur rectangulum sub DE in IH; fiet rectangulum sub DE in FH æquale rectangulo DIH; datur autem rectangulum sub DE in FH, quia utraque rectarum DE, FH datur; datur igitur rectangulum DIH et ad datam magnitudinem DIH applicatur deficiens figurâ quadratâ; ergo recta IH datur, ideoque reliqua IF. Datur autem punctum F positione; ergo datur et punctum I, et totum triangulum AIC. Non est difficilis ab analysi ad synthesein regressus.

Sed ut omne dubium tollatur, probatur facillimè triangulum quæsitum esse simile invento AIC in septimâ figurâ (triangulum autem AIC ex utraque parte puncti F verticem habere potest, in æquali à puncto F utrinque distantia; erit enim idem specie et magnitudine, et positio variabit.) Si enim triangulum quæsitum non est simile invento, manente eadem basi, ejus vertex vel ibi inter puncta F et I, vel inter puncta I et A. (Ex utraque parte nihil interest; nam de parte FC idem secundum triangulum AIC pari demonstratione concluditur.) Sit primùm, vertex inter A et I, et triangulum quæsitum ponatur, si fieri potest, simile triangulo AMC. Jungatur FN et demittatur perpendicularis FP; erit ratio perpendiculari MN ad MP data ex hypothesi, ideoque æqualis rationi IB ad IK, quam probavimus datæ æqualem: quod est absurdum; cum enim in triangulo FMP angulus ad M æquatur angulo ad I, triangulo IFK erunt similia triangula FIK, FMP; sed FM est major FI; ergo MP est major

major IK; est autem MN minor IB; non igitur eadem potest esse ratio MN ad MP quæ IB ad IK. Si punctum M sit inter I et F, probabitur augeri perpendicularum et minui differentiam laterum, idque eadem argumentatione. Idcirco varians proportionem si punctum M sit in positione FC, utemur secundo triangulo AIC, et erit eadem demonstratio; ut inutile sit diutius in his casibus immorari. Constat igitur triangulum quæsitum invento AIC esse simile, et patet propositio esse satisfactum.

Proponitur, si placet, tam Domino PASCAL quam Domino ROBERVAL solvendum hoc Problema:

Ad datum punctum in helice BALTANI, invenire tangentem?

Quænam autem sit hujusmodi helix, novit Dominus ROBERVAL.

Hujus problematis à nobis soluti, solutionem à viris eruditissimis expectamus; aut, si maluerint, ipsis impertiemur, imò et generalem de linearum curvarum contactibus methodum.

Sed, ne à præsentī materiā triangulārī vacuis manibus discessisse videamur, proponi possunt hæ quæstiones:

Datā basi, angulo verticis, et aggregato perpendiculari et differentie laterum: invenire triangulum?

Datā basi, angulo verticis, et differentia perpendiculari et differentie laterum: invenire triangulum?

Datā basi, angulo verticis, et rectangulo sub differentia laterum in perpendicularum: invenire triangulum?

Datā basi, angulo verticis, et summa quadratorum perpendiculari et differentie laterum: invenire triangulum?

Et multæ similes, quarum enodationem facilius inventuros viros doctissimos existimo, quàm de contactu helices BALTANI propositum problema aut theorema.

Sed observandum in quæstionibus de triangulis, quoties problema poterit solvi per plana, non recurrendum ad solida: quod cum nôrint viri doctissimi, super vacuum fortasse subit addidisse.

LETTRE DE M. SLUZE,

*Chanoine de la Cathédrale de Liège, traduite de l'Italien en Français, pour réponse à M. * * *.*

MONSIEUR,

J'AVOUE que j'ai grande obligation *alla gentilezza* de M. Pascal, et j'ai grande estime de sa science, par la solution du Problème que vous lui aviez proposé; mais je voudrois bien savoir s'il lui a été proposé avec toute son universalité: la raison qui m'en a fait douter, est que je vois qu'il considère tous les points donnés dans un même plan, et je les considère en quelques plans différents

	1	2	3	4	5	6	7	8
1	G	σ	π	∞	μ	δ	ϵ	
2	ρ	ψ	θ	R	S	N	7	8
3	A	B	C	ω	ζ			
4	D	E	F	P	Y			
5	H	M	K					
6	P	Q						
7								
8								
9								
10								

Triang

Reas Perpendiculus

ents qu'ils puissent être ; ce que vous pouvez lui demander comme de vous-même.

Pour ce qui est des Problèmes que vous m'avez envoyés, je dirai seulement que s'ils m'eussent été envoyés quand je les ai demandés, j'aurois tâché de lui donner satisfaction ; mais la multitude des affaires qui m'accablent, comme vous savez bien, les vacances étant finies, ne me permettent pas d'appliquer mon esprit à de semblables recherches. Mais voyant que vous le désirez, je n'ai pu m'empêcher de les considérer quelque peu ; et d'abord je me suis aperçu que le premier * Problème pouvoit recevoir très-aisément solution par les lieux solides, c'est-à-dire, avec l'intersection de deux hyperboles. Après avoir fait un petit griffonnement d'analyse, je reconnus que le Problème étoit *plan*, et que la résolution n'en étoit pas difficile, mais que la construction en seroit un peu longue et embrouillée. Ainsi, pour ne pas être obligé d'écrire beaucoup, j'ai choisi un cas seulement entre plusieurs qui sont dans le Problème ; et pour trouver une construction plus breve, je l'ai appliqué aux nombres, comme vous verrez dans le papier qui est dans cette Lettre. Par-là toutes les personnes intelligentes verront aisément que j'ai la construction universelle ; je vous l'enverrai si vous la désirez, bien que ma paresse s'y oppose. J'estime pourtant que M. Pascal sera satisfait.

Pour ce qui est de l'autre, je m'appergus d'abord qu'il prenoit son origine de cinq plans qui touchent un ou deux cônes opposés. La résolution en est longue, mais pourtant je ne la crois pas si difficile. Quoi qu'il en soit, l'embarras continuél des affaires qui se sont présentées et multipliées au triple depuis que vous n'avez été ici, ne me donne pas le temps d'y penser pour le présent.

Je souhaiterois bien que vous me fîssiez la faveur de me marquer les Livres qui ont été imprimés sur cette matiere, ou sur autre de Philosophie, qui soient de quelque considération. Nous avons ici les *Exercitationes Mathematicæ* de M. François Schooten, Professeur à Leyde : je crois qu'on les aura vues à Paris.

Je viens à ce que vous me dites de M. Descartes ; je l'estime un grand homme : c'est pourquoi je voudrois savoir particulièrement ce qu'on lui oppose. Je ne prétends pas le faire passer pour inépréhensible, même dans ses Ecrits de Géométrie, parce que j'ai remarqué en plusieurs endroits qu'il étoit homme, et que *quandoque bonus dormitat Homerus* : mais une petite tache ne rend pas difforme un beau visage, *atque opere in longo fas est obrepere feminum*.

* Ce Problème étoit ainsi proposé : *Etant donnés deux cercles et une ligne droite, trouver un cercle qui touche les deux cercles donnés, et qui laisse sur la ligne droite un arc capable d'un angle donné.* Le papier dont Sluze parle un peu plus bas, est collé en original au commencement de l'exemplaire des *Inventiones de A. De Witt* en Géométrie, qui appartient à la Bibliothèque du Roi. On n'a pas cru devoir imprimer ici la solution, ou plutôt la construction de Sluze, parce qu'elle n'est accompagnée d'aucune analyse, et qu'elle n'est d'ailleurs appliquée qu'à un cas particulier de la question. Nous n'avons pas besoin d'ajouter que ces sortes de Problèmes n'ont aujourd'hui aucune difficulté.

TRAITÉ DU TRIANGLE ARITHMÉTIQUE.

D E F I N I T I O N S.

J'APPELLE *Triangle Arithmétique*, une figure dont la construction est telle.

Je mène d'un point quelconque, G, deux lignes perpendiculaires, l'une à l'autre, G V, G Z, dans chacune desquelles je prends tant que je veux de parties égales et continues à commencer par G, que je nomme 1, 2, 3, 4, &c. ; et ces nombres sont les *exposants* des divisions des lignes.

Ensuite je joins les points de la première division qui sont dans chacune des deux lignes, par une autre ligne qui forme un triangle dont elle est la *base*.

Je joins ainsi les deux points de la seconde division par une autre ligne, qui forme un second triangle dont elle est la *base*.

Et joignant ainsi tous les points de division qui ont un même exposant, j'en forme autant de triangles et de *bases*.

Je mène par chacun des points de division, des lignes parallèles aux côtés, qui par leurs intersections forment de petits quarrés, que j'appelle *cellules*.

Et les cellules qui sont entre deux parallèles qui vont de gauche à droite, s'appellent *cellules d'un même rang parallèle*, comme les cellules G, σ , π , &c., ou ϕ , ψ , θ , &c.

Et celles qui sont entre deux lignes qui vont de haut en bas, s'appellent *cellules d'un même rang perpendiculaire*, comme les cellules G, ϕ , A, D, &c., et celles-ci, σ , ψ , B, &c.

Et celles qu'une même base traverse diagonalement, sont dites *cellules d'une même base*, comme celles qui suivent, D, B, θ , λ , et celles-ci, A, ψ , π .

Les cellules d'une même base également distantes de ses extrémités, sont dites *réiproques*, comme celles-ci, E, R, et B, θ ; parce que l'exposant du rang parallèle de l'une est le même que l'exposant du rang perpendiculaire de l'autre, comme il paroît en cet exemple, où E est dans le second rang perpendiculaire, et dans le quatrième parallèle ; et la réiproque R est dans le second rang parallèle, et dans le quatrième perpendiculaire réciproquement. Et il est bien facile de démontrer que celles qui ont leurs exposants réciproquement pareils, sont dans une même base, et également distantes de ses extrémités.

Il est aussi bien facile de démontrer que l'exposant du rang perpendiculaire de quelque cellule que ce soit, joint à l'exposant de son rang parallèle, surpasse de l'unité l'exposant de sa base. Par exemple, la cellule F est dans le troisième rang perpendiculaire, et dans le quatrième parallèle, et dans la sixième base, et les deux exposants des rangs $3 + 4$ surpassent de l'unité l'exposant de la base 6 ;

ce

ce qui vient de ce que les deux côtés du Triangle sont divisés en un pareil nombre de parties ; mais cela est plutôt compris que démontré.

Cette remarque est de même nature, que chaque base contient une cellule plus que la précédente, et chacune autant que son exposant d'unités ; ainsi la seconde ϕ a deux cellules, la troisième $A\psi\pi$ en a trois, &c.

Or les nombres qui se mettent dans chaque cellule se trouvent par cette méthode.

Le nombre de la première cellule qui est à l'angle droit est arbitraire ; mais celui-là étant placé, tous les autres sont forcés : et pour cette raison il s'appelle le *générateur* du triangle ; et chacun des autres est spécifié par cette seule règle :

“ Le nombre de chaque cellule est égal à celui de la cellule qui la précède dans son rang perpendiculaire, plus à celui de la cellule qui la précède dans son rang parallèle. Ainsi la cellule F, c'est-à-dire, le nombre de la cellule F, égale la cellule C, plus la cellule E ; et ainsi des autres.”

D'où se tirent plusieurs conséquences. En voici les principales, où je considère les triangles, dont le générateur est l'unité ; mais ce qui s'en dira conviendra à tous les autres.

CONSEQUENCE PREMIERE.

En tout Triangle Arithmétique, toutes les cellules du premier rang parallèle et du premier rang perpendiculaire sont pareilles à la génératrice.

Car par la construction du Triangle, chaque cellule est égale à celle qui la précède dans son rang perpendiculaire, plus à celle qui la précède dans son rang parallèle ; or les cellules du premier rang parallèle n'ont aucunes cellules qui les précèdent dans leurs rangs perpendiculaires, ni celles du premier rang perpendiculaire dans leurs rangs parallèles ; donc elles sont toutes égales entre elles, et partant au premier nombre générateur.

Ainsi ϕ égale G + zéro, c'est-à-dire, ϕ égale G.

Ainsi A égale ϕ + zéro, c'est-à-dire, ϕ .

Ainsi σ égale G + zéro, et π égale σ + zéro.

Et ainsi des autres.

CONSEQUENCE II.

En tout Triangle Arithmétique, chaque cellule est égale à la somme de toutes celles du rang parallèle précédent, comprises depuis son rang perpendiculaire jusqu'au premier inclusivement.

Soit une cellule quelconque ω : je dis qu'elle est égale à $R + \theta + \psi + \phi$, qui sont celles du rang parallèle supérieur depuis le rang perpendiculaire de ω jusqu'au premier rang perpendiculaire.

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Cela

Cela est évident par la seule interprétation des cellules, par celles d'où elles sont formées.

Car ω égale $R + C$.

$$\theta + B$$

$$\psi + A$$

Car A et ϕ sont égaux entre eux par la précédente.

Donc ω égale $R + \theta + \psi + \phi$.

CONSEQUENCE III.

En tout Triangle Arithmétique, chaque cellule égale la somme de toutes celles du rang perpendiculaire précédent, comprises depuis son rang parallèle jusqu'au premier inclusivement.

Soit une cellule quelconque C : je dis qu'elle est égale à $B + \psi + \sigma$, qui sont celles du rang perpendiculaire précédent, depuis le rang parallèle de la cellule C jusqu'au premier rang parallèle.

Cela paroît de même par la seule interprétation des cellules.

Car C égale $B + \theta$.

$$\psi + \pi$$

Car π égale σ par la première.

Donc C égale $B + \psi + \sigma$.

CONSEQUENCE IV.

En tout Triangle Arithmétique, chaque cellule diminuée de l'unité est égale à la somme de toutes celles qui sont comprises entre son rang parallèle et son rang perpendiculaire exclusivement.

Soit une cellule quelconque ξ : je dis que $\xi - G$ égale $R + \theta + \psi + \phi + \lambda + \pi + \sigma + G$, qui sont tous les nombres compris entre le rang ξ ω C B A et le rang ξ S μ exclusivement.

Cela paroît de même par l'interprétation :

Car ξ égale $\lambda + R + \omega$.

$$\pi + \theta + C$$

$$\sigma + \psi + B$$

$$G + \phi + A$$

$$G$$

Donc ξ égale $\lambda + R + \pi + \theta + \sigma + \psi + G + \phi + G$.

AYER-

A V E R T I S S E M E N T .

J'ai dit dans l'énonciation, *chaque cellule diminuée de l'unité*, parce que l'unité est le générateur ; mais si c'étoit un autre nombre, il faudroit dire, *chaque cellule diminuée du nombre générateur*.

C O N S E Q U E N C E V.

En tout Triangle Arithmétique, chaque cellule est égale à sa réciproque.

Car dans la seconde base $\phi \sigma$, il est évident que les deux cellules réciproques ϕ , σ sont égales entre elles et à G.

Dans la troisième A, ψ , π , il est visible de même que les réciproques π , A sont égales entre elles et à G.

Dans la quatrième, il est visible que les extrêmes D, λ sont encore égales entre elles et à G.

Et celles d'entre-deux, B, θ , sont visiblement égales, puisque B égale $A + \psi$, et θ égale $\psi + \pi$; or $\pi + \psi$ sont égales à $A + \psi$ par ce qui est montré ; donc, &c.

Ainsi l'on montrera dans toutes les autres bases que les réciproques sont égales, parce que les extrêmes sont toujours pareilles à G, et que les autres s'interpréteront toujours par d'autres égales dans la base précédente qui sont réciproques entre elles.

C O N S E Q U E N C E VI.

En tout Triangle Arithmétique, un rang parallèle et un perpendiculaire qui ont un même exposant, sont composés de cellules toutes pareilles les unes aux autres.

Car ils sont composés de cellules réciproques.

Ainsi le second rang perpendiculaire $\sigma \psi B E M Q$ est entièrement pareil au second rang parallèle $\phi \psi \theta R S N$.

C O N S E Q U E N C E VII.

En tout Triangle Arithmétique, la somme des cellules de chaque base est double de celles de la base précédente.

Soit une base quelconque D B θ λ : je dis que la somme de ses cellules est double de la somme des cellules de la précédente A ψ π .

Car les extrêmes	.	.	.	.	.	D,	λ ,
égale les extrêmes	.	.	.	.	.	A,	π ,
et chacune des autres	.	.	.	.	.	B,	θ ,
en égalent deux de l'autre base	.	.	.	.	.	A + ψ ,	ψ + π .

Donc D + λ + B + θ égalent 2A + 2 ψ + 2 π .

La même chose se démontre de même de toutes les autres.

CONSEQUENCE VIII.

En tout Triangle Arithmétique, la somme des cellules de chaque base est un nombre de la progression double, qui commence par l'unité, dont l'exposant est le même que celui de la base.

Car la première base est l'unité.

La seconde est double de la première; donc elle est 2.

La troisième est double de la seconde; donc elle est 4. Et ainsi à l'infini.

AVERTISSEMENT.

Si le générateur n'étoit pas l'unité, mais un autre nombre, comme 3, la même chose seroit vraie; mais il ne faudroit pas prendre les nombres de la progression double à commencer par l'unité, savoir, 1, 2, 4, 8, 16, &c.; mais ceux d'une autre progression double à commencer par le générateur 3, savoir, 3, 6, 12, 24, 48, &c.

CONSEQUENCE IX.

En tout Triangle Arithmétique, chaque base diminuée de l'unité est égale à la somme de toutes les précédentes.

Car c'est une propriété de la progression double.

AVERTISSEMENT.

Si le générateur étoit autre que l'unité, il faudroit dire, *chaque base diminuée du générateur.*

CONSEQUENCE X.

En tout Triangle Arithmétique, la somme de tant de cellules continues qu'on voudra de sa base, à commencer par une extrémité, est égale à autant de cellules de la base précédente, plus encore à autant, bornis une.

Soit prise la somme de tant de cellules qu'on voudra de la base D λ , par exemple, les trois premières, $D + B + \theta$; je dis qu'elle est égale à la somme des trois premières de la base précédente $A + \psi + \pi$, plus aux deux premières de la même base $A + \psi$.

Car	D.	B.	θ .
égale	A.	$A + \psi$.	$\psi + \pi$.
Donc	$D + B + \theta$ égale $2A + 2\psi + \pi$.		

DEFINITION.

J'appelle *cellules de la dividende*, celles que la ligne qui divise l'angle droit par la moitié, traverse diagonalement, comme les cellules G, ψ , C, ρ , &c.

CON.

CONSEQUENCE XI.

Chaque cellule de la dividende est double de celle qui la précède dans son rang parallèle ou perpendiculaire.

Soit une cellule de la dividende C: je dis qu'elle est double de θ , et aussi de B. Car C égale $\theta + B$, et θ égale B, par la cinquième conséquence.

AVERTISSEMENT.

Toutes ces conséquences sont sur le sujet des égalités qui se rencontrent dans le Triangle Arithmétique. On va en voir maintenant les proportions, dont la proposition suivante est le fondement.

CONSEQUENCE XII.

En tout Triangle Arithmétique, deux cellules contiguës étant dans une même base, la supérieure est à l'inférieure, comme la multitude des cellules depuis la supérieure jusqu'au haut de la base, à la multitude de celles depuis l'inférieure jusqu'en bas inclusivement.

Soient deux cellules contiguës quelconques d'une même base, E, C: je dis que

E est à C		comme	2	à	3
inférieure,	supérieure,	parce qu'il y a deux cellules depuis E jusqu'en bas, savoir, E, H,		parce qu'il y a trois cellules depuis C jusqu'en haut, savoir, C, R, μ .	

Quoique cette proposition ait une infinité de cas, j'en donnerai une démonstration bien courte, en supposant deux Lemmes.

Le premier, qui est évident de soi-même, que cette proportion se rencontre dans la seconde base; car il est bien visible que ϕ est à σ comme 1 à 1.

Le deuxième, que si cette proportion se trouve dans une base quelconque, elle se trouvera nécessairement dans la base suivante.

D'où il se voit qu'elle est nécessairement dans toutes les bases: car elle est dans la seconde base par le premier Lemme; donc par le second elle est dans la troisième base; donc dans la quatrième, et à l'infini.

Il faut donc seulement démontrer le second Lemme en cette sorte. Si cette proportion se rencontre en une base quelconque, comme en la quatrième D λ , c'est-à-dire, si D est à B comme 1 à 3; et B à θ comme 2 à 2; et θ à λ comme 3 à 1, &c. Je dis que la même proportion se trouvera dans la base suivante, H μ , et que, par exemple, E est à C comme 2 à 3.

Car D est à B comme 1 à 3, par l'hypothèse.

Donc D + B est à B comme 1 + 3 à 3.

E à B comme 4 à 3.

De

De même B est à θ comme 2 à 2, par l'hypothèse.

Donc B + θ à B, comme 2 + 2 à 2.

Mais $\begin{array}{c} C \text{ à B comme } 4 \text{ à } 2. \\ B \text{ à E comme } 3 \text{ à } 4, \end{array}$ comme il est montré. Donc par la proportion troublée, C est à E comme 3 à 2.

Ce qu'il falloit démontrer.

On le montrera de même dans tout le reste, puisque cette preuve n'est fondée que sur ce que cette proportion se trouve dans la base précédente, et que chaque cellule est égale à sa précédente, plus à sa supérieure; ce qui est vrai partout *.

CONSEQUENCE XIII.

En tout Triangle Arithmétique, deux cellules contiguës étant dans un même rang perpendiculaire, l'inférieure est à la supérieure, comme l'exposant de la base de cette supérieure à l'exposant de son rang parallèle.

Soient deux cellules quelconques dans un même rang perpendiculaire, F, C : je dis que

* Ce second Lemme peut être exprimé et démontré en termes généraux de la façon suivante.

Soit la multitude des cellules depuis l'inférieure de deux cellules contiguës sur aucune base du Triangle Arithmétique, en y comprenant cette cellule inférieure, p ; et la multitude des cellules depuis la cellule supérieure, en y comprenant cette cellule, q . Il faut démontrer, que, s'il est vrai que le nombre contenu dans la cellule supérieure soit au nombre contenu dans la cellule inférieure comme q à p , la même propriété aura lieu dans les nombres contenus dans les cellules contiguës de la prochaine base du Triangle Arithmétique qui correspondent aux deux dites cellules contiguës de la première base, ou qui remplissent les mêmes places dans la suite des cellules de la seconde base que les deux premières cellules contiguës remplissent dans la suite des cellules de la première base.

Le nombre des cellules dans la première base est $p + q$. Donc le nombre des cellules dans la seconde base est $p + q + 1$; et partant le nombre de cellules dans la seconde base depuis la cellule supérieure jusqu'au haut de la base inclusivement sera $q + 1$. Mais le nombre des cellules dans la même base depuis la cellule inférieure jusqu'en bas inclusivement est le même que dans la première base, c'est à dire, p . Il faut donc démontrer que le nombre contenu dans la cellule supérieure de la seconde base est au nombre contenu dans la cellule inférieure de la même base comme $q + 1$ est à p .

Supposons maintenant que le nombre contenu dans la cellule supérieure de la première base soit nommé θ , et que le nombre contenu dans la cellule inférieure de la même base, contiguë à la dite cellule supérieure, soit nommé B, et que le nombre contenu dans la cellule qui précède immédiatement la dite cellule inférieure soit nommé D; et que le nombre contenu dans la cellule supérieure de la seconde base soit nommé C, et que le nombre contenu dans la cellule inférieure de la même base soit nommé E. Après ces suppositions il faudra démontrer que le nombre C est au nombre E comme $q + 1$ à p .

DEMONSTRATION.

Or, par l'hypothèse, θ est à B comme q est à p . Donc, *componendo*, $\theta + B$ sera à B comme $q + p$ est à p . Mais C est $= \theta + B$. Donc C est à B comme $q + p$ est à p .

En outre, par l'hypothèse, B est à D comme $q + 1$ est à $p - 1$. Donc, *componendo*, B + D sera à B comme $q + 1 + p - 1$, ou $q + p$, est à $q + 1$. Mais E est $= B + D$. Donc E est à B comme $q + p$ est à $q + 1$; et, *invertendo*, B est à E comme $q + 1$ est à $q + p$.

Mais C est à B comme $q + p$ est à p .

Donc la proportion de C à E, qui est évanouée des deux proportions de C à B et de B à E, sera égale à une proportion composée des deux proportions de $q + p$ à p et de $q + 1$ à $q + p$, et, par conséquent, à la proportion de $q + 1$ à p ; ou, en d'autres mots, C sera à E comme $q + 1$ est à p .

C. E. D.

Le fameux théorème de Monsieur Newton, sur les puissances d'une quantité binôme, peut être déduit de cette XI^{ème} conséquence.

F est

F est à C comme 5 à l'inférieure, | la supérieure, | exposant de la base de C, | exposant du 3^e rang parallèle de C.

Car E est à C comme 2 à 3.

Donc E + C est à C comme 2 + 3 à 3.

F est à C comme 5 à 3.

CONSEQUENCE XIV.

En tout Triangle Arithmétique, deux cellules contiguës étant dans un même rang parallèle, la plus grande est à sa précédente, comme l'exposant de la base de cette précédente à l'exposant de son rang perpendiculaire.

Soient deux cellules dans un même rang parallèle F, E : je dis que

F est à E comme 5 à la plus grande, | précédente, | exposant de la base de E, | exposant du 2^e rang perpendiculaire de E.

Car E est à C comme 2 à 3.

Donc E + C est à E comme 2 + 3 à 2.

F est à E comme 5 à 2.

CONSEQUENCE XV.

En tout Triangle Arithmétique, la somme des cellules d'un quelconque rang parallèle est à la dernière de ce rang, comme l'exposant du triangle est à l'exposant du rang.

Soit un triangle quelconque, par exemple, le quatrième GDL ; je dis que, quelque rang qu'on y prenne, comme le second parallèle, la somme de ses cellules, savoir, $\phi + \psi + \theta$, est à θ comme 4 à 2. Car $\phi + \psi + \theta$ égale C, et C est à θ comme 4 à 2, par la treizième conséquence.

CONSEQUENCE XVI.

En tout Triangle Arithmétique, un quelconque rang parallèle est au rang inférieur, comme l'exposant du rang inférieur à la multitude de ses cellules.

Soit un triangle quelconque, par exemple, le cinquième, GH : je dis que, quelque rang qu'on y prenne, par exemple, le troisième, la somme de ses cellules est à la somme de celles du quatrième, c'est-à-dire, A + B + C est à D + E, comme 4 ; exposant du rang quatrième, à 2, qui est l'exposant de la multitude de ses cellules, car il en contient 2.

Car A + B + C égale F, et D + E égale M. Or F est à M comme 4 à 2, par la douzième conséquence.

AVR.

A V E R T I S S E M E N T.

On pourroit l'énoncer aussi de cette sorte : *chaque rang parallèle est au rang inférieur, comme l'exposant du rang inférieur à l'exposant du triangle, moins l'exposant du rang supérieur.* Car l'exposant d'un triangle moins l'exposant d'un de ses rangs, est toujours égal à la multitude des cellules du rang inférieur.

C O N S E Q U E N C E X V I I.

En tout Triangle Arithmétique, quelque cellule que ce soit, jointe à toutes celles de son rang perpendiculaire, est à la même cellule jointe à toutes celles de son rang parallèle, comme les multitudes des cellules prises dans chaque rang.

Soit une cellule quelconque B : je dis que $B + \psi + \sigma$ est à B + A, comme 3 à 2.

Je dis 3, parce qu'il y a trois cellules ajoutées dans l'antécédent ; et 2, parce qu'il y en a deux dans le conséquent.

Car $B + \psi + \sigma$ égale C, par la troisième conséquence ; et B + A égale E, par la seconde conséquence.

Or C est à E comme 3 à 2, par la douzième conséquence.

C O N S E Q U E N C E X V I I I.

En tout Triangle Arithmétique, deux rangs parallèles également distants des extrémités, sont entre eux comme la multitude de leurs cellules.

Soit un triangle quelconque G V Z, et deux de ses rangs également distants des extrémités, comme le sixième P + Q, et le second $\phi + \psi + \theta + R + S + N$: je dis que la somme des cellules de l'un est à la somme des cellules de l'autre, comme la multitude des cellules de l'un est à la multitude des cellules de l'autre.

Car, par la sixième conséquence, le second rang parallèle $\phi \psi \theta R S N$ est le même que le second rang perpendiculaire $\sigma \psi B E M Q$, duquel nous venons de démontrer cette proportion.

A V E R T I S S E M E N T.

On peut l'énoncer ainsi : *En tout Triangle Arithmétique, deux rangs parallèles, dont les exposants joints ensemble excèdent de l'unité l'exposant du Triangle, sont entre eux comme leurs exposants réciproquement.* Car ce n'est qu'une même chose que ce qui vient d'être énoncé.

C O N S E Q U E N C E D E R N I E R E.

En tout Triangle Arithmétique, deux cellules contiguës étant dans la dividende, l'inférieure est à la supérieure prise quatre fois, comme l'exposant de la base de cette supérieure, à un nombre plus grand de l'unité.

Soient deux cellules de la dividende p, C : je dis que p est à 4 C comme 5, exposant de la base de C, est à 6.

Car

Car ρ est double de ω , et C de θ ; donc 4 θ égalent 2 C.

Donc 4 θ sont à C comme 2 à 1.

Or ρ est à 4 C comme ω à 4 θ , ou en raison composée de $\frac{\omega}{C} \text{ à } \frac{C}{4\theta}$

Par les conséquences précédentes,	.	.	.	$\frac{\omega}{C} \text{ à } \frac{C}{4\theta}$	$\frac{C}{4\theta} \text{ à } \frac{4\theta}{C}$
				$5 \text{ à } 3.$	$1 \text{ à } 2$
					$\text{ou } 3 \text{ à } 6.$
				5	à 6.

Donc ρ est à 4 C comme 5 à 6. Ce qu'il falloit démontrer.

AVERTISSEMENT.

On peut tirer de-là beaucoup d'autres proportions; que je supprime, parce que chacun peut facilement les conclure, et que ceux qui voudront s'y attacher, en trouveront peut-être de plus belles que celles que je pourrais donner. Je finis donc par le Problème suivant, qui fait l'accomplissement de ce Traité.

P R O B L E M E.

Etant donnés les exposants des rangs perpendiculaire et parallèle d'une cellule, trouver le nombre de la cellule, sans se servir du Triangle Arithmétique ?

Soit, par exemple, proposé de trouver le nombre de la cellule ξ du cinquième rang perpendiculaire, et du troisième rang parallèle.

Ayant pris tous les nombres qui précèdent l'exposant du perpendiculaire 5, savoir, 1, 2, 3, 4; soient pris autant de nombres naturels, à commencer par l'exposant du parallèle 3, savoir, 3, 4, 5, 6.

Soient multipliés les premiers l'un par l'autre, et soit le produit 24. Soient multipliés les autres l'un par l'autre, et soit le produit 360, qui, divisé par l'autre produit 24, donne pour quotient 15: ce quotient est le nombre cherché.

Car ξ est à la première de sa base V, en raison composée de toutes les raisons des cellules d'entre-deux, c'est-à-dire, ξ est à V, en raison composée de $\frac{\xi}{3} \text{ à } \frac{3}{4} \text{ à } \frac{4}{5} \text{ à } \frac{5}{6} \text{ à } \frac{6}{V}$

ou par la 12 conséq.	.	:	$\frac{3}{4} \text{ à } \frac{4}{5} \text{ à } \frac{5}{6} \text{ à } \frac{6}{V}$	$\frac{V}{3} \text{ à } \frac{3}{4} \text{ à } \frac{4}{5} \text{ à } \frac{5}{6}$	$\frac{6}{V} \text{ à } 1.$
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Donc ξ est à V comme 3 en 4 en 5 en 6, à 4 en 3 en 2 en 1.

Mais V est l'unité; donc ξ est le quotient de la division du produit de 3 en 4 en 5 en 6, par le produit de 4 en 3 en 2 en 1.

AVERTISSEMENT.

Si le générateur n'étoit pas l'unité, il eût fallu multiplier le quotient par le générateur.

DIVERS USAGES DU TRIANGLE ARITHMETIQUE,

Dont le générateur est l'unité.

APRES avoir donné les proportions qui se rencontrent entre les cellules et les rangs des Triangles Arithmétiques, je passe à divers usages de ceux dont le générateur est l'unité ; c'est ce qu'on verra dans les Traités suivants. Mais j'en laisse bien plus que je n'en donne ; c'est une chose étrange combien il est fertile en propriétés ! Chacun peut s'y exercer ; j'avertis seulement ici, que dans toute la suite, je n'entends parler que des Triangles Arithmétiques, dont le générateur est l'unité.

USAGE DU TRIANGLE ARITHMETIQUE

POUR LES ORDRES NUMERIQUES.

ON a considéré dans l'Arithmétique les nombres des différentes progressions ; on a aussi considéré ceux des différentes puissances et des différents degrés ; mais on n'a pas, ce me semble, assez examiné ceux dont je parle, quoiqu'ils soient d'un très grand usage : et même ils n'ont pas de nom ; ainsi j'ai été obligé de leur en donner : et parce que ceux de *progression*, de *degré* et de *puissance* sont déjà employés, je me fers de celui d'*ordres*.

J'appelle donc *Nombres du premier ordre*, les simples unités,

1, 1, 1, 1, 1, &c.

J'appelle *Nombres du second ordre*, les naturels qui se forment par l'addition des unités,

1, 2, 3, 4, 5, &c.

J'appelle *Nombres du troisième ordre*, ceux qui se forment par l'addition des naturels, qu'on appelle *triangulaires*,

1, 3, 6, 10, &c.

C'est-à-dire, que le second des triangulaires, savoir, 3, égale la somme des deux premiers naturels, qui sont 1, 2 ; ainsi le troisième triangulaire 6 égale la somme des trois premiers naturels, 1, 2, 3, &c.

J'appelle *Nombres du quatrième ordre*, ceux qui se forment par l'addition des triangulaires, qu'on appelle *pyramidaux*,

1, 4, 10, 20, &c.

J'appelle *Nombres du cinquième ordre*, ceux qui se forment par l'addition des précédents, auxquels on n'a pas donné de nom exprès, et qu'on pourroit appeler * *triangulo-triangulaires* :

1, 5, 15, 35, &c.

* Il semble par cette expression, " qu'on pourroit appeler *triangulo-triangulaires*," que Pascal fut le premier auteur sur l'Arithmétique qui ait donné ce nom à ces nombres du cinquième ordre. Mais il me semble que le nom plus simple de *Nombres du cinquième ordre* les désigne plus commodément.

F. M.

J'appelle

J'appelle *Nombres du sixième ordre*, ceux qui se forment par l'addition des précédents :

1, 6, 21, 56, 126, 252, &c.
 1, 7, 28, 84, &c.
 1, 8, 36, 120, &c.
 Et ainsi à l'infini.

Or si on fait une table de tous les ordres des nombres, où l'on marque à côté les exposants des ordres, et au-dessus les racines, en cette sorte :

RACINES.

		1	2	3	4	5	&c.
Unités	Ordre 1	1	1	1	1	1	&c.
Naturels	Ordre 2	1	2	3	4	5	&c.
Triangul.	Ordre 3	1	3	6	10	15	&c.
Pyramid.	Ordre 4	1	4	10	20	35	&c.

On trouvera cette Table pareille au Triangle Arithmétique ; et le premier ordre des nombres sera le même que le premier rang parallèle du Triangle ; le second ordre des nombres sera le même que le second rang parallèle : et ainsi à l'infini.

Car dans le Triangle Arithmétique le premier rang est tout d'unités, et le premier ordre des nombres est de même tout d'unités.

Ainsi dans le Triangle Arithmétique, chaque cellule, comme la cellule F, égale $C + B + A$, c'est-à-dire, qu'elle égale sa supérieure, plus toutes celles qui précèdent cette supérieure dans son rang parallèle, comme il a été prouvé dans la deuxième conséquence du Traité de ce Triangle : et la même chose se trouve dans chacun des ordres des nombres ; car, par exemple, le troisième des pyramidaux 10 égale les trois premiers des triangulaires $1 + 3 + 6$, puisqu'il est formé par leur addition.

D'où il se voit manifestement, que les rangs parallèles du triangle ne sont autre chose que les ordres des nombres, et que les exposants des rangs parallèles sont les mêmes que les exposants des ordres, et que les exposants des rangs perpendiculaires sont les mêmes que les racines : et ainsi le nombre, par exemple, 21, qui dans le Triangle Arithmétique se trouve dans le troisième rang parallèle, et dans le sixième rang perpendiculaire, étant considéré entre les ordres numériques, il fera du troisième ordre, et le sixième de son ordre, ou de la sixième racine.

Ce qui fait connoître que tout ce qui a été dit des rangs et des cellules du Triangle Arithmétique, convient exactement aux ordres des nombres, et que les mêmes égalités et les mêmes proportions qui ont été remarquées aux uns, se trouveront aussi aux autres ; il ne faudra seulement que changer les énonciations, en substituant les termes qui conviennent aux ordres numériques, comme ceux de *racine* et d'*ordre*, à ceux qui convenoient au Triangle Arithmétique, comme de *rang parallèle* et *perpendiculaire*, j'en donnerai un petit Traité à part, où quelques exemples qui y sont rapportés, seront aisément apercevoir tous les autres.

USAGE DU TRIANGLE ARITHMETIQUE

POUR LES COMBINAISONS.

Le mot de *combinaison* a été pris en plusieurs sens différents ; de sorte que pour ôter l'équivoque, je suis obligé de dire comment je l'entends.

Lorsque de plusieurs choses on donne le choix d'un certain nombre, toutes les manières d'en prendre autant qu'il est permis entre toutes celles qui sont présentées, s'appellent ici les *différentes combinaisons*.

Par exemple, si de quatre choses exprimées par ces quatre lettres, A, B, C, D, on permet d'en prendre, par exemple, deux quelconque ; toutes les manières d'en prendre deux différentes dans les quatre qui sont proposées, s'appellent *combinaisons*.

Ainsi on trouvera par expérience, qu'il y a six manières différentes d'en choisir deux dans quatre ; car on peut prendre A et B, ou A et C, ou A et D, ou B et C, ou B et D, ou C et D.

Je ne compte pas A et A pour une des manières d'en prendre deux ; car ce ne sont pas des choses différentes, ce n'en est qu'une répétée.

Ainsi je ne compte pas A et B, et puis B et A, pour deux manières différentes ; car on ne prend en l'une et en l'autre manière que les deux mêmes choses, mais dans un ordre différent seulement, et je ne prends point garde à l'ordre ; de sorte que je pouvois m'expliquer en un mot à ceux qui ont accoutumé de considérer les combinaisons, en disant simplement que je parle seulement des combinaisons qui se font sans changer l'ordre.

On trouvera de même par expérience, qu'il y a quatre manières de prendre trois choses dans quatre ; car on peut prendre ABC, ou ABD, ou ACD, ou BCD.

Enfin on trouvera qu'on ne peut en prendre quatre dans quatre qu'en une manière, savoir, ABCD.

Je parlerai donc en ces termes :

1	dans	4	se combine	4 fois.
2	dans	4	se combine	6 fois.
3	dans	4	se combine	4 fois.
4	dans	4	se combine	1 fois.

Ou ainsi :

La multitude des combinaisons de 1 dans 4 est 4.

La multitude des combinaisons de 2 dans 4 est 6.

La multitude des combinaisons de 3 dans 4 est 4.

La multitude des combinaisons de 4 dans 4 est 1.

Mais la somme de toutes les combinaisons, en général, qu'on peut faire dans 4, est 15, parce que la multitude des combinaisons de 1 dans 4, de 2 dans 4, de 3 dans 4, et de 4 dans 4, étant jointes ensemble, font 15.

Ensuite

Ensuite de cette explication, je donnerai ces conséquences en forme de Lemmes.

L E M M E P R E M I E R.

Un nombre ne se combine point dans un plus petit, par exemple, 4 ne se combine point dans 2.

L E M M E II.

1	dans	1	se combine	1	fois.
2	dans	2	se combine	1	fois.
3	dans	3	se combine	1	fois.

Et généralement un nombre quelconque se combine une fois seulement dans son égal.

L E M M E III.

1	dans	1	se combine	1	fois.
1	dans	2	se combine	2	fois.
1	dans	3	se combine	3	fois.

Et généralement l'unité se combine dans quelque nombre que ce soit autant de fois qu'il contient d'unités.

L E M M E IV.

S'il y a quatre nombres quelconques, le premier tel qu'on voudra, le second plus grand de l'unité, le troisième tel qu'on voudra, pourvu qu'il ne soit pas moindre que le second, le quatrième plus grand de l'unité que le troisième : la multitude des combinaisons du premier dans le troisième, jointe à la multitude des combinaisons du second dans le troisième, égale la multitude des combinaisons du second dans le quatrième.

Soient quatre nombres tels que j'ai dit :

Le premier tel qu'on voudra, par exemple,	1.
Le second plus grand de l'unité, savoir,	2.
Le troisième tel qu'on voudra, pourvu qu'il ne soit pas moindre que le second, par exemple,	3.
Le quatrième plus grand de l'unité, savoir,	4.

Je dis que la multitude des combinaisons de 1 dans 3, plus la multitude des combinaisons de 2 dans 3, égale la multitude des combinaisons de 2 dans 4.

Soient trois lettres quelconques, B, C, D.

Soient les mêmes trois lettres, et une de plus, A, B, C, D.

Prenons, suivant la proposition, toutes les combinaisons d'une lettre dans les trois, B, C, D ; il y en aura trois, savoir, B, C, D.

Prenons dans les mêmes trois lettres toutes les combinaisons de deux, il y en aura trois, savoir, BC, BD, CD.

Prenons enfin dans les quatre lettres A, B, C, D toutes les combinaisons de deux, il y en aura six, savoir, AB, AC, AD, BC, BD, CD.

Il faut démontrer que la multitude des combinaisons de 1 dans 3 et celles de 2 dans 3, égalent celles de 2 dans 4.

Cela est aisé ; car les combinaisons de 2 dans 4 sont formées par les combinaisons de 1 dans 3, et par celles de 2 dans 3.

Pour le faire voir, il faut remarquer qu'entre les combinaisons de 2 dans 4, savoir, AB, AC, AD, BC, BD, CD, il y en a où la lettre A est employée, et d'autres où elle ne l'est pas.

Celles où elle n'est pas employée sont, BC, BD, CD, qui par conséquent sont formées de deux de ces trois lettres, B, C, D ; donc ce sont des combinaisons de 2 dans ces trois, B, C, D. Donc les combinaisons de 2 dans ces trois lettres, B, C, D, sont portion des combinaisons de 2 dans ces quatre lettres, A, B, C, D, puisqu'elles forment celles où A n'est pas employée.

Maintenant si des combinaisons de 2 dans 4 où A est employé, savoir AB, AC, AD, on ôte l'A, il restera une lettre seulement de ces trois, B, C, D, savoir, B, C, D, qui sont précisément les combinaisons d'une lettre dans les trois, B, C, D. Donc si aux combinaisons d'une lettre dans les trois, B, C, D, on ajoute à chacune la lettre A, et qu'ainsi on ait AB, AC, AD, on formera les combinaisons de 2 dans 4, où A est employé ; donc les combinaisons de 1 dans 3 sont portion des combinaisons de 2 dans 4.

D'où il se voit que les combinaisons de 2 dans 4 sont formées par les combinaisons de 2 dans 3, et de 1 dans 3 ; et partant que la multitude des combinaisons de 2 dans 4 égale celle de 2 dans 3, et de 1 dans 3.

On montrera la même chose de tous les autres exemples, comme :

La multitude des combinaisons de 29 dans 40, et la multitude des combinaisons de 30 dans 40, égalent la multitude des combinaisons de 30 dans 41. Ainsi la multitude des combinaisons de 15 dans 55, et la multitude des combinaisons de 16 dans 55, égalent la multitude des combinaisons de 16 dans 56. Et ainsi à l'infini. Ce qu'il falloit démontrer.

PROPOSITION PREMIERE

En tout Triangle Arithmétique, la somme des cellules d'un rang parallèle quelconque égale la multitude des combinaisons de l'exposant du rang dans l'exposant du Triangle.

Soit un triangle quelconque, par exemple, le quatrième GDA : je dis que la somme des cellules d'un rang parallèle quelconque, par exemple, du second, $\phi + \psi + \theta$, égale la somme des combinaisons de ce nombre 2, qui est l'exposant de ce second rang, dans ce nombre 4, qui est l'exposant de ce triangle.

Ainsi la somme des cellules du cinquième rang du huitième triangle égale la somme des combinaisons de 5 dans 8, &c.

La démonstration en sera courte, quoiqu'il y ait une infinité de cas, par le moyen de ces deux Lemmes.

Le premier, qui est évident de lui-même, que dans le premier triangle cette égalité se trouve, puisque la somme des cellules de son unique rang, savoir G,
ou

ou l'unité, égale la somme des combinaisons de 1, exposant du rang, dans 1, exposant du Triangle.

Le deuxième, que s'il se trouve un Triangle Arithmétique dans lequel cette proportion se rencontre, c'est-à-dire, dans lequel quelque rang que l'on prenne, il arrive que la somme des cellules soit égale à la multitude des combinaisons de l'exposant du rang dans l'exposant du Triangle : je dis que le Triangle suivant aura la même propriété.

D'où il s'ensuit que tous les Triangles Arithmétiques ont cette égalité ; car elle se trouve dans le premier Triangle par le premier Lemme, et même elle est encore évidente dans le second, donc par le second Lemme, le suivant l'aura de même, et partant le suivant encore ; et ainsi à l'infini.

Il faut donc seulement démontrer le second Lemme.

Soit un triangle quelconque, par exemple, le troisième, dans lequel on suppose que cette égalité se trouve, c'est-à-dire, que la somme des cellules du premier rang $G + \sigma + \pi$ égale la multitude des combinaisons de 1 dans 3 ; et que la somme des cellules du deuxième rang $\phi + \psi$ égale les combinaisons de 2 dans 3 ; et que la somme des cellules du troisième rang A égale les combinaisons de 3 dans 3 : je dis que le quatrième Triangle aura la même égalité, et que, par exemple, la somme des cellules du second rang $\phi + \psi + \theta$ égale la multitude des combinaisons de 2 dans 4.

Car $\phi + \psi + \theta$ égale	$\phi + \psi$	+	θ
	$\phi + \psi$		$+ G + \sigma + \pi$
Par l'hypothese	ou la multitude des combinaif. de 2 dans 3.	+	ou la multitude des combinaif. de 1 dans 3.

Ou la multitude des combinaisons
de 2 dans 4.

Par le quatrième Lemme

On le montrera de même de tous les autres. Ce qu'il falloit démontrer.

PROPOSITION II.

Le nombre de quelque cellule que ce soit, égale la multitude des combinaisons d'un nombre moindre de l'unité que l'exposant de son rang parallèle, dans un nombre moindre de l'unité que l'exposant de sa base.

Soit une cellule quelconque, F, dans le quatrième rang parallèle et dans la sixième base : je dis qu'elle égale la multitude des combinaisons de 3 dans 5, moindres de l'unité que 4 et 6 ; car elle égale les cellules $A + B + C$. Donc par la précédente, &c.

PROBLEME

PROBLEME I. PROPOSITION III.

Etant proposés deux nombres, trouver combien de fois l'un se combine dans l'autre, par le Triangle Arithmétique ?

Soient les nombres proposés 4, 6, il faut trouver combien 4 se combine dans 6.

PREMIER MOYEN.

Soit prise la somme des cellules du quatrième rang du sixième triangle : elle satisfera à la question.

SECOND MOYEN.

Soit prise la cinquième cellule de la septième base, parce que ces nombres 5, 7 excèdent de l'unité les donnés 4, 6 ; son nombre est celui qu'on demande.

CONCLUSION.

Par le rapport qu'il y a des cellules et des rangs du Triangle Arithmétique aux combinaisons, il est aisé de voir que tout ce qui a été prouvé des uns convient aux autres suivant leur manière ; c'est ce que je montrerai en peu de discours dans un petit Traité que j'ai fait des Combinaisons.

USAGE DU TRIANGLE ARITHMETIQUE,

Pour déterminer les partis qu'on doit faire entre deux Joueurs qui jouent en plusieurs parties.

POUR entendre les règles des partis, la première chose qu'il faut considérer, est que l'argent que les Joueurs ont mis au jeu ne leur appartient plus ; car ils en ont quitté la propriété : mais ils ont reçu en revanche le droit d'attendre ce que le hasard peut leur en donner, suivant les conditions dont ils sont convenus d'abord.

Mais, comme c'est une loi volontaire, ils peuvent la rompre de gré à gré : et ainsi, en quelque terme que le jeu se trouve, ils peuvent le quitter, et, au contraire de ce qu'ils ont fait en y entrant, renoncer à l'attente du hasard, et rentrer chacun en la propriété de quelque chose ; et en ce cas, le règlement de ce qui doit leur appartenir doit être tellement proportionné à ce qu'ils avoient droit d'espérer de la fortune, que chacun d'eux trouve entièrement égal de prendre ce qu'on lui assigne, ou de continuer l'aventure du jeu : et cette juste distribution s'appelle *le parti*.

Le premier principe qui fait connoître de quelle sorte on doit faire les partis, est celui-ci.

Si

Si un des Joueurs se trouve en telle condition, que, quoi qu'il arrive, une certaine somme doit lui appartenir en cas de perte et de gain, sans que le hasard puisse la lui ôter; il ne doit en faire aucun parti, mais la prendre entière comme assurée, parce que le parti devant être proportionné au hasard, puisqu'il n'y a nul hasard de perdre, il doit tout retirer sans parti.

Le second est celui-ci. Si deux Joueurs se trouvent en telle condition, que si l'un gagne, il lui appartiendra une certaine somme, et s'il perd, elle appartiendra à l'autre; si le jeu est de pur hasard, et qu'il y ait autant de hasards pour l'un que pour l'autre, et par conséquent non plus de raison de gagner pour l'un que pour l'autre, s'ils veulent se séparer sans jouer, et prendre ce qui leur appartient légitimement, le parti est qu'ils séparent la somme qui est au hasard par la moitié, et que chacun prenne la sienne.

COROLLAIRE PREMIER.

Si deux Joueurs jouent à un jeu de pur hasard, à condition que, si le premier gagne, il lui reviendra une certaine somme, et s'il perd, il lui en reviendra une moindre; s'ils veulent se séparer sans jouer, et prendre chacun ce qui leur appartient: le parti est, que le premier prenne ce qui lui revient en cas de perte, et de plus la moitié de l'excès, dont ce qui lui reviendrait en cas de gain, surpasse ce qui lui revient en cas de perte.

Par exemple, si deux Joueurs jouent à condition que, si le premier gagne, il emportera 8 pistoles, et s'il perd, il en emportera 2: je dis que le parti est qu'il prenne ces 2, plus la moitié de l'excès de 8 sur 2, c'est-à-dire, plus 3; car 8 surpasse 2 de 6, dont la moitié est 3.

Car par l'hypothèse, s'il gagne, il emporte 8, c'est-à-dire, 6 + 2, et s'il perd, il emporte 2; donc ces 2 lui appartiennent en cas de perte et de gain: et par conséquent par le premier principe, il ne doit en faire aucun parti, mais les prendre entières. Mais pour les 6 autres, elles dépendent du hasard; de sorte que, s'il lui est favorable, il les gagnera, sinon, elles reviendront à l'autre; et, par l'hypothèse, il n'y a pas plus de raison qu'elles reviennent à l'un qu'à l'autre: donc le parti est qu'ils les séparent par la moitié, et que chacun prenne la sienne, qui est ce que j'avois proposé.

Donc, pour dire la même chose en d'autres termes, il lui appartient le cas de la perte, plus la moitié de la différence des cas de perte et de gain.

Et partant, si, en cas de perte, il lui appartient A, et en cas de gain A + B, le parti est qu'il prenne A + $\frac{1}{2}$ B.

COROLLAIRE II.

Si deux Joueurs sont en la même condition que nous venons de dire: je dis que le parti peut se faire de cette façon, qui revient au même, que l'on assemble les deux sommes de gain et de perte, et que le premier prenne la moitié de cette somme; c'est-à-dire, qu'on joigne 2 avec 8, et ce sera 10, dont la moitié 5 appartiendra au premier.

Car la moitié de la somme de deux nombres est toujours la même que, la moindre plus la moitié de leur différence. Et cela se démontre ainsi.

Soit A ce qui revient en cas de perte, et $A + B$ ce qui revient en cas de gain : je dis que le parti se fait en assemblant ces deux nombres, qui font $A + A + B$, et en donnant la moitié au premier, qui est $\frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}B$. Car cette somme égale $A + \frac{1}{2}B$, qui a été prouvée faire le parti juste.

Ces fondemens étant posés, nous passerons aisément à déterminer le parti entre deux Joueurs qui jouent en tant de parties qu'on voudra, en quelque état qu'ils se trouvent ; c'est-à-dire, quel parti il faut faire quand ils jouent en deux parties, et que le premier en a une à point ; ou qu'ils jouent en trois, et que le premier en a une à point, ou quand il en a deux à point, ou quand il en a deux à une. Et généralement en quelque nombre de parties qu'ils jouent, et en quelque gain de parties qu'ils soient, et l'un, et l'autre.

Sur quoi la première chose qu'il faut remarquer, est que deux Joueurs qui jouent en deux parties, dont le premier en a une à point, sont en même condition que deux autres qui jouent en trois parties, dont le premier en a deux, et l'autre une : car il y a cela de commun, que, pour achever, il ne manque qu'une partie au premier, et deux à l'autre, et c'est en cela que consiste la différence des avantages, et qui doit régler les partis ; de sorte qu'il ne faut proprement avoir égard qu'au nombre des parties qui restent à gagner à l'un et à l'autre, et non pas au nombre de celles qu'ils ont gagnées, puisque, comme nous avons déjà dit, deux Joueurs se trouvent en même état, quand jouant en deux parties, l'un en a une à point, que deux qui jouant en douze parties, l'un en a onze à dix.

Il faut donc proposer la question en cette sorte.

Etant proposés deux Joueurs, à chacun desquels il manque un certain nombre de parties pour achever, faire le parti ?

J'en donnerai ici la méthode, que je poursuivrai seulement en deux ou trois exemples, qui seront si aisés à continuer, qu'il ne sera pas nécessaire d'en donner davantage.

Pour faire la chose générale sans rien omettre, je la prendrai par le premier exemple, qu'il est peut-être mal à propos de toucher, parce qu'il est trop clair ; je le fais pourtant pour commencer par le commencement : c'est celui-ci.

P R E M I E R C A S.

Si à un des Joueurs il ne manque aucune partie, et à l'autre quelques-unes, la somme entière appartient au premier ; car il l'a gagnée, puisqu'il ne lui manque aucune des parties dans lesquelles il devoit la gagner.

S E C O N D C A S.

Si à un des Joueurs il manque une partie, et à l'autre une, le parti est qu'ils séparent l'argent par la moitié, et que chacun prenne la sienne : cela est évident par le second principe. Il en est de même s'il manque deux parties à l'un, et deux à l'autre ; et de même quelque nombre de parties qui manque à l'un, s'il en manque autant à l'autre,

TROI-

TROISIEME CAS.

Si à un des Joueurs il manque une partie, et à l'autre deux, voici l'art de trouver le parti.

Considérons ce qui appartiendrait au premier Joueur (à qui il ne manque qu'une partie) en cas de gain de la partie qu'ils vont jouer, et puis ce qui lui appartiendrait en cas de perte.

Il est visible que si celui à qui il ne manque qu'une partie, gagne cette partie qui va se jouer, il ne lui en manquera plus ; donc tout lui appartiendra par le premier cas. Mais ; au contraire, si celui à qui il manque deux parties, gagne celle qu'ils vont jouer, il ne lui en manquera plus qu'une ; donc ils seront en telle condition, qu'il en manquera une à l'un, et une à l'autre. Donc ils doivent partager l'argent par la moitié, par le deuxième cas.

Donc si le premier gagne cette partie qui va se jouer, il lui appartient tout ; et s'il la perd, il lui appartient la moitié : donc en cas qu'ils veuillent se séparer sans jouer cette partie, il lui appartient $\frac{1}{2}$ par le second Corollaire.

Et si on veut proposer un exemple de la somme qu'ils jouent, la chose sera bien plus claire.

Posons que ce soit 8 pistoles : donc le premier en cas de gain, doit avoir le tout, qui est 8 pistoles ; et en cas de perte, il doit avoir la moitié, qui est 4 ; donc il lui appartient en cas de parti la moitié de $8 + 4$, c'est-à-dire, 6 pistoles de 8 ; car $8 + 4$ font 12, dont la moitié est 6.

QUATRIEME CAS.

Si à un des Joueurs il manque une partie, et à l'autre trois, le parti se trouvera de même, en examinant ce qui appartient au premier en cas de gain et de perte.

Si le premier gagne, il aura toutes ses parties, et partant tout l'argent ; qui est, par exemple, 8.

Si le premier perd, il ne faudra plus que deux parties à l'autre à qui il en falloit trois. Donc ils seront en tel état, qu'il faudra une partie au premier, et deux à l'autre ; et partant, par le cas précédent, il appartiendra 6 pistoles au premier.

Donc en cas de gain, il lui en faut 8, et en cas de perte 6 ; donc en cas de parti, il lui appartient la moitié de ces deux sommes, savoir, 7 ; car $6 + 8$ font 14, dont la moitié est 7.

CINQUIEME CAS.

Si à un des Joueurs il manque une partie, et à l'autre quatre, la chose est de même.

Le premier en cas de gain, gagne tout, qui est, par exemple, 8 ; et en cas de perte, il manque une partie au premier, et trois à l'autre ; donc il lui appartient 7 pistoles de 8 ; donc en cas de parti, il lui appartient la moitié de 8, plus la moitié 7, c'est-à-dire, $7\frac{1}{2}$.

SIXIÈME CAS.

Ainsi, s'il manque une partie à l'un, et cinq à l'autre; et à l'infini.

SEPTIÈME CAS.

De même s'il manque deux parties au premier, et trois à l'autre; car il faut toujours examiner les cas de gain et de perte.

Si le premier gagne, il lui manquera une partie, et à l'autre trois; donc par le quatrième cas il lui appartient $\frac{7}{8}$ de 8.

Si le premier perd, il lui manquera deux parties, et à l'autre deux; donc par le deuxième cas il appartient à chacun la moitié, qui est 4; donc en cas de gain, le premier en aura 7, et en cas de perte, il en aura 4; donc en cas de parti, il aura la moitié de ces deux ensemble, savoir, 5 $\frac{1}{2}$.

Par cette méthode on fera les partis sur toutes sortes de conditions, en prenant toujours ce qui appartient en cas de gain et ce qui appartient en cas de perte, et assignant pour le cas de parti la moitié de ces deux sommes.

Voilà une des manières de faire les partis.

Il y en a deux autres, l'une par le Triangle Arithmétique, et l'autre par les Combinaisons.

Méthode pour faire les partis entre deux Joueurs qui jouent en plusieurs parties, par le moyen du Triangle Arithmétique.

Avant que de donner cette méthode, il faut faire ce Lemme.

L E M M E.

Si deux Joueurs jouent à un jeu de pur hasard, à condition que si le premier gagne, il lui appartiendra une portion quelconque sur la somme qu'ils jouent, exprimée par une fraction, et que s'il perd, il lui appartiendra une moindre portion sur la même somme, exprimé par une autre fraction: s'ils veulent se séparer sans jouer, la condition des partis se trouvera en cette sorte. Soient réduites les deux fractions à même dénomination, si elles n'y sont pas; soit prise une fraction dont le numérateur soit la somme des deux numérateurs, et le dénominateur double des précédents: cette fraction exprime la portion qui appartient au premier sur la somme qui est au jeu.

Par exemple, qu'en cas de gain, il appartienne les $\frac{3}{5}$ de la somme qui est au jeu, et qu'en cas de perte, il lui en appartienne $\frac{1}{5}$: je dis que ce qui lui appartient en cas de parti, se trouvera en prenant la somme des numérateurs, qui est 4, et le double du dénominateur, qui est 10, dont on fait la fraction $\frac{4}{10}$.

Car

Car par ce qui a été démontré au deuxième Corollaire, il falloit assembler les cas de gain et de perte, et en prendre la moitié; or la somme des deux fractions $\frac{3}{5} + \frac{1}{5}$ est $\frac{4}{5}$, qui se fait par l'addition des numérateurs, et sa moitié se trouve en doublant le dénominateur, et ainsi l'on a $\frac{4}{10}$. Ce qu'il falloit démontrer.

Or ces regles sont générales et sans exception, quoi qu'il revienne en cas de perte ou de gain; car si, par exemple, en cas de gain, il appartient $\frac{1}{2}$, et en cas de perte, rien, en réduisant les deux fractions à même dénominateur, on aura $\frac{1}{2}$ pour le cas de gain, et $\frac{0}{2}$ pour le cas de perte; donc en cas de parti, il faut cette fraction $\frac{1}{4}$, dont le numérateur égale la somme des autres, et le dénominateur est double du précédent.

Ainsi si en cas de gain, il appartient tout, et en cas de perte $\frac{1}{3}$, en réduisant les fractions à même dénomination, on aura $\frac{3}{3}$ pour le cas de gain, et $\frac{1}{3}$ pour celui de la perte; donc en cas de parti, il appartient $\frac{4}{6}$.

Ainsi si en cas de gain, il appartient tout, et en cas de perte rien, le parti sera visiblement $\frac{1}{2}$; car le cas de gain est $\frac{1}{1}$, et le cas de perte $\frac{0}{1}$; donc le parti est $\frac{1}{2}$.

Et ainsi de tous les cas possibles.

PROBLEME I. PROPOSITION I.

Etant proposés deux Joueurs, à chacun desquels il manque un certain nombre de parties pour achever, trouver par le Triangle Arithmétique le parti qu'il faut faire (s'ils veulent se séparer sans jouer) ou égard aux parties qui manquent à chacun?

Soit prise dans le Triangle la base dans laquelle il y a autant de cellules qu'il manque de parties aux deux ensemble: ensuite soient prises dans cette base autant de cellules continues à commencer par la première, qu'il manque de parties au premier Joueur; et qu'on prenne la somme de leurs nombres. Donc il reste autant de cellules, qu'il manque de parties à l'autre. Qu'on prenne encore la somme de leurs nombres: ces sommes sont l'une à l'autre comme les avantages des Joueurs réciproquement: de sorte que si la somme qu'ils jouent est égale à la somme des nombres de toutes les cellules de la base, il en appartiendra à chacun ce qui est contenu en autant de cellules qu'il manque de parties à l'autre; et s'ils jouent une autre somme, il leur en appartiendra à proportion.

Par exemple, qu'il y ait deux Joueurs, au premier desquels il manque deux parties, et à l'autre quatre: il faut trouver le parti.

Soient ajoutés ces deux nombres 2 et 4, et soit leur somme 6; soit prise la

lixième

fixième base du Triangle Arithmétique P δ , dans laquelle il y a par conséquent six cellules P, M, F, ω , S, δ . Soient prises autant de cellules à commencer par la première P, qu'il manque de parties au premier joueur, c'est-à-dire, les deux premières P, M; donc il en reste autant que de parties à l'autre, c'est-à-dire, quatre, F, ω , S, δ : je dis que l'avantage du premier est à l'avantage du second, comme $F + \omega + S + \delta$ à $P + M$, c'est-à-dire, que si la somme qui se joue est égale à $P + M + F + \omega + S + \delta$, il en appartient à celui à qui il manque deux parties la somme des quatre cellules $\delta + S + \omega + F$; et à celui à qui il manque quatre parties, la somme des deux cellules $P + M$: et s'ils jouent une autre somme, leur en appartient à proportion.

Et pour le dire généralement, quelque somme qu'ils jouent, il en appartient au premier une portion exprimée par cette fraction $\frac{F + \omega + S + \delta}{F + \omega + S + \delta + P + M + F + \omega + S + \delta}$, dont le numérateur est la somme des quatre cellules de l'autre, et le dénominateur la somme de toutes les cellules; et à l'autre une portion exprimée par cette fraction, $\frac{P + M}{F + \omega + S + \delta + P + M + F + \omega + S + \delta}$, dont le numérateur est la somme des deux cellules de l'autre, et le dénominateur la même somme de toutes les cellules.

Et s'il manque une partie à l'un, et cinq à l'autre, il appartient au premier la somme des cinq premières cellules $P + M + F + \omega + S$, et à l'autre la somme de la cellule δ .

Et s'il manque six parties à l'un, et deux à l'autre, le parti s'en trouvera dans la huitième base, dans laquelle les six premières cellules contiennent ce qui appartient à celui à qui il manque deux parties, et les deux autres ce qui appartient à celui à qui il en manque six. Et ainsi à l'infini.

Quoique cette proposition ait une infinité de cas, je la démontrerai néanmoins en peu de mots par le moyen de deux Lemmes.

Le premier, que la seconde base contient les partis des Joueurs, auxquels il manque deux parties en tout. Le deuxième, que si une base quelconque contient les partis de ceux auxquels il manque autant de parties qu'elle a de cellules, la base suivante fera de même, c'est-à-dire, qu'elle contiendra aussi les partis des Joueurs, auxquels il manque autant de parties qu'elle a de cellules.

D'où je conclus en un mot que toutes les bases du Triangle Arithmétique ont cette propriété: car la seconde l'a par le premier Lemme; donc par le second Lemme, la troisième l'a aussi, et par conséquent la quatrième; et ainsi à l'infini. Ce qu'il falloit démontrer.

Il faut donc seulement démontrer ces 2 Lemmes.

Le premier est évident de lui-même; car s'il manque une partie à l'un et une à l'autre, il est évident que leurs conditions sont comme ϕ à σ , c'est-à-dire, comme 1 à 1, et qu'il appartient à chacun cette fraction, $\frac{\sigma}{\phi + \sigma}$ qui est $\frac{1}{2}$.

Le deuxième se démontrera de cette sorte.

Si une base quelconque comme la quatrième D λ contient les partis de ceux à qui il manque quatre parties, c'est-à-dire, que s'il manque une partie au premier, et trois au second, la portion qui appartient au premier sur la somme qui se joue, soit celle qui est exprimée par cette fraction $\frac{\phi + \psi + \theta}{\phi + \psi + \theta + \lambda}$ qui a pour déno-

dénominateur la somme des cellules de cette base, et pour numérateur les trois premières ; et que s'il manque deux parties à l'un, et deux à l'autre, la fraction qui appartient au premier soit $\frac{D + B}{D + B + \delta + \lambda}$; et que s'il manque trois parties au premier, et une à l'autre, la fraction du premier soit $\frac{D}{D + B + \delta + \lambda}$, &c.

Je dis que la cinquième base contient aussi les partis de ceux auxquels il manque cinq parties ; et que s'il manque, par exemple, deux parties au premier, et trois à l'autre, la portion qui appartient au premier sur la somme qui se joue, est exprimée par cette fraction, $\frac{D + B + C}{D + B + C + A + \mu}$.

Car pour savoir ce qui appartient à deux Joueurs à chacun desquels il manque quelques parties, il faut prendre la fraction qui appartiendrait au premier en cas de gain, et celle qui lui appartiendrait en cas de perte, les mettre à même dénomination, si elles n'y sont pas, et en former une fraction, dont le numérateur soit la somme des deux autres, et le dénominateur double de l'autre, par le Lemme précédent.

Examinons donc les fractions qui appartiendroient à notre premier Joueur en cas de gain et de perte.

Si le premier à qui il manque deux parties gagne celle qu'ils vont jouer, il ne lui manquera plus qu'une partie, et à l'autre toujours trois ; donc il leur manque quatre parties en tout ; donc, par l'hypothèse, leur parti se trouve en la base quatrième, et il appartiendra au premier cette fraction $\frac{D + B + \delta}{D + B + \delta + \lambda}$.

Si au contraire le premier perd, il lui manquera toujours deux parties, et deux seulement à l'autre ; donc par l'hypothèse la fraction du premier sera

$$\frac{D + B}{D + B + \delta + \lambda}. \text{ Donc en cas de parti il appartiendra au premier cette fraction } \frac{D + B + \delta + D + B}{2D + 2B + 2\delta + 2\lambda}; \text{ c'est-à-dire, } \frac{D + B + C}{D + B + C + A + \mu}.$$

Ce qu'il falloit démontrer.

Ainsi cela se démontre en toutes les autres bases sans aucune différence, parce que le fondement de cette preuve est qu'une base est toujours double de la précédente par la septième conséquence, et que, par la dixième conséquence, tant de cellules qu'on voudra d'une même base sont égales à autant de la base précédente (qui est toujours le dénominateur de la fraction en cas de gain) plus encore aux mêmes cellules, excepté une (qui est le numérateur de la fraction en cas de perte ;) ce qui étant vrai généralement par-tout, la démonstration sera toujours sans obstacle et universelle.

PROBLEME.

PROBLEME II. PROPOSITION II.

Etant proposés deux Jouveurs qui jouent chacun une même somme en un certain nombre de parties proposé, trouver dans le Triangle Arithmétique la valeur de la dernière partie sur l'argent du perdant ?

Par exemple, que deux Jouveurs jouent chacun 3 pistoles en quatre parties : on demande la valeur de la dernière partie sur les 3 pistoles du perdant.

Soit prise la fraction qui a l'unité pour numérateur, et pour dénominateur la somme des cellules de la base quatrième, puisqu'on joue en quatre parties : je dis que cette fraction est la valeur de la dernière partie sur la mise du perdant.

Car si deux Jouveurs jouant en quatre parties, l'un en a trois à point, et qu'ainsi il en manque une au premier, et quatre à l'autre, il a été démontré que ce qui appartient au premier pour le gain qu'il a fait de ses trois premières parties, est exprimé par cette fraction, $\frac{n + e + c + a}{n + e + c + a + p}$, qui a pour dénominateur la somme des cellules de la cinquième base, et pour numérateur ses quatre premières cellules ; donc il ne reste sur la somme totale des deux mises que cette fraction $\frac{p}{n + e + c + a + p}$, laquelle seroit acquise à celui qui a déjà les trois premières parties en cas qu'il gagnât la dernière ; donc la valeur de cette dernière sur la somme des deux mises est $\frac{p}{n + e + c + a + p}$, c'est-à-dire,

$$\frac{\text{l'unité}}{2D + 2B + 2\theta + 2\lambda}$$

Or puisque la somme totale des mises est $2D + 2B + 2\theta + 2\lambda$, la somme de chaque mise est $D + B + \theta + \lambda$; donc la valeur de la dernière partie sur la seule mise du perdant est cette fraction, $\frac{1}{D + B + \theta + \lambda}$, double de la précédente, et laquelle a pour numérateur l'unité, et pour dénominateur la somme des cellules de la quatrième base. Ce qu'il falloit démontrer.

PROBLEME III. PROPOSITION III.

Etant proposés deux Jouveurs qui jouent chacun une même somme en un certain nombre de parties donné, trouver dans le Triangle Arithmétique la valeur de la première partie sur la mise du perdant ?

Par exemple, que deux Jouveurs jouent chacun 3 pistoles en quatre parties ; on demande la valeur de la première sur la mise du perdant.

Soit ajouté au nombre 4 le nombre 3, moindre de l'unité, et soit la somme 7 ; soit prise la fraction qui ait pour dénominateur toutes les cellules de la septième base, et pour numérateur la cellule de cette base qui se rencontre dans la dividende, savoir, cette fraction, $\frac{p}{v + a + x + e + \xi + n + \zeta}$: je dis qu'elle satisfait au problème.

Car

Car si deux Joueurs jouant en quatre parties, le premier en a une à point, il en restera trois à gagner au premier, et quatre à l'autre; donc il appartient au premier sur la somme des deux mises cette fraction, $\frac{v+q+k+p}{v+q+k+p+\xi+n+\zeta}$ qui a pour dénominateur toutes les cellules de la septième base, et pour numérateur les quatre premières cellules.

Donc il lui appartient $V + Q + K + p$ sur la somme totale des deux mises, exprimée par $V + Q + K + p + \xi + N + \zeta$; mais cette dernière somme étant l'assemblage des deux mises, il en avoit mis au jeu la moitié, favior, $V + Q + K + \frac{1}{2}p$, car $V + Q + K$ sont égaux à $\zeta + N + \xi$.

Donc il a $\frac{1}{2}p$, c'est-à-dire, $\frac{p}{2}$, plus qu'il n'avoit en entrant au jeu; donc il a gagné sur la somme totale des deux mises une portion exprimée par cette fraction, $\frac{p}{v+q+k+p+\xi+n+\zeta}$; donc il a gagné sur la mise du perdant une portion qui sera double de celle-là, favior, celle qui est exprimée par cette fraction, $\frac{p}{v+q+k+p+\xi+n+\zeta}$.

Donc le gain de la première partie lui a acquis cette fraction; donc sa valeur est telle.

COROLLAIRE.

Donc la valeur de la première partie de deux sur la mise du perdant, est exprimée par cette fraction, $\frac{1}{2}$.

Car en prenant cette valeur suivant la règle qui vient d'en être donnée, il faut prendre la fraction qui a pour dénominateur les cellules de la troisième base (parce que le nombre des parties en quoi on joue est 2, et le nombre moindre de l'unité est 1, qui avec 2 fait 3) et pour numérateur la cellule de cette base qui est dans la dividende; donc on aura cette fraction, $\frac{\psi}{a+\psi+\pi}$.

Or le nombre de la cellule ψ est 2, et les nombres des cellules $A + \psi + \pi$, sont, $1 + 2 + 1$. Donc on a cette fraction, $\frac{2}{1+2+1}$ c'est-à-dire, $\frac{2}{4}$, c'est-à-dire, $\frac{1}{2}$.

Donc le gain de la première partie lui a acquis cette fraction; donc sa valeur est telle. Ce qu'il falloit démontrer.

PROBLEME IV. PROPOSITION IV.

Etant proposés deux Joueurs qui jouent chacun une même somme en un certain nombre de parties donné, trouver par le Triangle Arithmétique la valeur de la seconde partie sur la mise du perdant ?

Soit le nombre donné des parties dans lesquelles on joue, 4; il faut trouver la valeur de la deuxième partie sur la mise du perdant.

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Soit

Soit prise la valeur de la première partie par le Problème précédent : je dis qu'elle est la valeur de la seconde.

Car deux Joueurs jouant en quatre parties, si l'un en a deux à point, la fraction qui lui appartient est celle-ci, $\frac{r+n+r+w}{r+m+r+w+s+d}$, qui a pour dénominateur la somme des cellules de la sixième base, et pour numérateur la somme des quatre premières ; mais il en avoit mis au jeu cette fraction,

$\frac{r+n+r}{r+m+r+w+s+d}$, savoir, la moitié du tout. Donc il lui reste de gain cette fraction, $\frac{w}{r+m+r+w+s+d}$, qui est la même chose que celle-ci . . .

$\frac{p}{v+q+k+r+\xi+n+d}$; donc il a gagné sur la moitié de la somme entière, c'est-à-dire, sur la mise du perdant, cette fraction $\frac{2p}{v+q+k+r+\xi+n+d}$, double de la précédente.

Donc le gain des deux premières parties lui a acquis cette fraction sur l'argent du perdant, qui est le double de ce que la première partie lui avoit acquis, par la précédente ; donc la seconde partie lui en a autant acquis que la première.

CONCLUSION.

On peut aisément conclure, par le rapport qu'il y a du Triangle Arithmétique aux partis qui doivent se faire entre deux Joueurs, que les proportions des cellules qui ont été données dans le Traité du Triangle, ont des conséquences qui s'étendent à la valeur des partis, qui sont bien aisées à tirer, et dont j'ai fait un petit discours en traitant des partis, qui donne l'intelligence et le moyen de les étendre plus avant.

USAGE DU TRIANGLE ARITHMETIQUE,

Pour trouver les puissances des Binomes et des Apotomes.

S'IL est proposé de trouver la puissance quelconque, comme le quatrième degré d'un binome, dont le premier nombre soit A, l'autre l'unité, c'est-à-dire, qu'il faille trouver le quarré-quarré de A + 1 ; il faut prendre dans le Triangle Arithmétique la base cinquième, savoir, celle dont l'exposant 5 est plus grand de l'unité que 4, exposant de l'ordre proposé : les cellules de cette cinquième base sont, 1, 4, 6, 4, 1, dont il faut prendre le premier nombre 1 pour

pour co-efficient de A au degré proposé, c'est-à-dire, de A^4 ; ensuite il faut prendre le second nombre de la base, qui est 4, pour co-efficient de A au degré prochainement inférieur, c'est-à-dire, de A^3 , et prendre le nombre suivant de la base, savoir, 6, pour co-efficient de A au degré inférieur, savoir, A^2 , et le nombre suivant de la base, savoir, 4, pour co-efficient de A au degré inférieur, savoir, A racine, et prendre le dernier nombre de la base 1 pour nombre absolu: et ainsi on aura $1A^4 + 4A^3 + 6A^2 + 4A + 1$, qui sera la puissance carré-carrée du binome $A + 1$. De sorte que, si A (qui représente tout nombre) est l'unité, et qu'ainsi le binome $A + 1$ soit le binaire, cette puissance

$$\begin{array}{r} 1A^4 + 4A^3 + 6A^2 + 4A + 1 \\ 1.1^4 + 4.1^3 + 6.1^2 + 4.1 + 1. \end{array}$$

C'est-à-dire, une fois le carré-carré de l'unité A, c'est-à-dire,	1
Quatre fois le cube de 1, c'est-à-dire,	4
Six fois le carré de 1, c'est-à-dire,	6
Quatre fois l'unité, c'est-à-dire,	4
Plus l'unité,	1

qui ajoutés font 16

Et en effet le carré-carré de 2 est 16.

Si A est un autre nombre, comme 4, et partant que le binome $A + 1$ soit 5, alors son carré-carré sera toujours suivant cette méthode, $1A^4 + 4A^3 + 6A^2 + 4A + 1$, qui signifie maintenant $1.4^4 + 4.4^3 + 6.4^2 + 4.4 + 1$.

C'est-à-dire, une fois le carré-carré de 4, savoir	256
Quatre fois le cube de 4, savoir	256
Six fois le carré de 4	96
Quatre fois la racine 4	16
Plus l'unité	1

dont la somme 625

fait le carré-carré de 5: et en effet le carré-carré de 5 est 625. Et ainsi des autres exemples.

Si on veut trouver le même degré du binome $A + 2$, il faut prendre de même $1A^4 + 4A^3 + 6A^2 + 4A + 1$, et ensuite écrire ces quatre nombres 2, 4, 8, 16, qui sont les quatre premiers degrés de 2, sous les nombres 4, 6, 4, 1, c'est-à-dire, sous chacun des nombres de la base, en laissant le premier: et cette

$$\begin{array}{r} 1A^4 + 4A^3 + 6A^2 + 4A + 1 \\ 2 4 8 16 \end{array}$$

et multiplier les nombres qui se répondent l'un par l'autre

$$\begin{array}{r} 1A^4 + 4A^3 + 6A^2 + 4A + 1 \\ 2 4 8 16 \end{array}$$

$$\begin{array}{r} \hline \text{en cette sorte } 1A^4 + 8A^3 + 24A^2 + 32A + 16 \\ \phantom{\text{en cette sorte }} 3 Y 2 \end{array}$$

Et

Et ainsi on aura le carré-carré du binôme $A + 2$; de sorte que si A est l'unité, ce carré-carré sera tel.

Une fois le carré-carré de l'unité	.	.	.	.	.	.	.	1
Huit fois le cube de l'unité	.	.	.	.	.	.	.	8
Vingt quatre fois le carré de 1	.	.	.	.	.	.	.	24
Trente-deux fois 1	.	.	.	.	.	.	.	32
Plus le carré-carré de 2	.	.	.	.	.	.	.	16
dont la somme	.	.	.	.	.	.	.	81

fera le carré-carré de 3 : et en effet 81 est le carré-carré de 3.

Et si A est 2, alors $A + 2$ sera 4, et son carré-carré sera

Une fois le carré-carré de A ou de 2, savoir	.	.	.	.	.	.	.	16
8. 2'	.	.	.	.	.	.	.	64
24. 2"	.	.	.	.	.	.	.	96
32. 2	.	.	.	.	.	.	.	64
Plus le carré-carré de 2	.	.	.	.	.	.	.	16
dont la somme	.	.	.	.	.	.	.	256

fera le carré-carré de 4.

De la même manière on trouvera le carré-carré de $A + 3$, en mettant de la même sorte $A^4 + 4A^3 + 6A^2 + 4A + 1$ et au-dessous les nombres 3 9 27 81 qui sont les quatre premiers degrés de 3 ; et, multipliant les nombres correspondants, on trouvera que le carré-carré de $A + 3$ est

$$1A^4 + 12A^3 + 54A^2 + 108A + 81.$$

Et ainsi à l'infini.

Si au lieu du carré-carré on veut le carré-cube, ou le cinquième degré, il faut prendre la base sixième, et en user comme j'ai dit de la cinquième ; et ainsi de tous les autres degrés.

On trouvera de même les puissances des Apotomes $A - 1$, $A - 2$, &c. La méthode en est toute semblable, et ne diffère qu'aux signes ; car les signes de + et de - se suivent toujours alternativement, et le signe de + est toujours le premier.

Ainsi le carré-carré de $A - 1$ se trouvera de cette sorte. Le carré-carré de $A + 1$ est, par la règle précédente, $1A^4 + 4A^3 + 6A^2 + 4A + 1$. Donc, en changeant les signes comme j'ai dit, on aura

$$1A^4 - 4A^3 + 6A^2 - 4A + 1.$$

Ainsi le cube de $A - 2$ se trouvera de même.

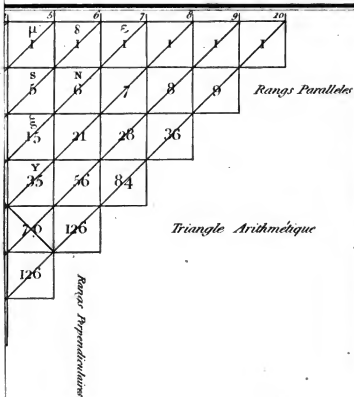
Car le cube de $A + 2$, par la règle précédente, est $A^3 + 6A^2 + 12A + 8$.

Donc le cube de $A - 2$ se trouvera, en changeant les signes,

$$A^3 - 6A^2 + 12A - 8.$$

Et ainsi à l'infini.

Je



Je ne donne point la démonstration de tout cela, parce que d'autres en ont déjà traité, comme Hérigogne *, outre que la chose est évidente d'elle-même.

TRAITE DES ORDRES NUMERIQUES.

JE présuppose qu'on a vu le Traité du Triangle Arithmétique, et son usage pour les Ordres numériques; autrement j'y renvoie ceux qui veulent voir ce discours, qui en est proprement une suite. J'y ai donné la définition des ordres numériques, et je ne la répéterai pas. J'y ai montré aussi que le Triangle Arithmétique n'est autre chose que la table des ordres numériques; ensuite de quoi il est évident que toutes les propriétés qui ont été données dans le Triangle Arithmétique entre les cellules ou entre les rangs, conviennent aux ordres numériques: de sorte que, si peu qu'on ait l'art d'appliquer les propriétés des uns aux autres, il n'y a point de proposition dans le Traité du Triangle qui n'ait ses conséquences touchant les divers ordres; et cela est tout ensemble, et si facile, et si abondant, que je suis fort éloigné de vouloir tout donner expressément; j'aimerois mieux laisser tout à faire, puisque la chose est si aisée; mais, pour me tenir entre ces deux extrêmes, j'en donnerai seulement quelques exemples, qui ouvriront le moyen de trouver tous les autres.

Par exemple: de ce qui a été dit dans une des conséquences du Traité du Triangle, que chaque cellule égale celle qui la précède dans son rang parallèle, plus celle qui la précède dans son rang perpendiculaire, j'en forme cette proposition touchant les Ordres numériques.

PROPOSITION PREMIÈRE.

Un nombre, de quelque ordre que ce soit, égale celui qui le précède dans son ordre, plus son corradical de l'ordre précédent: et par conséquent, le quatrième, par exemple, des pyramidaux égale le troisième pyramidal, plus le quatrième triangulo-triangular: ainsi le cinquième triangulo-triangular égale le quatrième triangle-triangular, plus le cinquième pyramidal, &c.

Autre exemple: de ce qui a été montré dans le Triangle, que chaque cellule, comme F, égale $E + B + \psi + \sigma$, c'est-à-dire, celle qui la précède dans son rang

* Hérigogne, Mathématicien du siècle passé, publia, pour la première fois, en 1635, un Cours de Mathématiques, latin et français, en 5 volumes in-8°. et pour la seconde fois, en 1644, le même Ouvrage en 7 volumes. J'ai sous la main la seconde édition. Dans le Traité d'Algèbre (Chap. V. pag. 31) l'Auteur détermine immédiatement par le calcul, ou par des multiplications effectives, le carré, le cube, &c., d'un binôme tel que $a + b$ ou $a - b$. On voit ici que Pascal trouve, au moyen de son Triangle Arithmétique, les co-efficients des différents termes d'un binôme élevé à une puissance quelconque entière et positive.

parallèle,

parallèle, plus toutes celles qui précèdent cette précédente dans son rang perpendiculaire, je forme cette proposition.

PROPOSITION II.

Un nombre, de quelque ordre que ce soit, égale sous eux tant de son ordre que de tous les précédents, dont la racine est moindre de l'unité que la sienne; et partant le quatrième des pyramidaux, par exemple, égale le troisième des pyramidaux, plus le troisième des triangulaires, plus le troisième des naturels, plus le troisième des unités, c'est à dire, l'unité.

D'où on peut maintenant tirer d'autres conséquences, comme celle-ci, que je donne pour ouvrir le chemin à d'autres pareilles.

PROPOSITION III.

Chaque nombre, de quelque cellule que ce soit, est composé d'autant de nombres qu'il y a d'ordres depuis le sien jusqu'au premier inclusivement, chacun desquels nombres est de chacun de ces ordres: ainsi un triangulo-triangulaire est composé d'un autre triangulo-triangulaire, d'un pyramidal, d'un triangulaire, d'un naturel et de l'unité.

Et si on veut en faire un problème, il pourra s'énoncer ainsi.

PROPOSITION IV. PROBLÈME.

Étant donné un nombre d'un ordre quelconque, trouver un nombre dans chacun des ordres depuis le premier jusqu'au sien inclusivement, dont la somme égale le nombre donné.

La solution en est facile: il faut prendre dans tous ces ordres les nombres dont la racine est moindre de l'unité que celle du nombre donné.

Autre exemple: de ce que les cellules correspondantes sont égales entre elles, il se conclut:

PROPOSITION V.

Que deux nombres de différents ordres sont égaux entre eux, si la racine de l'un est le même nombre que l'exposant de l'ordre de l'autre: et partant le troisième pyramidal est égal au quatrième triangulaire: le cinquième du huitième ordre est le même que le huitième du cinquième ordre, &c.

On n'auroit jamais achevé: par exemple,

PROPOSITION VI.

Tous les quatrièmes nombres de tous les ordres sont les mêmes que tous les nombres du quatrième ordre, &c.

Parce que les rangs parallèles et perpendiculaires qui ont un même exposant, sont composés de cellules toutes pareilles. Par cette méthode, on trouvera un rapport admirable en tout le reste, comme celui-ci:

PRO.

PROPOSITION VII.

Un nombre, de quelque ordre que ce soit, est au prochainement plus grand dans le même ordre, comme la racine du moindre est à cette même racine jointe à l'exposant de l'ordre, moins l'unité.

Ce qui s'ensuit de la quatorzième conséquence du Triangle, où il est montré que chaque cellule est à celle qui la précède dans son rang parallèle, comme l'exposant de la base de cette précédente à l'exposant de son rang perpendiculaire. Et, afin de ne rien cacher de la manière dont se tirent ces correspondances, j'en montrerai le rapport à découvert : il est un peu plus difficile ici que tantôt, parce qu'on ne voit point de rapport de la base des triangles avec les ordres des nombres ; mais voici le moyen de le trouver. Au lieu de l'exposant de la base, dont j'ai parlé dans cette quatorzième conséquence, il faut substituer l'exposant du rang parallèle, plus l'exposant du rang perpendiculaire, moins l'unité : ce qui produit le même nombre, et avec cet avantage, qu'on connoît le rapport qu'il y a de ces exposants avec les ordres numériques : car on fait qu'en ce nouveau langage, il faut dire, l'exposant de l'ordre, plus la racine, moins l'unité. Je dis tout ceci, afin de faire toucher la méthode pour faire et pour faciliter ces réductions. Ainsi on trouvera que

PROPOSITION VIII.

Un nombre, de quelque ordre que ce soit, est à son corradical de l'ordre suivant, comme l'exposant de l'ordre du moindre est à ce même exposant joint à leur racine commune, moins l'unité.

C'est la treizième conséquence du Triangle : ainsi on trouvera encore que

PROPOSITION IX.

Un nombre, de quelque ordre que ce soit, est à celui de l'ordre précédent, dont la racine est plus grande de l'unité que la sienne, comme la racine du premier à l'exposant de l'ordre du second.

Ce n'est que la même chose que la douzième conséquence du Triangle Arithmétique. J'en laisse beaucoup d'autres, chacune desquelles, aussi-bien que de celles que je viens de donner, peut encore être augmentée de beaucoup par de différentes énonciations : car au lieu d'exprimer ces proportions comme j'ai fait, en disant qu'un nombre est à un autre comme un troisième à un quatrième, ne peut-on pas dire que le rectangle des extrêmes est égal à celui des moyens ? Et ainsi multiplier les propositions, et non sans utilité ; car étant regardées d'un autre côté, elles donnent d'autres ouvertures. Par exemple, si on veut tourner autrement cette dernière proposition, on peut l'énoncer ainsi :

PRO-

PROPOSITION X.

Un nombre, de quelque ordre que ce soit, étant multiplié par la racine précédente, égale l'exposant de son ordre, multiplié par le nombre de l'ordre suivant procédant de cette racine.

Et parce que, quand quatre nombres sont proportionnels, le rectangle des extrêmes ou des moyens étant divisé par un des deux autres, donne pour quotient le dernier, on peut dire aussi :

PROPOSITION XI.

Un nombre, de quelque ordre que ce soit, étant multiplié par la racine précédente, et divisé par l'exposant de son ordre, donne pour quotient le nombre de l'ordre suivant qui procède de cette racine.

Les manières de tourner une même chose sont infinies : en voici un illustre exemple, et bien glorieux pour moi. Cette même proposition que je viens de rouler en plusieurs sortes, est tombée dans la pensée de notre célèbre Conseiller de Toulouse M. de Fermat ; et, ce qui est admirable, sans qu'il n'en eût donné la moindre lumière, ni moi à lui, il écrivoit dans sa Province ce que j'inventois à Paris, heure pour heure, comme nos Lettres écrites et reçues en même-temps le témoignent. Heureux d'avoir concouru en cette occasion, comme j'ai fait encore en d'autres d'une manière tout-à-fait étrange, avec un homme si grand et si admirable, et qui, dans toutes les recherches de la plus sublime Géométrie, est dans le plus haut degré d'excellence, comme ses Ouvrages, que nos longues prières ont enfin obtenus de lui, le feront bientôt voir à tous les Géomètres de l'Europe, qui les attendent ! La manière dont il a pris cette même proposition est telle.

En la progression naturelle qui commence par l'unité, un nombre quelconque étant mené dans le prochainement plus grand, produit le double de son triangle.

Le même nombre étant mené dans le triangle du prochainement plus grand, produit le triple de sa pyramide.

Le même nombre mené dans la pyramide du prochainement plus grand, produit le quadruple de son triangulo-triangular ; et ainsi à l'infini, par une méthode générale et uniforme.

Voilà comment on peut varier les énonciations. Ce que je montre en cette proposition s'entendant de toutes les autres, je ne m'arrêterai plus à cette manière accommodante de traiter les choses, laissant à chacun d'exercer son génie en ces recherches, où doit consister toute l'étude des Géomètres : car, si on ne fait pas tourner les propositions à tous sens, et qu'on ne se serve que du premier biais qu'on a envisagé, on n'ira jamais bien loin : ce sont ces diverses routes qui ouvrent les conséquences nouvelles, et qui, par des énonciations assorties au sujet, lient des propositions qui sembloient n'avoir aucun rapport dans les termes où elles étoient conçues d'abord. Je continuerai donc ce sujet en la manière dont on a accoutumé de traiter la Géométrie, et ce que j'en dirai sera comme un nouveau Traité des Ordres numériques ; et même je le donnerai en latin, parce qu'il se rencontre que je l'ai écrit ainsi en l'inventant.

D E

DE NUMERICIS ORDINIBUS TRACTATUS.

TRIANGULI Arithmetici tractatum, ipsiusque circa numericos ordines usum, supponit tractatus iste, ut et plerique è sequentibus : huc ergò mittitur lector horum cupidus ; ibi noscet quid sint ordines numerici, nempe, unitates, numeri naturales, trianguli, pyramides, triangulo-trianguli, &c. Quæ cum perlegerit, facillè hæc assequetur.

Hic propriè ostenditur connexio inter numerum cuiusvis ordinis cum sua radice et exponente sui ordinis, quæ talis est, ut ex his tribus, datis duobus quibuscumque, tertius inveniatur. Verbi gratiâ, datâ radice et exponente ordinis, numerus ipse datur ; sic dato numero et sui ordinis exponente, radix elicitur ; nec non ex dato numero et radice, exponens ordinis invenitur : hæc constituunt tria priora problemata ; quantum de summâ ordinum agit.

DE NUMERICORUM ORDINUM COMPOSITIONE :

P R O B L E M A P R I M U M .

Datis numeri cujuscumque radice et exponente ordinis, componere numerum ?

PRODUCTUS numerorum qui præcedunt radicem, dividat productum totidem numerorum quorum primus sit exponens ordinis : quotiens erit quæsitus numerus.

Propositum sit invenire numerum ordinis, verbi gratiâ, tertii, radice verò quintæ.

Productus numerorum 1, 2, 3, 4, qui præcedunt radicem 5, nempe 24, dividat productum totidem numerorum continuorum 3, 4, 5, 6, quorum primus sit exponens ordinis 3, nempe 360 : quotiens 15 est numerus quæsitus.

Nec difficilis demonstratio : eadem enim prorsus constructione, inventa est, ad finem Tractatus Trianguli Arithmetici, cellula quintæ seriei perpendicularis, tertię verò seriei parallelæ ; cujus cellulæ numerus idem est ac numerus quintus ordinis tertii, qui quæritur.

Potest autem et sic resolveri idem problema.

Productus numerorum qui præcedunt exponentem ordinis, dividat productum totidem numerorum continuorum quorum primus sit radix : quotiens est numerus quæsitus.

Sic in proposito exemplo, productus numerorum 1, 2, qui præcedunt exponentem ordinis 3, nempe 2, dividat productum totidem numerorum 5, 6, quorum primus sit radix 5, nempe 30 : quotiens 15, est numerus quæsitus.

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Nec

Nec differt hæc constructio à præcedente, nisi in hoc solo, quod in alterâ idem fit de radice, quod fit in alterâ de exponente ordinis: perinde ac si idem esset invenire, quintum numerum ordinis tertii, ac tertium numerum ordinis quinti; quod quidem verum esse jam ostendimus.

Hinc autem obiter colligere possumus arcanum numericum; cùm enim ambo illi quotientes 15, sint iidem, constat, divisores esse inter se ut dividendos. Animadvertemus itaque:

Si sint duo quilibet numeri, productus omnium numerorum primum ex ambobus propositis præcedentium, est ad productum totidem numerorum quorum primus est secundus ex his ambobus, ut productus ex omnibus qui præcedunt secundum ex illis ambobus, ad productum totidem numerorum continuorum quorum primus est primus ex iis ambobus propositis.

Hæc qui prosequeretur et demonstraret, novi fortassis tractatûs materiam reperiret: nunc autem, quia extrâ rem nostram sunt, sic pergamus.

DE NUMERICORUM ORDINUM RESOLUTIONE:

P R O B L E M A II.

Dato numero, ac exponente sui ordinis, invenire radicem?

POTEST autem et sic enuntiari.

Dato quolibet numero, invenire radicem maximi numeri ordinis numerici cujuscunque propositi, qui in dato numero contineatur?

Sit datus numerus quilibet, verbi gratiâ, 58, ordo verò numericus quicunque propositus, verbi gratiâ, sextus. Oportet igitur invenire radicem sexti ordinis numeri 58.

Exhibeatur ex unâ parte exponens ordinis . . . 6	et continuò	{ Exponatur ex alterâ parte numerus datus . . . 58
Multiplicetur ipse 6, per nu- merum 7, proximè majorem; sitque productus . . . 42	et continuò	{ Multiplicetur ipse numerus per 2, sitque productus 116
Multiplicetur iste productus per proximè sequentem mul- tiplicatorem 8, sitque produc- tus . . . 336	et continuò	{ Multiplicetur ipse productus per proximè sequentem mul- tiplicatorem 3, sitque produc- tus . . . 348
Multiplicetur iste productus per proximè sequentem mul- tiplicatorem 9, sitque produc- tus . . . 3024	et continuò	{ Multiplicetur iste productus per proximè sequentem mul- tiplicatorem 4, sitque produc- tus . . . 1392 Et.

Et sic in infinitum, donec ultimus productus exponentis 6, nempe 3024, major evadat quàm ultimus productus numeri dati, nempe 1392; et tunc absoluta est operatio: ultimus enim multiplicator dati numeri, nempe 4, est radix quæ quærebatur.

Igitur dico numerum sexti ordinis cujus radix est 4, nempe 56, maximum esse ejus ordinis qui in numero dato contineatur; seu dico numerum sexti ordinis cujus radix est 4, nempe 56, non esse majorem dato numero 58; numerum verò ejusdem ordinis proximè majorem, seu cujus radix est 5, nempe 126, esse majorem numero dato 58.

Etenim productus ille ultimus numeri dati, nempe 1392, factus est ex numero dato 58, multiplicato per productum numerorum 1, 2, 3, 4, nempe 24; productus verò præcedens hunc ultimum, nempe 348, factus est ex numero dato 58, multiplicato per productum numerorum 1, 2, 3, nempe 6.

Ergo productus numerorum 6, 7, 8, non est major producto numerorum 1, 2, 3, multiplicato per 58. Productus verò numerorum 6, 7, 8, 9, est major producto numerorum 1, 2, 3, 4, multiplicato per 58, *ex constructione*.

Jam numerus ordinis *sexti* cujus radix est 4, nempe 56, multiplicatus per numeros 1, 2, 3, æquatur producto numerorum 6, 7, 8, ex demonstratis in Tractatu de Ordinibus numericis.

Sed productus numerorum 6, 7, 8, non est major, *ex offensis*, producto numerorum 1, 2, 3, multiplicato per datum 58; igitur productus numerorum 1, 2, 3, multiplicatus per 56, non est major quàm idem productus numerorum 1, 2, 3, multiplicatus per datum 58. Igitur 56 non est major quàm 58.

Jam sit 126, numerus ordinis *sexti* cujus radix est 5. Igitur ipse 126, multiplicatus per productum numerorum 1, 2, 3, 4, æquatur producto numerorum 6, 7, 8, 9, ex Tractatu de Ordinibus numericis. Sed productus ille numerorum 6, 7, 8, 9, est major quàm numerus datus 58 multiplicatus per productum numerorum 1, 2, 3, 4, *ex offensis*. Igitur, numerus 126, multiplicatus per productum numerorum 1, 2, 3, 4, est major quàm numerus datus 58 multiplicatus per eundem productum numerorum 1, 2, 3, 4. Igitur numerus 126 est major quàm numerus datus 58.

Ergo numerus 56 *sexti* ordinis cujus radix est 4, non est major quàm numerus datus; numerus verò 126, ejusdem ordinis cujus radix 5 est proximè major, major est quàm datus numerus.

Ergo ipse numerus 56, maximus est ejus ordinis qui in dato contineatur, et ejus radix 4 inventa est.

Q. E. F. E. D.

DE NUMERICORUM ORDINUM RESOLUTIONE:

P R O B L E M A III.

Dato quolibet numero, et ejus radice, invenire ordinis exponentem?

NON differt hoc problema à præcedente; radix enim, et exponens ordinis, reciprocè convertuntur, ita ut dato numero, verbi gratiâ, 58, et ejus radice 4, reperietur exponens sui ordinis 6, eâdem methodo ac si dato numero ipso 58, et exponente ordinis 4, radix 6 esset invenienda: *quartus* enim numerus *sexti* ordinis idem est ac *sextus quarti*; ut jam demonstratum est.

DE NUMERICORUM ORDINUM SUMMA:

P R O B L E M A IV.

Propositi cujuslibet ordinis numerici, tot quot imperabitur, priorum numerorum summam invenire?

PROPOSITUM sit invenire summam *quinque*, verbi gratiâ, priorum numerorum ordinis, verbi gratiâ, *sexti*.

Inveniat ex præcedente numerus *quintus* (quia *quinque* priorum numerorum summa requiritur) ordinis *septimi*, nempe ejus qui propositum *sextum* proximè sequitur: ipse satisfaciet problemati.

Numericorum enim ordinum generatio talis est, ut numerus cujusvis ordinis æquetur summæ eorum omnium ordinis præcedentis quorum radices non sunt sua majores; ita ut quintus septimi ordinis æquetur, ex naturâ et generatione ordinum, quinque prioribus numeris sexti ordinis: quod difficultate caret.

C O N C L U S I O.

Methodus quâ ordinum resolutionem expedio est generalissima; verùm ipsam didi quæsi; quæ primò sese obtulit ea est.

Si dati numeri quærebatur radix tertii ordinis, ita procedebam. Sumatur *duplum numeri propositi*, istius *dupli radix quadrata* inveniat: hæc *quæsitæ est*, aut *saltem ea quæ unitate minor erit*.

Si dati numeri quæritur radix quarti ordinis, *multiplicetur numerus datus per 6*, nempe *per productum numerorum 1, 2, 3*; *producti inveniatur radix cubica*, ipsa, aut ea quæ unitate minor est, satisfaciet.

Si dati numeri quæritur radix quinti ordinis, *multiplicetur datus numerus per 24*, nempe *per productum numerorum 1, 2, 3, 4*, *productique inveniatur radix quartæ gradus*: ipsa, unitate minuta, satisfaciet problemati.

Et

Et ita reliquorum ordinum radices quærebam, constructione non generali, sed cuique ordini propriâ. Nec tamen idem mihi omnino displicebat: illa enim quæ resolvuntur potestates non generalior est: aliter enim extrahitur radix quadrata, aliter cubica, &c., quamvis ab eodem principio viæ illæ differentes procedant. Ut ergo nondum generalis potestatum resolutio data erat, sic et vix generalem ordinum resolutionem assequi sperabam: conatus tamen expectationem superantes eam quam tradidi præbuerunt generalissimam, et quidem amicis meis, universalium solutionum amatoribus doctissimis, graïssimam; à quibus excitatus et generalem potestatum purarum resolutionem tentare, ad instar generalis ordinum resolutionis, obtemperans quæsi, et satis feliciter mihi contigit reperisse, ut infra videbitur.

DE NUMERORUM CONTINUORUM PRODUCTIS,

Seu de numeris qui producuntur ex multiplicatione numerorum serie naturali procedentium.

NUMERI qui producuntur ex multiplicatione numerorum continuorum à nemine, quod sciam, examinati sunt. Idem nomen eis impono, nempe, *producti continuorum.*

Sunt autem qui ex duorum multiplicatione formantur, ut iste 20, qui ex 4 in 5 oritur; et possent dici *secundæ speciei.*

Sunt qui ex trium multiplicatione formantur, ut iste 120, qui ex 4 in 5 in 6 oritur; et dici possent *tertiæ speciei.*

Sic *quartæ speciei* dici possent qui ex quatuor numerorum continuorum multiplicatione formantur, et sic in infinitum: ita ut, ex multitudine *multiplicatorum*, species nominationem exponentis sortiretur; et sic nullus esset productus primæ speciei, nullus est enim productus ex uno tantum numero.

Primum hujus tractatuli theorema, illud est quod obiter in præcedente tractatu annotavimus, quod quærendo, reliqua invenimus, imò et generalem potestatum resolutionem; adeò strictâ connexionione sibi mutuo cohærent veritates!

PROPOSITIO PRIMA.

Si sint duo numeri quilibet, productus omnium numerorum primum præcedentium, est ad productum totidem numerorum continuorum à secundo incipientium, ut productus omnium numerorum secundum præcedentium, ad productum totidem numerorum continuorum à primo incipientium.

Sint duo numeri quilibet 5, 8: dico productum numerorum 1, 2, 3, 4, qui præcedunt 5, nempe 24, esse ad productum totidem continuorum numerorum 8, 9, 10, 11, nempe 7920: ut productum numerorum 1, 2, 3, 4, 5, 6, 7, qui præcedunt

precedunt 8, nempe 5040, ad productum totidem continuorum numerorum 5, 6, 7, 8, 9, 10, 11, nempe 1663200.

Etenim productus numerorum 5, 6, 7, ductus in productum istorum 1, 2, 3, 4, efficit productum horum 1, 2, 3, 4, 5, 6, 7; et idem productus numerorum 5, 6, 7, ductus in productum numerorum 8, 9, 10, 11, efficit productum horum 5, 6, 7, 8, 9, 10, 11; ergo, ut productus numerorum 1, 2, 3, 4, ad productum numerorum 1, 2, 3, 4, 5, 6, 7: ita productus numerorum 8, 9, 10, 11, ad productum numerorum 5, 6, 7, 8, 9, 10, 11. Q. E. D.

PROPOSITIO II.

Omnis productus à quotlibet numeris continuis, est multiplex producti à totidem numeris continuis quorum primus est unitas; et quotiens est numerus figuratus.

Sit productus quilibet, à tribus, verbi gratiâ, numeris continuis 5, 6, 7, nempe 210, et productus totidem numerorum ab unitate incipientium 1, 2, 3, nempe 6: dico ipsum 210 esse multiplicem ipsius 6, et quotientem esse numerum figuratum.

Etenim ipse 6, ductus in quintum numerum ordinis quarti, nempe 35, sequatur ipsi producto ex 5, 6, 7, ex demonstratis in Tractatu de Ordinibus numericis.

PROPOSITIO III.

Omnis productus à quotlibet numeris continuis est multiplex numeri cujusdam figurati, nempe ejus cujus radix est minimus ex his numeris, exponens verò ordinis est unitate major quàm multitudo eorum numerorum.

Hoc patet ex præcedente. Et unica utrique convenit demonstratio.

M O N I T U M.

Ambo divisores in his duabus propositionibus ostensi, tales sunt, ut alter alterius sit quotiens. Ita ut, si quilibet productus à quotlibet numeris continuis, sit divisus per productum totidem numerorum ab unitate incipientium, (quod secunda propositio docet fieri posse,) quotiens sit numerus figuratus in tertiâ propositione enuntiatus.

PROPOSITIO IV.

Omnis productus à quotlibet numeris continuis ab unitate incipientibus, est multiplex producti à quotlibet numeris continuis etiâ ab unitate incipientibus quorum multitudo minor est.

Sint quotlibet numeri continui ab unitate 1, 2, 3, 4, 5, quorum productus 120; quotlibet autem ex ipsis ab unitate incipientes 1, 2, 3, quorum productus 6: dico 120 esse multiplicem 6.

Etenim productus numerorum 1, 2, 3, 4, 5, sit ex producto numerorum 1, 2, 3, multiplicato per productum numerorum 4, 5.

PRO-

PROPOSITIO V.

Omnis productus à quolibet numeris continuus est multiplex producti à quolibet numeris continuus ab unitate incipientibus quorum multitudo minor est.

Etenim productus continuorum quorumlibet est multiplex totidem continuorum ab unitate incipientium *ex secundâ*; sed *ex quartâ* productus continuorum ab unitate est multiplex producti continuorum ab unitate quorum multitudo minor est. Ergo, &c.

PROPOSITIO VI.

Productus quolibet continuorum, est ad productum totidem proximè majorum, ut minimus multiplicatorum ad maximum.

Sint quolibet numeri 4, 5, 6, 7, quorum productus 840; et totidem proximè majores 5, 6, 7, 8, quorum productus 1680: dico 840 esse ad 1680, ut 4 ad 8.

Etenim productus numerorum 4, 5, 6, 7, est factus ex producto continuorum 5, 6, 7, multiplicato per 4; productus verò continuorum 5, 6, 7, 8, factus est ex eodem producto continuorum 5, 6, 7, multiplicato per 8. Ergo, &c.

PROPOSITIO VII.

Minimus productus continuorum cujuslibet speciei, ille est cujus multiplicatores ab unitate incipiunt.

Verbi gratiâ, minimus productus ex quatuor continuis factus, ille est qui producitur ex quatuor his continuis 1, 2, 3, 4, qui quidem multiplicatores 1, 2, 3, 4, ab unitate incipiunt. Hoc ex se et ex præcedentibus patet.

PRODUCTA:

PRODUCTA CONTINUORUM RESOLVERE,

*Seu Resolutio numerorum qui ex numeris progressionē naturali procedentibus
producuntur.*

P R O B L E M A.

*Dato quocunque numero, invenire tot quot imperabitur, numeros continuos ex quorum
multiplicatione factus numerus, sit maximus ejus speciei qui in dato numero con-
tineatur?*

*Oportet autem datum numerum non esse minorem producto totidem numerorum ab uni-
tate continuorum.*

DATUS sit numerus, verbi gratiā, 4335, oporteatque reperire, verbi gratiā, quatuor numeros continuos ex quorum multiplicatione factus numerus sit maximus, qui in dato 4335 contineatur, eorum omnium qui producuntur ex multiplicatione quatuor numerorum continuorum:

Sumantur ab unitate tot numeri continui quot sunt numeri inveniendi, nempe quatuor in hoc exemplo, 1, 2, 3, 4; quorum per productum 24, dividatur numerus datus; sitque quotiens 180. Ipsi quotientis invenitur radix ordinis numerici non quidem *quarti*, sed sequentis, nempe *quinti*, sitque ea 6: ipse 6 est primus numerus, secundus 7, tertius 8, quartus 9.

Dico itaque productum quatuor numerorum 6, 7, 8, 9, esse maximum numerum qui in dato contineatur, id est: dico productum quatuor numerorum 6, 7, 8, 9, nempe 3024, non esse majorem quam numerum datum 4335; productum verò quatuor proximè majorum numerorum 7, 8, 9, 10, nempe 5040, esse majorem numero dato 4335.

Etenim ex demonstratis in Tractatu de Ordinibus numericis, constat productum numerorum 1, 2, 3, 4, seu 24, ductum in numerum quinti ordinis cujus radix est 6, nempe 126, efficere numerum æqualem producto numerorum 6, 7, 8, 9, nempe 3024; similiter, et eundem productum numerorum 1, 2, 3, 4, nempe 24, ductum in numerum ejusdem ordinis quinti cujus radix est 7, efficere numerum æqualem producto numerorum 7, 8, 9, 10, nempe 5040.

Jam verò numerus quinti ordinis cujus radix est 6, nempe 126, cum sit maximus ejus ordinis qui in 180 contineatur, ex constructione patet ipsum 126 non esse majorem quam 180, numerum verò quinti ordinis cujus radix est 7, nempe 210, esse majorem quam ipsum 180.

Cum verò numerus 4335, divisus per 24, dederit 180, quotientem patet 180 ductum in 24, seu 4320, non esse majorem quam 4335, sed aut æqualem esse, aut differre numero minore quam 24.

Itaque cum sit 210 major quam 180 ex constructione, patet 210 in 24, seu 5040, majorem esse quam 180 in 24, seu 4320, et excessum esse ad minimum 24; numerus verò datus 4335, aut non excedit ipsum 4320, aut excedit numero minore

minore quàm 24. Ergò numerus 5040 major est quàm datus 4335; id est, productus numerorum 7, 8, 9, 10, major est dato numero.

Jam numerus 126 non est major quàm 180, ex constructione. Igitur 26 in 24, non est major quàm 180 in 24; sed 180 in 24, non est major dato numero, ex ostensis. Ergò 126 in 24, seu productus numerorum 6, 7, 8, 9, non est major numero dato: productus autem numerorum 7, 8, 9, 10, ipso major est. Ergò, &c. Q. E. F. E. D.

Sic ergò exprimi potest et enuntiatio, et generalis constructio.

Invenire tot quot imperabitur numeros progressionē naturali continuos, ex quorum multiplicatione ortus numerus, sit maximus ejus speciei qui in dato numero contineatur?

• Dividatur numerus datus, per productum totidem numerorum ab unitate serie naturali procedentium quot sunt numeri inveniendi; inventoque quotiente, assumatur ipsius radix ordinis numerici cujus exponents est unitate major quàm multitudo numerorum inveniendorum: ipsa radix est primus numerus, reliqui per incrementum unitatis in promptu habentur.

M O N I T U M.

Hæc omnia ex naturâ rei demonstrari poterant, absque Trianguli Arithmetici aut ordinum numericorum auxilio; non tamen fugienda illa connexio mihi visâ est, præsertim cùm ea sit quæ lumen primum dedit. Et quod amplius est, alia demonstratio laboriosior esset et prolixior.

NUMERICARUM POTESTATUM GENERALIS RESOLUTIO.

GENERALEM numericarum potestatum resolutionem inquirenti, hæc mihi venit in mentem observatio: nihil aliud esse querere radicem, *verbi gratia*, quadratam dati numeri, quàm querere duos numeros æquales quorum productus æquetur numero dato. Sic et querere radicem cubicam numeri dati nihil aliud esse quàm querere tres numeros æquales quorum productus sit numerus datus: et sic de cæteris.

Itaque potestatis cujuslibet resolutio, est indagatio totidem numerorum æqualium, quot exponents potestatis continet unitates, quorum productus æquetur dato numero; potestates enim ipse nihil aliud sunt quàm æqualium numerorum producti.

Sicut enim in præcedenti tractatu egimus de numeris qui producuntur ex multiplicatione numerorum naturali progressionē procedentium, sic et in hoc de potestatibus tractatu agitur de numeris qui producuntur ex multiplicatione numerorum æqualium.

Visum est itaque quàm proximos esse ambos hos tractatus, et nihil esse vicinius, producto ex æqualibus, quàm productum ex continuis solius unitatis incremento differentibus.

Quapropter potestatum resolutionem generalem, seu *productorum ex equalibus* resolutionem, non mediocriter provectam esse censui, cum eam *productorum ex continuis* generalis resolutio præcefferit.

Dato enim numero, cujus radix cujusvis gradûs quaeritur, verbi gratiâ, *quarti*, quaeruntur *quatuor* numeri æquales quorum productus æquetur dato; si ergò inveniantur ex præcedente tractatu *quatuor* continui quorum productus æquetur dato, quis non videt, inventam esse radicem quaesitam, cum ea sit unus ex his *quatuor* continuis? minimus enim ex his *quatuor*, *quater* sumptus et toties multiplicatus, manifestè minor est producto continuorum; maximus verò ex his *quatuor*, *quater* sumptus ac toties multiplicatus, manifestè major est producto continuorum; radix ergò quaesita unus ex illis est.

Verùm latet adhuc ipsa in multitudine; reliquum est igitur ut eligatur, et discernatur quis ex continuis satisfaciat quaestioni.

Huic perquisitioni nondum fortè satis incubui; crudam tamen meditationem proferam, aliâ, si digna videatur, diligentius elaborandam.

POSTULATUM.

Hoc autem prænotum esse postulo; quæ sit radix *quadrata* numeri 2, nempe 1; etenim 1 est radix maximi quadrati in 2 contenti. Sic et quæ sit radix *cubica* numeri 6, scilicet, qui ex *multiplicatione trium numerorum* 1, 2, 3, oritur, nempe 1. Sic et quæ sit radix *quarti* gradûs numeri 24, scilicet, qui ex *multiplicatione quatuor numerorum* 1, 2, 3, 4, oritur, nempe 2, et sic de cæteris gradibus. In unoquoque enim peto nosci radicem, istius gradûs, numeri qui producit ex *multiplicatione* tot numerorum continuorum ab unitate quot exponens gradûs propositi continet unitates. Sic ergò in investigatione radicis, verbi gratiâ, *decimi* gradûs postulo notam esse radicem istius *decimi* gradûs, numeri 3628800, qui producit ex *multiplicatione decem* priorum numerorum 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, nempe 5. Et hoc uno verbo dici potest. In unoquoque gradu, postulo notam esse radicem istius gradûs minimi producti totidem continuorum quot exponens gradûs continet unitates; minimus enim productus continuorum quotlibet, ille est cujus multiplicatores ab unitate sumunt exordium.

Nec sanè molesta hæc petitio est; in unoquoque enim gradu *unius* tantum numeri radicem suppono, in vulgari autem methodo, multò gravius in unoquoque gradu, *novem* priorum characterum, potestates exiguntur.

Notum sit ergo:

Producti numerorum	1, 2, nempe	2, rad. quadr. esse	1
Producti numeror.	1, 2, 3, nempe	6, rad. cub. esse	1
Producti num.	1, 2, 3, 4, nempe	24, rad. 4 gradûs esse	2
Producti num.	1, 2, 3, 4, 5, nempe	120, rad. 5 gradûs esse	2
Prod. num.	1, 2, 3, 4, 5, 6, nempe	720, rad. 6 gradûs esse	2
Prod. num.	1, 2, 3, 4, 5, 6, 7, nempe	5040, rad. 7 gradûs esse	3
&c.			

PROBLEMA

P R O B L E M A.

Dato quolibet numero, invenire radicem propositæ potestatis maximæ quæ in dato contineatur?

Sit datus numerus, verbi gratiâ, 4335, et invenienda sit radix gradûs, verbi gratiâ, *quarti* maximi numeri *quartii* gradûs, seu *quadrato-quadrati*, qui in dato numero contineatur.

Inveniantur, ex præcedente tractatu, *quatuor* numeri continui (quia *quartus* gradus proponitur) quorum productus sit maximus ejus speciei qui in 4335 contineatur; sintque ipsi 6, 7, 8, 9.

Radix quæsita est unus ex his numeris. Ut verò discernatur, sic procedendum est.

Sumatur ex postulato radix *quarti* gradûs numeri qui producitur ex multiplicatione *quatuor* priorum numerorum 1, 2, 3, 4, nempe radix *quadrato-quadrata* numeri 24, quæ est 2; ipse 2 cum minimo continuorum inventorum 6 unitate minuto, nempe 5, efficit 7.

Hic 7 est minimus qui radix quæsita esse possit; omnes enim inferiores sunt necessariò minores radice quæsita.

Jam triangulus *numeri* 4, qui exponens est propositi gradûs *quarti*, nempe 10, dividatur per ipsum exponentem 4, sitque quotiens 2 (*superfluum divisionis non curo*): ipse quotiens 2, eum minimo continuorum 6 junctus, efficit 8.

Ipsæ 8 est maximus qui radix esse possit; omnes enim superiores sunt necessariò majores radice quæsita.

Denique constituantur in *quarto* gradu ipsi extremi numeri 7, 8, nempe 2401, 4096; necnon et omnes qui inter ipsos interjecti sunt: quod ad generalem methodum dictum sit; hic enim nulli inter 7 et 8 interjacent, sed in remotissimis potestatibus quidam, quamvis perpauci, contingunt.

Harum potestatum, illa quæ æqualis erit dato numero, si ita eveniat, aut saltè quæ proximè minor erit dato numero, nempe 4096, satisfaciet problemati. Radix enim 8, unde orta est, ea est quæ quaeritur.

Sic ergò institui potest et enuntiatio et generalis constructio.

Invenire numerum qui in gradu proposito constitutus maximus sit ejus gradûs qui in dato numero contineatur?

Inveniantur ex tractatu præcedenti tot numeri continui, quot sunt unitates in exponente gradûs propositi, quorum productus sit maximus ejus speciei qui in dato numero contineatur. Et assumpto producto totidem continuorum ab unitate, inveniantur ejus radix gradûs propositi, ex postulato ipsa radix jungatur eum minimo continuorum inventorum unitate minuto: hic erit minimus extremus.

Jam triangulus exponentis ordinis per ipsum exponentem divisus quemlibet præbeat quotientem, qui cum minimo continuorum inventorum jungatur: hic erit maximus extremus.

Ambo hi extremi, ac numeri inter eos interpositi, in gradu proposito consistunt.

Harum potestatum, ea quæ dato numero erit aut æqualis aut proximè minor, satisfacit problemati; radix enim, unde orta est, radix quæsitæ est.

Horum demonstrationem, paratam quidem, sed prolixam etsi facilem, ac magis tædiofam quàm utilem suppressimus, ad illa, quæ plus afferunt fructus quàm laboris, vergentes.

COMBINATIONES.

DEFINITIONES.

COMBINATIONIS nomen diversè à diversis usurpatur; dicam itaque quo sensu intelligam.

Si exponatur multitudo quævis rerum quarumlibet, ex quibus liceat aliquam multitudinem assumere; verbi gratiâ, si ex *quatuor* rebus per litteras A, B, C, D, expressis, liceat *duas* quasvis ad libitum assumere: singuli modi quibus possunt eligi *duæ* differentes ex his *quatuor* oblatis, vocantur hic *combinationes*.

Experimento igitur patebit, *duas* posse assumi, inter *quatuor*, sex modis; potest enim assumi A et B, vel A et C, vel A et D, vel B et C, vel B et D, vel C et D.

Non constituo A et A, inter modos eligendi *duas*; non enim essent differentes: nec constituo A et B, et deinde B et A, tanquam differentes modos; ordine enim solummodò differunt; *ad ordinem autem non attendo*: ita ut uno verbo dixisse poteram, combinationes hic considerari quæ nec mutato ordine procedunt.

Similiter experimento patebit, *tria* inter *quatuor*, *quatuor* modis assumi posse, nempe ABC, ABD, ACD, BCD.

Sic et *quatuor* in *quatuor*, unico modo assumi posse, nempe ABCD.

His igitur verbis utar:

- 1 in 4 combinatur 4 modis seu combinationibus.
- 2 in 4 combinatur 6 modis seu combinationibus.
- 3 in 4 combinatur 4 modis seu combinationibus.
- 4 in 4 combinatur 1 modo seu combinatione.

Summa autem omnium combinationum quæ fieri possunt in 4 est 15; summa enim combinationum 1 in 4, et 2 in 4, et 3 in 4, et 4 in 4, est 15.

LEMMA PRIMUM.

Numerus quilibet non combinatur in minore.

Verbi gratiâ, 4 non combinatur in 2.

LEMMA

L E M M A II.

- 1 in 1 combinatur 1 combinatione.
 2 in 2 combinatur 1 combinatione.
 3 in 3 combinatur 1 combinatione.

Et sic generaliter omnis numerus semel tantum in equali combinatur.

L E M M A III.

- 1 in 1 combinatur 1 combinatione.
 1 in 2 combinatur 2 combinationibus.
 1 in 3 combinatur 3 combinationibus.

Et generaliter unitas in quovis numero toties combinatur quoties ipse continet unitatem.

L E M M A IV.

Si sint quatuor numeri; primus ad libitum; secundus unitate major quam primus; tertius ad libitum, modò non sit minor secundo; quartus unitate major quam tertius: multitudo combinationum primi in tertio, plus multitudine combinationum secundi in tertio, æquatur multitudini combinationum secundi in quarto.

Sint quatuor numeri, ut dictum est:

Primus ad libitum, verbi gratiâ	1.
Secundus unitate major, nempe	2.
Tertius ad libitum, modò non sit minor quam secundus, verbi gratiâ	3.
Quartus unitate major quam tertius, nempe	4.

Dico multitudinem combinationum 1 in 3, plus multitudine combinationum 2 in 3, æquari multitudini combinationum 2 in 4. *Quod ut paradigmata fiat evidentius:*

Assumantur tres characteres, nempe B, C, D; jam verò assumantur iidem tres characteres et unus præterea, A, B, C, D; deinde assumantur combinationes unius litteræ in tribus, B, C, D, nempe B, C, D; assumantur quoque omnes combinationes duarum litterarum in tribus B, C, D, nempe BC, BD, CD; denique assumantur omnes combinationes duarum litterarum in quatuor A, B, C, D, nempe AB, AC, AD, BC, BD, CD.

Dico itaque, tot esse combinationes duarum litterarum in quatuor A, B, C, D, quot sunt duarum in tribus B, C, D, et insuper quot unius in tribus B, C, D.

Hoc manifestum est ex generatione combinationum; combinationes enim duarum in quatuor formantur, partim ex combinationibus duarum in tribus, partim ex combinationibus unius in tribus; quod ita evidens fiet.

Ex combinationibus duarum in quatuor, nempe AB, AC, AD, BC, BD, CD, quedam sunt in quibus ipsa littera A usurpatur, ut istæ, AB, AC, AD; quedam quæ ipsa A carent, ut istæ, BC, BD, CD.

Porrò, combinationes illæ BC, BD, CD, duarum in quatuor A, B, C, D, quæ ipso A carent, constant ex residuis tribus B, C, D; sunt ergò combinationes
duarum

duarum in *tribus* B, C, D; igitur combinationes *duarum* in *tribus* B, C, D, sunt quoque combinationes *duarum* in *quatuor* A, B, C, D, nempe illæ quæ carent ipso A.

Illæ verò combinationes AB, AC, AD, *duarum* in *quatuor* A, B, C, D, in quibus A usurpatur, si ipso A spoliuntur, relinquent residuas litteras B, C, D, quæ sunt ex *tribus* litteris B, C, D, sũntque combinationes *unius* litteræ in *tribus* B, C, D; igitur combinationes *unius* litteræ in *tribus* B, C, D, nempe B, C, D, ascito A, efficiunt AB, AC, AD, quæ constituunt combinationes *duarum* litterarum in *quatuor* A, B, C, D, in quibus A usurpatur.

Igitur combinationes *duarum* litterarum in *quatuor* A, B, C, D, formantur partim ex combinationibus *unius* in *tribus* B, C, D, partim ex combinationibus *duarum* in *tribus* B, C, D; quare multitudo primarum æquatur multitudini reliquarum.

Q. E. D.

Eodem prorsus modo in reliquis ostendetur exemplis; verbi gratià:

Tot esse combinationes numeri	.	.	.	.	.	29 in 40.
quot sunt combinationes numeri	.	.	.	.	.	29 in 39.
et insuper quot sunt combinationes numeri	.	.	.	.	.	28 in 39.

Quatuor enim numeri 28, 29, 39, 40, conditionem requisitam habent.

Sic tot sunt combinationes numeri	.	.	.	.	.	16 in 56.
quot sunt combinationes numeri	.	.	.	.	.	16 in 55.
ac insuper quot sunt combinationes numeri	.	.	.	.	.	15 in 55.

&c.

LEMMA V.

In omni Triangulo Arithmetico summa cellularum seriei cujuscunque, æquatur multitudini combinationum exponentis seriei, in exponente trianguli.

Sic triangulus quilibet, verbi gratià, *quartus* G D λ: dico summam cellularum seriei cujuscunque, verbi gratià, *secundæ* $\Phi + \Psi + \theta$, æquari multitudini combinationum numeri 2, *exponentis secundæ seriei*, in numero 4, *exponente quarti trianguli*.

Sic dico summam cellularum seriei, verbi gratià, *quintæ* trianguli, verbi gratià, *octavi*, æquari multitudini combinationum numeri 5 in numero 8, &c.

Quamvis infiniti sint hujus propositionis casus, (sunt enim infiniti trianguli,) breviter tamen demonstrabo, positis duobus assumptis.

Primò, quod ex se patet, in *primo triangulo eam proportionem contingere*: summa enim cellularum unicæ suæ seriei, nempe numerus primæ cellulæ G, id est unitas, æquatur multitudini combinationum exponentis seriei, in exponente trianguli; hi enim exponentes sunt unitates; unitas verò in unitate unico modo *ex Lemmate 2 hujus* combinatur.

Secundò, *si ea proportio in aliquo triangulo contingat; id est, si summa cellularum uniuscujuscunque seriei trianguli cujuscunque, æquetur multitudini combinationum exponentis seriei in exponente trianguli: dico et eandem proportionem in triangulo proxime sequenti contingere.*

Hic

His assumptis, facile ostendetur in singulis triangulis eam proportionem contingere; contingit enim in primo, *ex primo assumpto*; immò et manifesta quoque ipsa est in secundo triangulo; ergò *ex secundo assumpto* et in sequenti triangulo contingit; quare et in sequenti, et in infinitum.

Totum ergò negotium in secundi assumpti demonstratione consistit; quod ita expeditur.

Sit triangulus quilibet, verbi gratiâ, *tertius*, in quo supponitur hæc proportio, id est, summam cellularum seriei *primæ* $G + \sigma + \pi$ æquari multitudini combinationum numeri 1, *exponentis seriei*, in numero 3, *exponente trianguli*; summam verò cellularum *secundæ seriei* $\phi + \psi$ æquari multitudini combinationum numeri 2, *exponentis seriei*, in numero 3, *exponente trianguli*; summam verò cellularum *tertiæ seriei*, nempe cellulam A, æquari combinationibus numeri 3, *exponentis seriei*, in 3 *exponente trianguli*: dico et eandem proportionem contingere et in sequenti triangulo *quarto*, id est, summam cellularum, verbi gratiâ, *secundæ seriei* $\phi + \psi + \theta$ æquari multitudini combinationum numeri 2, *exponentis seriei*, in numero 4, *exponente trianguli*.

Etenim $\phi + \psi$ æquatur multitudini combinationum numeri 2 in 3 *ex hypothesisi*; cellula verò θ æquatur, *ex generatione trianguli arithmetici*, cellulis $G + \sigma + \pi$; hæ verò cellulae æquantur *ex hypothesisi* multitudini combinationum numeri 1 in 3. Ergò cellulae $\phi + \psi + \theta$ æquantur multitudini combinationum numeri 2 in 3, plus multitudine combinationum numeri 1 in 3; hæ autem multitudines æquantur, *ex quarto Lemmate hujus*, multitudini combinationum numeri 2 in 4. Ergò summa cellularum $\phi + \psi + \theta$ æquatur multitudini combinationum numeri 2 in 4. Q. E. D.

Idem LEMMA V. problematicè enuntiatum.

Datis duobus numeris inæqualibus, invenire in triangulo arithmetico quot modis minor in majore combinetur?

Propositi sint duo numeri, verbi gratiâ, 4 et 6, oportet reperire in triangulo arithmetico quot modis 4 combinetur in 6.

PRIMA METHODUS.

Summa cellularum *quartæ seriei sexti* trianguli, satisfacit, *ex precedente, nempe cellulae* $D + E + F$.

Hoc est numeri $1 + 4 + 10$, seu 15: ergò 4 in 6 combinatur 15 modis.

SECUNDA METHODUS.

Cellula *quinta*, basis *septimæ* K, satisfacit; illi numeri 5, 7, sunt proximè majores bis 4, 6.

Etenim illa cellula, nempe K, seu 15, æquatur summæ cellularum *quartæ seriei sexti* trianguli $D + E + F$, ex generatione.

MONITUM.

MONITUM.

In basi *septimâ* sunt septem cellulae, nempe V, Q, K, ρ, ξ, N, ζ, ex quibus *quinta* assumenda est. Potest autem ipsa duplici modo assumi; sunt enim duae basis extremitates V, ζ: si ergo ab extremo V inchoaveris, erit V prima, Q secunda, K tertia, ρ quarta, ξ quinta quaesita; si verò à ζ incipias, erit ζ prima, N secunda, ξ tertia, ρ quarta, K quinta quaesita. Sunt igitur duae quae possunt dici, *quintae*: sed quoniam ipsae sunt aequè ab extremitatibus remotae, ideòque reciprocae, sunt ipsae eadem; quare indifferenter assumi alterutra potest, et ab alterutra basis extremitate inchoari.

MONITUM.

Jam satis patet, quam benè conveniant combinationes et triangulus arithmeticus, et ideò, proportionem inter series, aut inter cellulas trianguli, observatas, ad combinationum rationes protendi, ut in sequentibus videre est.

PROPOSITIO PRIMA.

Duo quilibet numeri aequè combinantur in eo quod amborum aggregatum est.

Sint duo numeri quilibet 2, 4; quorum aggregatum 6: dico numerum 2 toties combinari in 6, quoties ipse 4 in eodem 6 combinatur, nempe singulos, modis 15.

Hoc nihil aliud est quam consecutio 4 trianguli arithmetici, et potest hoc uno verbo demonstrari; *cellulae enim reciprocae sunt eadem*. Si verò ampliori demonstratione egere videatur, hæc satisfaciunt.

Multitudo combinationum numeri 2 in 6 æquatur, ex 5 *Lemmate*, seriei *secundæ* trianguli *sexti*, nempe cellulis $\phi + \psi + \theta + R + S$, seu cellulae ξ; sic multitudo quoque combinationum numeri 4 in 6 æquatur, ex eodem, seriei *quartæ* trianguli *sexti*, nempe cellulis $D + E + F$, seu cellulae K; ipsa verò K, est reciproca ipsius ξ, ideòque ipsi æqualis; quare et multitudo combinationum numeri 2 in 6, æquatur multitudini combinationum numeri 4 in 6. Q. E. D.

COROLLARIUM.

Ergo omnis numerus toties combinatur in proximè majori quot sunt unitates in ipso majori.

Verbi gratiâ, numerus 6 in 7 combinatur *septiès*, et 4 in 5 *quinguiès*, &c. Ambo enim numeri 1, 6, aequè combinantur in aggregato eorum 7, ex *propositione hæc* 1; sed 1 in 7 combinatur *septiès*, ex *Lemmate* 3. Igitur 6 in 7 combinatur quoque *septiès*.

PRO-

PROPOSITIO II.

Si duo numeri combinentur in numero quod amborum aggregatum est unitate minuto; multitudines combinationum erunt inter se, ut ipsi numeri reciprocè.

Hoc nihil aliud est quàm confessatio 17 trianguli arithmetici.

Sint duo quilibet numeri 3, 5, quorum summa 8, unitate minuta, est 7: dico multitudinem combinationum numeri 3 in 7, esse ad multitudinem combinationum numeri 5 in 7, ut 5 ad 3.

Multitudo enim combinationum numeri 3 in 7 æquatur, ex 5 *Lemmate*, *tertiæ seriei septimi trianguli arithmetici*, nempe $A + B + C + \omega + \xi$, seu 35: Multitudo autem combinationum numeri 5 in 7, æquatur, ex eodem, *quintæ seriei ejusdem septimi trianguli*, nempe $H + M + K$, seu 21. In triangulo autem *septimo*, series *quinta* et *tertia* sunt inter se ut 3 ad 5, ex *confessione* 17 *trianguli arithmetici*: aggregatum enim exponentium serierum 5, 3, nempe 8, æquatur exponenti trianguli 7 unitate aucto.

PROPOSITIO III.

Si numerus combinetur, primò in numero qui sui duplus est, deindè in ipsomet numero duplo unitate minuto, prima combinationum multitudo secundæ dupla erit.

Hoc nihil aliud est quàm confessatio 10 trianguli arithmetici.

Sit numerus quilibet 3, cujus duplus 6, qui, unitate minutus, est 5: dico multitudinem combinationum numeri 3 in 6, duplam esse multitudinis combinationum numeri 3 in 5.

Possèm uno verbo dicere omnis enim cellula dividendis dupla est præcedentis *corradicalis*: sic autem demonstro.

Multitudo enim combinationum numeri 3 in 6 æquatur, ex 5 *Lemmate*, cellulæ 4 basis 7, nempe e , seu 20; quæ quidem e , medium basis occupat locum, quod inde procedit quòd 3 sit dimidium 6, unde sit ut 4, proximè major quàm 3, medium occupet locum in numero 7 proximè majori quàm 6. Igitur ipsa cellula quarta e est in dividende; quare dupla est cellulæ F, seu ω , ex 10 *confessione* trianguli arithmetici; quæ quidem ω est quoque quarta cellula basis sextæ; ideòque, ex *Lemmate* 5, ipsa ω , seu F, æquatur multitudini combinationum numeri 3 in 5. Ergò multitudo combinationum 3 in 6 dupla est multitudinis combinationum 3 in 5. Q. E. D.

PROPOSITIO IV.

Si sint duo numeri proximi, et alius quilibet in utroque combinetur, multitudo combinationum quæ sunt in majore, erit ad alteram multitudinem, ut major numerus, ad ipsamet majorem dempto eo qui combinatus est.

Sint duo numeri unitate differentes 5, 6; et alius quilibet 2 combinetur in 5, et deindè in 6: dico multitudinem combinationum ipsius 2 in 6, esse ad multitudinem combinationum ipsius 2 in 5, ut 6 ad 5 — 2.

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Hoc

Hoc ex 13 consecratione trianguli arithmetici est manifestum, et sic ostenditur.

Multitudo enim combinationum ipsius 2 in 6, æquatur summæ cellularum seriei 2 trianguli 6, nempe $\phi + \psi + \theta + R + S$, ex *Lemmate* 5, hoc est, cellulæ ξ , seu 15. Sed, ex eodem, multitudo combinationum ejusdem 2 in 5, æquatur summæ cellularum seriei 2 trianguli 5, nempe $\phi + \psi + \theta + R$, seu cellulæ ω , seu 10; est autem cellula ξ ad ω ut 6 ad 4, hoc est, ut 6 ad 6 — 2, ex 13 consecratione trianguli arithmetici.

PROPOSITIO V.

Si duo numeri proximi in alio quolibet combinentur, erit multitudo combinationum minoris ad alteram, ut major numerus combinatus ad numerum in quo ambo combinati sunt dempto minore numero combinato.

Sint duo quilibet numeri proximi 3, 4, et alius quilibet 6: dico multitudinem combinationum minoris 3 in 6, esse ad multitudinem combinationum majoris 4 in 6, ut 4 ad 6 — 3.

Hæc cum 11 consecratione trianguli arithmetici convenit, et sic ostenditur.

Multitudo enim combinationum numeri 3 in 6 æquatur, ex *Lemmate* 5, summæ cellularum seriei 3 trianguli 6, nempe $A + B + C + \omega$, seu cellulæ p , seu 20. Multitudo verò combinationum numeri 4 in 6 æquatur, ex eodem, summæ cellularum seriei 4 trianguli 6, nempe $D + E + F$, seu cellulæ K , seu 15; est autem p ad K ut 4 ad 3, seu ut 4 ad 6 — 3, ex consecratione 11 trianguli arithmetici.

PROPOSITIO VI.

Si sint duo numeri quilibet, quorum minor in majore combinetur; sint autem et alii duo his proximè majores, quorum minor in majore quoque combinetur: erunt multitudines combinationum inter se, ut bi ambo ultimi numeri.

Sint duo quilibet numeri 2, 4, alii verò his proximè majores 3, 5: dico multitudinem combinationum numeri 2 in 4, esse ad multitudinem combinationum numeri 3 in 5, ut 3 ad 5.

Consecratio 12 trianguli arithmetici hanc continet, et sic demonstratur.

Multitudo enim combinationum ipsius 2 in 4 æquatur, ex *Lemmate* 5, summæ cellularum seriei 2 trianguli 4, nempe $\phi + \psi + \theta$, seu cellulæ C , seu 6. Multitudo verò combinationum numeri 3 in 5 æquatur, ex eodem, summæ cellularum seriei 3 trianguli 5, nempe $A + B + C$, seu cellulæ F , seu 10; est autem C ad F , ut 3 ad 5, ex 12 consecratione trianguli arithmetici.

LEMMA

L E M M A VI.

Summa omnium cellularum basis trianguli cujuslibet arithmetici, unitate minuta, æquatur summæ omnium combinationum quæ fieri possunt in numero qui proximè minor est quàm exponens basis.

Sit triangulus quilibet arithmeticus, verbi gratiâ, quintus $GH\mu$: dico summam cellularum suæ basis $H + E + C + R + \mu$, minus unitate, seu minus undè ex extremis H vel μ , æquari summæ omnium combinationum quæ fieri possunt in numero 4 qui proximè minor est quàm exponens basis 5. Id est: dico summam cellularum $R + C + E + H$ (*supprimo enim extremam μ*) id est, $4 + 6 + 4 + 1$, seu 15, æquari multitudini combinationum numeri 1 in 4, nempe 4; plus multitudine combinationum numeri 2 in 4, nempe 6; plus multitudine combinationum numeri 3 in 4, nempe 4; plus multitudine combinationum numeri 4 in 4, nempe 1. Quæ quidem sunt omnes combinationes quæ fieri possunt in 4; superiores enim numeri 5, 6, 7, &c. non combinantur in numero 4: major enim numerus in minore non combinatur.

Multitudo enim combinationum numeri 1 in 4 æquatur, ex 5 Lemmate, cellulæ 2 basis 5, nempe R, seu 4. Multitudo verò combinationum numeri 2 in 4 æquatur cellulæ 3 basis 5, nempe C, seu 6. Multitudo quoque combinationum numeri 3 in 4 æquatur cellulæ 4 basis 5, nempe E, seu 4. Multitudo denique combinationum numeri 4 in 4 æquatur cellulæ 5 basis 5, nempe H, seu 1. Igitur summa cellularum basis quintæ, demptâ extremâ, seu unitate, æquatur summæ omnium combinationum quæ possunt fieri in 4.

P R O P O S I T I O VII.

Summa omnium combinationum quæ fieri possunt in numero quolibet, unitate aucta, est numerus progressionis duplæ quæ ab unitate sumit exordium, quippè ille cujus exponens est numerus proximè major quàm datus.

Sit numerus quilibet, verbi gratiâ, 4: dico summam omnium combinationum quæ fieri possunt in 4, nempe 15, unitate auctam, nempe 16, esse numerum quintum (nempe proximè majorem quàm quartum) progressionis duplæ quæ ab unitate sumit exordium.

Hoc nihil aliud est quàm 7 confectio trianguli arithmetici, et sic uno verbo demonstrari posset, *omnis enim basis est numerus progressionis duplæ*: sic tamen demonstrstro.

Summa enim combinationum omnium quæ fieri possunt in 4, unitate aucta, æquatur, ex Lemmate 6, summæ cellularum basis quintæ; ipsa verò basis est quintus numerus progressionis duplæ quæ ab unitate sumit exordium, ex 7 confectatione trianguli arithmetici.

PROPOSITIO VIII.

Summa omnium combinationum quæ fieri possunt in numero quolibet, unitate aucta, dupla est summæ omnium combinationum quæ fieri possunt in numero proximè minore, unitate aucta.

Hoc convenit cum 6 consecratione trianguli arithmetici, nempe *omnis basis dupla est præcedentis*: sic autem ostendimus.

Sint duo numeri proximi 4, 5: dico summam combinationum quæ fieri possunt in 5, nempe 31, unitate auctam, nempe 32, esse duplam summæ combinationum quæ fieri possunt in 4, nempe 15, unitate auctæ, nempe 16.

Summa enim combinationum quæ fieri possunt in 5, unitate aucta, æquatur, *ex præcedente*, sexto numero progressionis duplæ. Summa verò combinationum quæ fieri possunt in 4, unitate aucta, æquatur, *ex eisdem*, quinto numero progressionis duplæ. Sextus autem numerus progressionis duplæ, duplus est proximè præcedentis, nempe *quinti*.

PROPOSITIO IX.

Summa omnium combinationum quæ fieri possunt in quovis numero, unitate minuta, dupla est summæ combinationum quæ fieri possunt in numero proximè minori.

Hæc cum præcedente omnino convenit.

Sint duo numeri proximi 4, 5: dico summam omnium combinationum quæ fieri possunt in 5, nempe 31, unitate minutam, nempe 30, esse duplam omnium combinationum quæ fieri possunt in 4, nempe 15.

Etenim *ex præcedente* summa combinationum quæ fiunt in 5, unitate aucta, dupla est summæ combinationum quæ fiunt in 4, unitate auctæ: si ergo ex *minori* summâ auferatur unitas, et ex *duplâ summâ* auferantur duæ unitates, reliquum summæ *duplæ*, nempe *summæ combinationum quæ fiunt in 5, unitate minuta*, remanebit *dupla residui alterius summæ*, nempe *summæ combinationum quæ fiunt in 4*.

PROPOSITIO X.

Summa omnium combinationum quæ fieri possunt in quolibet numero, minuta ipsomet numero, æquatur summæ omnium combinationum quæ fieri possunt in singulis numeris præfisso minoribus.

Hæc cum 8 consecratione trianguli arithmetici concurret, quæ sic habet, *basis quælibet, unitate minuta, æquatur summæ omnium præcedentium*. Sic autem ostendo.

Sit numerus quilibet 5: dico summam omnium combinationum quæ possunt fieri in 5, nempe 31, ipso 5 minutam, nempe 26, æquari summæ omnium combinationum quæ possunt fieri in 4, nempe 15; plus summâ omnium quæ possunt fieri in 3, nempe 7; plus summâ omnium quæ possunt fieri in 2, nempe 3; plus eâ quæ potest fieri in 1, nempe 1, quarum aggregatus est 26.

Etenim proprium numerorum hujus progressionis duplæ illud est, ut quilibet

cx

ex ipsis, verbi gratiâ, sextus 32, exponente suo minutus, nempe 6, id est 26, æquetur summæ inferiorum numerorum hujus progressionis, nempe $16 + 8 + 4 + 2 + 1$ unitate minorum, nempe $15 + 7 + 3 + 1 + 0$, nempe 26. Unde facilis est demonstratio hujus propositionis.

PROBLEMA PRIMUM.

Dato quovis numero, invenire summam omnium combinationum quæ in ipso fieri possunt.

Absque triangulo arithmetico.

Numerus progressionis duplæ quæ ab unitate sumit exordium cujus exponents proximè major est quàm numerus datus, satisfaciet problemati, modò unitate minuatur.

Sit numerus datus, verbi gratiâ, 5; quæritur summa omnium combinationum quæ in 5 fieri possunt?

Numerus sextus progressionis duplæ quæ ab unitate incipit, nempe 32, unitate minutus, nempe 31, satisfacit, ex *Lemmate* 6; ergò possunt fieri 31 combinationes in numero 5.

PROBLEMA II.

Datis duobus numeris inæqualibus, invenire quot modis minor in majore combinetur.

Absque triangulo arithmetico.

Hoc est propriè ultimum problema Tractatus Trianguli Arithmetici, quod sic resolvo.

Productus numerorum qui præcedunt differentiam datorum unitate auctam, dividat productum totidem numerorum continuorum quorum primus sit minor datorum unitate auctus: quotiens est quæsitus.

Sint dati numeri 2, 6: oportet invenire quot modis 2 combinetur in 6.

Assumatur eorum differentia 4, quæ, unitate aucta, est 5. Jam assumantur omnes numeri qui præcedunt ipsum 5, nempe 1, 2, 3, 4, quorum productus sit 24. Assumantur totidem numeri continui quorum primus sit 3, nempe proximè major quàm 2, qui minor est ex ambobus datis, nempe 3, 4, 5, 6, quorum productus 360, dividatur per præcedentem productum 24: quotiens 15 est numerus quæsitus. Ita ut numerus 2 combinetur in numero 6 modis 15 differentibus.

Nec difficilis demonstratio. Si enim queratur in triangulo arithmetico quot modis 2 combinetur in 6, assumenda est cellula 3 basis 7, ex *Lemmate* 5, nempe cellula ξ , et ipsius numerus exponet multitudinem combinationum numeri 2 in 6. Ut autem inveniat numerus cellulæ ξ cujus radix est 5, et exponents seriei 3, oportet, ex *problemate trianguli arithmetici*, ut productus numerorum qui præcedunt 5, dividat productum totidem numerorum continuorum quorum primus sit 3, et quotiens erit numerus cellulæ ξ : sed idem divisor ac idem dividendus in constructione hujus propositus est; quare et eundem quotientem sortita est divisio. Ergò in hac constructione repertus est numerus cellulæ ξ ; quare et exponents multitudinis combinationum numeri 2 in 6, quæ quærebatur. Q. E. P. E. D.

M O N I T U M.

Hoc problemate tractatum hunc absolvere constitueram; non tamen omnino sine molestia, cum multa alia parata haberem: sed ubi tanta ubertas, vi moderanda est fames. His ergo pauca hæc subijciam.

Eruditissimus ac mihi charissimus, D. D. de Ganieres, circa combinationes, assiduo ac perutili labore, more suo, incumbens, ac indigens facili constructione ad inveniendum quotiens numerus datus in alio dato combinetur, hanc ipse sibi praxim instituit.

Datis numeris, verbi gratia, 2, 6, invenire quot modis 2 combinetur in 6.

Assumatur, *inquit*, progressio duorum terminorum quia minor numerus est 2, inchoando à majore 6, ac retrogrediendo, seu detrahendo unitatem ex unoquoque termino, hoc modo, 6, 5; deinde assumatur altera progressio, inchoando ab ipso minore 2, ac similiter retrogrediendo, hoc modo, 2, 1. Multiplicentur invicem numeri primæ progressionis 6, 5; sitque productus 30. Multiplicentur et numeri secundæ progressionis 2, 1; sitque productus 2. Dividatur major productus per minorem: quotiens est quæsitus.

Excellentem hanc solutionem ipse mihi ostendit, ac etiam demonstrandam proposuit; ipsam ego sanè miratus sum, sed difficultate territus vix opus suscepi, et ipsi authori relinquendum existimavi; attamen, trianguli arithmetici auxilio, sic proclivis facta est via.

In 5 Lemmate hujus, ostendi numerum cellulæ ξ , exponere multitudinem combinationum numeri 2 in 6; quare ipsius reciproca cellula K eundem numerum continebit. Verùm cellula ipsa K est quotiens divisionis in quâ productus numerorum 1, 2, qui præcedunt 3 radicem cellula K, dividit productum totidem numerorum continuorum quorum primus est 5 exponens seriei cellula K, nempe numerorum 5, 6. Sed ille divisor ac dividendus sunt iidem ac illi qui in constructione amici sunt propositi; igitur eundem quotientem sortitur divisio: quare ipse exponit multitudinem combinationum numeri 2 in 6, quæ querebatur.

Q. E. D.

Hac demonstratione affectutâ, jam reliqua, quæ invitus supprimebam, libenter omitto; adeò dulce est amicorum memorari.

POTESTATUM

POTESTATUM NUMERICARUM SUMMA.

M O N I T U M.

DATIS, ab unitate, quocunque numeris continuis, verbi gratiâ, 1, 2, 3, 4, invenire summam quadratorum eorum, nempe $1 + 4 + 9 + 16$, id est 30, tradiderunt veteres; imò etiam et summam cuborum eorundem; ad reliquas verò potestates non protraxerunt suas methodos, his solummodò gradibus proprias. Hic autem exhibetur, non solum summa quadratorum, et cuborum, sed et quadrato-quadratorum, et reliquarum in infinitum potestatum. Ex non solum à radicibus ab unitate continuis, sed à quolibet numero initium summentibus, verbi gratiâ, numerorum 8, 9, 10, &c. Et non solum numerorum qui progressionē naturali procedunt, sed et eorum omnium qui progressionē, verbi gratiâ, cujus differentiā est 2, aut 3, aut 4, aut alius quilibet numerus, formantur; ut istorum, 1, 3, 5, 7, &c.; vel horum, 2, 4, 6, 8, &c. qui per incrementum binarii augentur; aut horum, 1, 4, 7, &c. qui per incrementum ternarii; et sic de cæteris: sed, *et quod amplius est*, à quolibet numero exordium sumat illa progressio, sive incipiat ab unitate, ut isti 1, 4, 7, 10, 13, &c. qui sunt ejus progressionis quæ per incrementum ternarii procedit, et ab unitate sumit exordium; sive ab aliquo hujus progressionis numero incipiat, ut isti, 7, 10, 13, 16, 19; sive, *quod ultimum est*, à numero qui non sit ejus progressionis, ut isti 5, 8, 11, 14, quorum progressio per ternarii differentiam procedit, et à numero 5, ipsi progressionī extraneos, exordium sumit. Et, *quod sanè feliciter inventum est*, tam multos differentes casus, unica ac generalissima resolvit methodus; adeò simplex, ut absque litterarum auxilio, quibus difficiliore egent enuntiationes, paucis lineis contineatur: ut ad finem problematis sequentis patebit.

D E F I N I T I O.

Si binomium, cujus alterum nomen sit A, alterum verò numerus quilibet ut 3, nempe $A + 3$, ad quamlibet constituatur potestatem, ut, verbi gratiâ, ad quartum gradum, cujus hæc sit expositio

$$A^4 + 12.A^3 + 54.A^2 + 108.A + 81:$$

Ipsi numeri 12, 54, 108, per quos ipse A multiplicator in singulis gradibus, quique partim ex numeris figuratis, partim ex numero 3, qui binomii est secundum nomen, formantur, vocabuntur *co-efficientes* potestatum ipsius A.

Erit ergo in hoc exemplo 12 *co-efficientes* A cubi, et 54 *co-efficientes* A quadrati, et 108 *co-efficientes* A radices.

Numerus verò 81 *numerus absolutus* dicitur.

LEMMA.

L E M M A.

Sit radix quælibet 14; altera verò sit binomium $14 + 3$, cujus primum nomen sit 14, alterum verò alius quilibet numerus 3; ita ut harum radicum 14, et $14 + 3$, differentia sit 3. Constituantur ipsæ in quolibet gradu, ut, verbi gratiâ, in quarto: ergò quartus gradus radicis 14 est 14^4 ; quartus verò gradus binomii $14 + 3$ est . . . $14^4 + 12 \cdot 14^3 + 54 \cdot 14^2 + 108 \cdot 14 + 81$.

Cujus quidem binomii primam nomen 14, eisdem co-efficientes sortitur in singulis gradibus, quos A sortitus est in similibus gradibus in expositione ejusdem gradus binomii $A + 3$; quod rationi consentaneum est: harum verò potestatum, nempe, hujus, 14^4 , et hujus, $14^4 + 12 \cdot 14^3 + 54 \cdot 14^2 + 108 \cdot 14 + 81$, differentia est $12 \cdot 14^3 + 54 \cdot 14^2 + 108 \cdot 14 + 81$: quæ quidem constat primò, ex radice 14 constitutâ in singulis gradibus proposito gradui quarto inferioribus, nempe in *tertia*, in *secundo*, et in *primo*, et in unoquoque multiplicatâ per co-efficientes quos A sortitur in similibus gradibus, in expositione ejusdem gradus binomii $A + 3$; deindè ex ipso numero 3, qui est differentia radicum, constituto in proposito quarto gradu; numerus enim absolutus 81 est quartus gradus radicis 3. Hinc igitur elicietur Canon iste:

Duarum similium potestatum differentia, æquatur, differentie radicum constitutæ in eodem gradu in quo sunt potestates propositiæ; plus minori radice constitutâ in singulis gradibus proposito gradui inferioribus ac in unoquoque multiplicatâ per co-efficientes quos A sortiretur in similibus gradibus, si binomium cujus primum nomen esset A, alterum verò esset differentia radicum, constitueretur in eadem potestate propositiâ.

Sic ergò differentia inter 14^4 et 11^4 , erit
 $12 \cdot 11^3 + 54 \cdot 11^2 + 108 \cdot 11 + 81$.

Differentia enim radicum est 3. Et sic de cæteris.

Ad summam potestatum cujuslibet progressionis invenientiam unica ac generalis methodus.

Datis quocunque numeris, in qualibet progressionem, à quovis numero incœpante, invenire quarumvis potestatum eorum summam?

Quilibet numerus 5, sit initium progressionis quæ per incrementum cujusvis numeri, verbi gratiâ, ternarii, procedat, et in eâ progressionem dati sint quolibet numeri, verbi gratiâ, isti, 5, 8, 11, 14, qui omnes in quâcunque potestate constituantur, ut, verbi gratiâ, in tertio gradu, seu cubo. Oportet invenire summam horum cuborum, nempe $5^3 + 8^3 + 11^3 + 14^3$.

Cubi illi sunt 125 + 512 + 1331 + 2744, quorum summa est 4712, quæ quæritur; et sic invenitur.

Exponatur

Exponatur binomium $A + 3$, cujus primum nomen sit A , alterum verò sit numerus 3, qui est differentia progressionis.

Constituatur binomium hoc $A + 3$ in gradu *quarto*, qui proximè superior est proposito *tertio*; sique hæc ejus expositio

$$A^4 + 12. A^3 + 54. A^2 + 108. A + 81.$$

Jam assumatur numerus 17, qui in progressionē propositā proximè sequitur ultimum progressionis terminum datum 14. Et constituto ipso 17 in eodem gradu *quarto*, nempe 83521, auferantur ab eo, hæc:

Primò, summa numerorum propositorum $5 + 8 + 11 + 14$, nempe 38 multiplicata per numerum 108, qui est co-efficiens ipsius A radicis.

Secundò, summa quadratorum eorundem numerorum 5, 8, 11, 14, multiplicata per numerum 54, qui est co-efficiens A quadrati.

Et sic deinceps procedendum esset si superessent gradus alii inferiores ipsi gradui tertio qui propositus est.

Deinde auferatur primus terminus *propositus* 5 in *quarto gradu constitutus*.

Denique auferatur numerus 3, qui est differentia progressionis, in eodem gradu *quarto constitutus*, ac toties sumptus quot sunt numeri *propositi*, nempe quater in hoc exemplo.

Residuum erit multiplex summæ quæsitæ, eamque toties continebit, quoties numerus 12, qui est co-efficiens ipsius A cubi, seu A in gradu *tertio* proposito, continet unitatem.

Si ergo ad *proxim* methodus reducat, numerus 17 constituendus est in 4 gradu, nempe 83521, et ab eo hæc auferenda sunt:

Primò, summa numerorum propositorum $5 + 8 + 11 + 14$, nempe 38, multiplicata per 108, unde oritur productus 4104.

Deinde summa quadratorum numerorum propositorum, id est, $5^2 + 8^2 + 11^2 + 14^2$, nempe $25 + 64 + 121 + 196$, quorum summa est 406, quæ multiplicata per 54 efficit 21924.

Deinceps auferendus est numerus 5 in *quarto* gradu, nempe 625.

Denique auferendus est numerus 3 in *quarto* gradu, nempe 81, quater sumptus, nempe 324. Numeri ergò auferendi sunt hi, nempe, 4104, 21924, 625, 324; quorum summa est 26977, quæ cum fuerit ablata à numero 83521, superest 56544.

Hoc ergò residuum continebit summam quæsitam, nempe 4712, multiplicatam per 12; et profectò 4712 per 12 multiplicata efficit 56544.

Paradigma facilè est construere; hoc autem sic demonstrabitur.

Etenim numerus 17 in 4 gradu constitutus, qui quidem sic exprimitur, 17^4 , æquatur $17^4 - 14^4 + 14^4 - 11^4 + 11^4 - 8^4 + 8^4 - 5^4 + 5^4$.

Solus enim 17^4 signum affirmationis solum sortitur, reliqui autem affirmantur ac negantur.

Sed differentia radicum 17, 14, est 3; eadèmq; est differentia radicum 14, 11; eadèmq; radicum 11, 8, ac etiam radicum 8, 5. Igitur ex præmissò Lemmate:

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17*

$$17^4 - 14^4 \text{ æquatur } 12 \cdot 14^3 + 54 \cdot 14^2 + 108 \cdot 14 + 81.$$

$$\text{Sic } 14^4 - 11^4 \text{ æquatur } 12 \cdot 11^3 + 54 \cdot 11^2 + 108 \cdot 11 + 81.$$

$$\text{Sic } 11^4 - 8^4 \text{ æquatur } 12 \cdot 8^3 + 54 \cdot 8^2 + 108 \cdot 8 + 81.$$

$$\text{Sic } 8^4 - 5^4 \text{ æquatur } 12 \cdot 5^3 + 54 \cdot 5^2 + 108 \cdot 5 + 81.$$

Non interpretor 5^4 .

Igitur 17^4 æquatur his omnibus:

$$\begin{aligned} &12 \cdot 14^3 + 54 \cdot 14^2 + 108 \cdot 14 + 81 \\ &+ 12 \cdot 11^3 + 54 \cdot 11^2 + 108 \cdot 11 + 81 \\ &+ 12 \cdot 8^3 + 54 \cdot 8^2 + 108 \cdot 8 + 81 \\ &+ 12 \cdot 5^3 + 54 \cdot 5^2 + 108 \cdot 5 + 81 \\ &+ 5^4. \end{aligned}$$

Hoc est *mutato ordine*, 17^4 æquatur his

$$\begin{aligned} &5 + 8 + 11 + 14 \text{ multiplicatis per } 108; \\ &+ 5^2 + 8^2 + 11^2 + 14^2 \text{ multiplicatis per } 54; \\ &+ 5^3 + 8^3 + 11^3 + 14^3 \text{ multiplicatis per } 12; \\ &+ 81 + 81 + 81 + 81; \\ &+ 5^4. \end{aligned}$$

Ablatis utrinque his

$$\begin{aligned} &5 + 8 + 11 + 14 \text{ multiplicatis per } 108; \\ &+ 5^2 + 8^2 + 11^2 + 14^2 \text{ multiplicatis per } 54; \\ &+ 81 + 81 + 81 + 81; \\ &+ 5^4; \end{aligned}$$

Remanet 17^4 minus his, nempe

$$\begin{aligned} &- 5 - 8 - 11 - 14 \text{ multiplicatis per } 108; \\ &- 5^2 - 8^2 - 11^2 - 14^2 \text{ multiplicatis per } 54; \\ &- 81 - 81 - 81 - 81; \\ &- 5^4; \end{aligned}$$

æqualis $5^4 + 8^4 + 11^4 + 14^4$ multiplicatis per 12. Q. E. D.

Sic ergò potest institui enuntiatio et generalis constructio.

SUMMA POTESTATUM.

Datis quocunque numeris, in quâlibet progressionē, à quovis numero initium sumente, invenire summam quarumvis potestatum eorum?

Exponatur binomium, cujus primum nomen sit A, alterum verò sit numerus qui differentia progressionis est; et constituatur hoc binomium in gradu qui proximè superior est gradui proposito, et in expositione potestatis ejus notentur co-efficientes quos A sortitur in singulis gradibus.

Constituatur et in eodem gradu superiori numerus qui in eadem progressionē propositâ proximè sequitur ultimum progressionis terminum propositum. Et ab eo auferatur hæc:

Primū;

Primò, primus terminus *progressionis datus*, seu minimus numerus datorum, in eodem superiori gradu constitutus.

Secundò, numerus qui differentia est progressionis, in eodem superiori gradu constitutus, ac toties sumptus quot sunt termini dati.

Tertiò, auferantur singuli numeri dati, in singulis gradibus *proposito gradu* inferioribus constituti, ac in unoquoque gradu multiplicati per jam notatos co-efficientes quos A fortitur in iisdem gradibus in expositione hujus superioris gradus binomii primò assumpti.

Reliquum est *multiplex summa quæsitæ*, eamque toties continet quoties co-efficientes quem A in gradu proposito fortitur continet unitatem.

M O N I T U M.

Præxes jam particulares sibi quisque pro genio suppeditabit: verbi gratiâ, si quæris summam quotlibet numerorum progressionis naturalis à quolibet inchoantis, hic, ex methodo generali, elicietur Canon:

In progressionē naturali à quovis numero inchoante, differentia inter quadratum minimi termini et quadratum numeri qui proximè major est ultimo termino, minuta numero qui exponit multitudinem, dupla est aggregati ex omnibus.

Sint quotlibet numeri naturali progressionē continui, quorum primus sit ad libitum; verbi gratiâ, quatuor isti 5, 6, 7, 8: dico $9^2 - 5^2 = 4$ æquari duplo $5 + 6 + 7 + 8$.

Similes canones et reliquarum potestatum summis inveniendis et reliquis progressionibus faciliè aptabuntur; quos quisque sibi comparet.

C O N C L U S I O.

Quantum hæc notitia ad spatiorum curvilineorum dimensiones conferat, satis nōrunt qui in Indivisibilium doctrinâ tantisper versati sunt. Omnes enim omnium generum Parabolæ illicò quadrantur, et alia innumera facillimè mensurantur.

Si ergo illa, quæ hæc methodo in numeris reperimus, ad quantitatem continuam applicare libet, hi possunt institui canones.

Canones ad naturalem progressionem quæ ab unitate sumit exordium.

Summa linearum est ad quadratum maximæ, ut . . . 1 ad 2.

Summa quadratorum est ad cubum maximæ, ut . . . 1 ad 3.

Summa cuborum est ad 4 gradum maximæ, ut . . . 1 ad 4.

Canon generalis ad progressionem naturalem quæ ab unitate sumit exordium.

Summa omnium in quolibet gradu, est ad maximam in proximè superiori gradu, ut unitas, ad exponentem superioris gradus.

Non de reliquis differam, quia hic locus non est: hæc obiter notavi, reliqua facili negotio penetrantur, eo posito principio, *in continuâ quantitate, quolibet quantitates cujusvis generis quantitati superioris generis additas, nihil ei superaddere.* Sic puncta lineis, lineæ superficiebus, superficies solidis, nihil adjiciunt: seu *ut numericis, in numerico tractatu, verbis utar,* radices quadratis, quadrata cubis, cubi quadrato-quadratis, &c. nihil apponunt. Quare inferiores gradus, nullius valoris existentes, non considerandi sunt. Hæc, quæ Indivisibilibus studiosis familiaria sunt, subungere placuit, ut nunquam satis mirata connexio, quæ ea etiâ quæ remotissima videntur, in unum addicat unitatis amatrix natura, ex hoc exemplo prodeat, in quo, *quantitatis continuæ dimensionem cum numericarum potestatum summâ conjunctam contemplari licet.*

DE NUMERIS MULTIPLICIBUS

Ex solâ characterum numericorum additione agnoscendis.

M O N I T U M.

NIHIL tritius est apud Arithmeticos, quàm numeros numeri 9 multiplices, constare characteribus, quorum aggregatum est quoque ipsius 9 multiplex. Si enim ipse, verbi gratiâ, dupli 18, characteres numericos 1 + 8, jungas, aggregatum erit 9. Ita ut ex solâ additione characterum numericorum numeri cujuslibet, liceat agnoscere, utrùm sit ipse 9 multiplex: verbi gratiâ, si numeri 1719 characteres numericos jungas 1 + 7 + 1 + 9, aggregatum 18 est ipse 9 multiplex; unde certò colligitur, et ipsum 1719 ejusdem 9 esse multiplicem. Vulgata sanè illa observatio est; verùm ejus demonstratio à nemine, quod sciam, data est, nec ipsa norio ulterius provecta. In hoc autem Tractatulo non solum istius, sed et variarum aliarum observationum generalissimam demonstrationem dedi, ac methodum universalem agnoscendi ex solâ additione characterum numericorum propositi cujusvis numeri, utrùm ille sit alterius propositi numeri multiplex; et non solum in progressionem denariâ, quâ numeratio nostra procedit (denaria enim ex instituto hominum, non ex necessitate naturæ, ut vulgus arbitrat, et sanè satis ineptè, posita est;) sed in quâcunque progressionem instituat numeratio, non fallit hic tradita methodus, ut in paucis mox videbitur paginis.

PRO-

PROPOSITIO UNICA.

Agnoscere ex solâ additione characterum dati cujuslibet numeri, an ipse sit alterius dati numeri multiplex?

Ut hæc solutio fiat generalis, litteris utemur vice numerorum. Sit ergo divisor, numerus quilibet expressus per litteram A; dividendus autem, numerus expressus per litteras T V N M, quarum ultima M exprimit numerum quemlibet in unitatum columnâ collocatum; N verò, numerum quemlibet in denariorum columnâ; V numerum quemlibet in columnâ centenariorum; T autem numerum quemlibet in columnâ millenariorum, et sic deinceps in infinitum: ita ut, si litteras in numeros convertere velis, assumere possis loco ipsius M quemlibet ex novem primis characteribus, verbi gratiâ, 4; loco N quemlibet numerum, ut 3; loco V quemlibet numerum, ut 5; et loco T, quemlibet numerum, ut 6; et collocando singulos illos characteres numericos in propriâ columnâ, prout collocatæ sunt litteræ quæ illos exprimunt, proveniet hic numerus 6534, divisor autem A erit numerus quilibet, ut 7. Missis autem peculiaribus his exemplis generali istâ enunciatione omnia amplectimur.

Dato quocumque dividendo T V N M, et quocumque divisore A, agnoscere ex solâ additione characterum numericorum T V N M, utrùm ipse numerus T V N M exactè dividatur per ipsum numerum A?

Ponantur seorsim numeri serie naturali continui 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, et dexteri; à dextrâ ad sinistram, sic

&c. 10 9 8 7 6 5 4 3 2 1

&c. K I H G F E D C B A.

Jam ipsi primo numero 1, subscribatur unitas.

Ex ipsâ unitate *decies* sumptâ, seu ex 10 auferatur A quotiès fieri poterit, et supersit B; qui sub 2 subscribatur.

Ex B *decies* sumpto seu ex 10 B, auferatur A quotiès poterit, et supersit C; qui ipsi 3 subscribatur.

Ex 10 C, auferatur A quoties poterit, et supersit D; qui ipsi 4 subscribatur.

Ex 10 D, auferatur A, &c. in continuum.

Nunc sumatur ultimus character dividendi M, (qui quidem et primus est à dextrâ ad sinistram,) scribatûrque seorsim semel; *primo enim numero 1 subjacet unitas.*

M
N in B
V in C
T in D

Jam sumatur secundus character N, et totiès repetatur quot sunt unitates in B, *qui secundo numero subjacet*, hoc est, multiplicetur N per B; et sub M ponatur productus.

Jam sumatur tertius character V, et totiès repetatur quot sunt unitates in C, *qui tertio numero subjacet*, seu multiplicetur V per C; et productus sub primis ponatur.

Sic denique multiplicetur quartus T per D, et sub aliis scribatur. Et sic in infinitum.

Dico, prout summa horum numerorum M + N in B + V in C + T in D, est ipsius A multiplex aut non, et quoque ipsum numerum T V N M esse ejusdem multiplicem, vel non.

Etenim

Etenim si propositus dividendus *unicum* haberet characterem M, sanè prout ipse esset multiplex ipsius A, numerus quoque M esset ejusdem A multiplex, cum sit ipse numerus totus.

Si verò constet *duobus* characteribus NM: dico quoque, prout M + N in B, est multiplex A, et ipsum numerum NM ejusdem multiplicem esse.

Etenim character N in columnâ denarii, æquatur 10N.

Verùm, ex constructione, est 10 — B multiplex A.
 Quare, ducendo 10 — B in N, est 10N — B in N multiplex A.
 Si ergò contingit et esse M + B in N multiplex A.
 Ergò ambo ultimi multiplices juncti 10N + M erunt mult. A.
 Id est, N in columnâ denarii et M in columnâ unitatis, seu numerus NM est multiplex A.
 Q. E. D.

Si numerus dividendus constet *tribus* characteribus VNM: dico quoque ipsum esse aut non esse multiplicem A, prout M + N in B + V in C, erit ipsius A multiplex, vel non.

Etenim character V, in columnâ centenarii, æquatur 100V.

At ex constructione est 10 — B, multiplex A.
 Quare, multiplicando 10 — B per 10, erit 100 — 10B, multiplex A.
 Et, ducendo ipsos in V, erit 100V — 10B in V, multiplex A.
 Sed est etiam, ex constructione, ... 10B — C, multiplex A.
 Quare, ducendo in V, erit 10B in V — C in V, multiplex A.
 Sed ex ostensis 100V — 10B in V, multiplex A.
 Ergò juncti duo ultimi 100V — C in V, multiplex A.
 Jam verò ostendemus, ut in secundo casu, ... 10N — B in N, multiplex A.
 Ergò juncti duo ultimi, 100V + 10N — C in V — B in N, multiplex A.
 Ergò si contingat hos numeros C in V + B in N + M, esse multiplex A.
 ambo ultimi juncti, nempe 100V + 10N + M, est multiplex A.
 Seu V in columnâ centenarii, et N in columnâ denarii, et M in columnâ unitatis, hoc est, numerus VNM, est multiplex A.
 Q. E. D.

Non secus demonstrabitur de numeris ex *pluribus* characteribus compcitis.
 Quare prout, &c. Q. E. D.

Exemplis gaudeamus.

Quæro, qui sint numeri multiplices numeri 7? Scriptis continuis 1, 2, 3, 4, 5, &c. subscribo 1 sub 1

10	9	8	7	6	5	4	3	2	1
6	2	3	1	5	4	6	2	3	1

Ex unitate decies sumptâ, seu ex 10, aufero 7 quotiès potest; superest 3; quem pono sub 2.

Ex 3 decies sumpto, seu ex 30, aufero 7 quotiès potest; superest 2; quem pono sub 3.

Ex

Ex 20 aufero 7 quotiès potest; superest 6; et pono sub 4.

Ex 60 aufero 7 quotiès potest; superest 4; et pono sub 5.

Ex 40 aufero 7 quotiès potest; superest 5; et pono sub 6.

Ex 50 aufero 7 quotiès potest; superest 1; et pono sub 7.

Ex 10 aufero 7 quotiès potest; et redit 3; et pono sub 8.

Ex 30 aufero 7 quotiès potest; et redit 2; et pono sub 9.

Et sic redit series numerorum 1, 3, 2, 6, 4, 5, in infinitum.

Jam proponatur numerus quilibet 287,542,178, de quo quaeritur utrùm exactè dividatur per 7; hoc sic agnoscetur.

Sumatur *semèl* ejus character qui primus est à dextrà ad sinistram, nempe 8; *primo enim numero seriei continuæ subjacet unitas*: quare ponatur 8. *semèl*

Secundus, qui est 7, *tër* sumatur, seu per 3 multiplicetur, *secundo enim numero seriei subjacet 3*; sitque productus 21.

Tertius *bls* sumatur; *subjacet enim 2 ipsi 3*: quare terti character qui est 1, per 2 multiplicatus, sit 2.

Quartus eadem ratione per 6 multiplicatus 12.

Quintus per 4 multiplicatus 16.

Sextus per 5 multiplicatus 25.

Septimus *semèl*, *septimo enim subjacet 1*, 7.

Octavus *tër* sumptus 24.

Nonus *bls* sumptus 4.

Et sic deinceps, si alii numeri superessent. Jungantur hi numeri 119.

Si ipse aggregatus 119 est multiplex ipsius 7, numerus quoque propositus 287,542,178 ejusdem 7 multiplex erit.

Potest autem dignosci eadem methodo, utrùm ipse 119 sit multiplex 7, scilicet, sumendo *semèl* primum characterem 9

secundum characterem *tër* 3

et præcedentem *bls* 2

14

Si enim summa 14 est multiplex 7, erit et 119 ejusdem multiplex.

Sed et, si, curiositate potiùs quàm necessitate moti, velimus agnoscere utrùm 14 sit multiplex 7, sumatur character ultimus *semèl* 4

et præcedens *tër* 3

7

Si summa est multiplex ipsius 7, erit et 14 multiplex 7: quare et 14, et 119, et 287,542,178 sunt multiplices ipsius 7.

Vis agnoscere quinam numeri dividantur per 6? Scriptis, ut sæpiùs dictum est, numeris naturalibus 1, 2, 3, 4, 5, &c., et 1 sub 1 posito

&c. 4 3 2 1

&c. 4 4 4 1

Ex

Ex 10 aufer 6; reliquum 4 sub 2 ponito.

Ex 40 aufer 6; reliquum 4 sub 3 ponito.

Ex 40 aufer 6; reliquum 4 sub 4 ponito.

Et sic semper redibit 4; quod agnosci potuit ubi semel rediit.

Ergo, si proponatur numerus quilibet, de quo quærebatur utrùm sit dividendus per 6, nempe 248,742? sume ultimam ejus figuram semel

præcedentem quatèr	2
præcedentem quatèr, &c.	16
et uno verbo, primam semel, reliquarum verò	28
superam quatèr	32
	16
	8

102

Si summa 102 dividatur per 6, dividetur et ipse numerus propositus 248,742 per eundem 6.

Vis agnoscere utrùm numerus dividatur per 3? Scriptis ut priùs numeris naturalibus, et 1 sub 1 posito

5	4	3	2	1
1	1	1	1	1

Ex 10 aufer 3, quotiès potest; reliquum 1 sub 2 ponito.

Ex 10 aufer 3, quotiès potest; reliquum 1 sub 3 ponito; et sic in infinitum.

Ergo si proponatur numerus quilibet 2451, ut scias utrùm dividatur per 3,

sume semel ultimam figuram

præcedentem semel	1
et semel singulas	5
	4
	2

12

Si summa dividatur per 3, dividetur et numerus propositus per 3.

Vis agnoscere utrùm numerus dividatur per 9? Scriptis numeris 1, 2, 3, &c., et 1 sub 1 posito.

Ex 10 aufer 9, et quoniam superest 1, patet unitatem contingere singulis numeris. Ergo, si numeri propositi singuli characteres simul sumpti dividantur per 9, dividetur et ipse.

Vis agnoscere utrùm numerus dividatur per 4? Scriptis numeris naturalibus, ut mos est, et posito 1 sub 1

4	3	2	1
0	0	2	1

Ex 10 aufer 4 quantum potest; reliquum 2 pone sub 2.

Ex 20 aufer 4 quantum potest; reliquum 0 pone sub 3.

Ex 00 aufer 4; superest semper 0.

Quare si proponatur numerus dividendus 2486, pono ultimum characterem semel

præcedentem bis; subjacet enim 2 sub 2

	6
	16

22

Præcedens

Præcedens per 0 multiplicatus facit zero, et sic de reliquis; quare ad ipsos non attendio: et si summa priorum, nempe 22, per 4 dividatur, dividetur et ipse; secus autem, non.

Sic numeri quorum ultimus character semel, præcedens bis, præcedens quater, (reliquis neglectis; zero enim fortuitur:) simul juncti numerum efficiunt multiplicem numeri 8, sunt ipsi et ejusdem 8 multiplices; secus autem, non.

In exemplum autem dabimus et illud.

Agnoscere qui numeri dividantur per 16? Scriptis, ut dictum est, numeris naturalibus 1, 2, 3, 4, 5, 6, 7, &c., et 1 sub 1 posito

7	6	5	4	3	2	1
0	0	0	8	4	10	1

Ex 10 aufer 16 quantum potest, superest ipse 10. Ex minore enim numero major numerus subtrahi non potest; quare ipsemet numerus 10 ponatur sub 2.

Ex ipso 10 decies sumpto, ut mos est, seu ex 100, aufero 16 quantum potest; superest 4, quem pono sub 3.

Ex 40 aufero 16 quantum potest; reliquum 8 pono sub 4.

Ex 80 aufero 16 quantum potest; superest 0.

Idem omnis numerus cujus ultimus character semel sumptus, penultimus decies, præcedens quater, et præcedens octies, efficiunt numerum multiplicem 16, erit et ipse ipsius 16 multiplex.

Sic reperies omnes numeros, quorum penultimus character decies, reliqui autem omnes, scilicet, ultimus, ante-penultimus, præante-penultimus, et reliqui semel sumpti, efficiunt numerum divisibilem per 45, vel 18, vel 15, vel 30, vel 90, et uno verbo omnes divisores numeri 90, duobus constantes characteribus, dividi quoque et ipsos per hos divisores.

Non difficilis est inde ad alia progressus; sed intentatam huc usque materiam aperuisse, et satis obscuram lucidissimâ demonstratione illustrasse, sufficit. Ars etenim illa, quâ, ex additione characterum numeri, noscitur per quos sit divisibilis, ex imâ numerorum naturâ, et ex eorum denariâ progressionem vim suam fortitur: si enim aliâ progressionem procederent, verbi gratiâ, duodenariâ (quod sanè gratum foret) et sic ultrâ primas novem figuras, aliæ duæ institutæ essent, quarum altera denarium, altera undenarium exhiberet; tunc non amplius contingeret, numeros quorum omnes characteres simul sumpti efficiunt numerum multiplicem numeri 9, esse et ipsos ejusdem 9 multiplices.

Sed methodus nostra, necnon et demonstratio, et huic progressioni, et omnibus possibilibus convenit.

Si enim in hac duodenariâ progressionem, proponitur agnoscere an numerus dividatur per 9.

Instituemus, ut antea, numeros naturali serie continuos 1, 2, 3, 4, 5, &c., et 1 sub 1 posito

4	3	2	1
0	0	3	1

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4 D

Ex

Ex unitate jam duodeciès sumptâ seu ex 10, (qui jam potest *duodecim*, non autem *decem*) auferendo 9 quantum potest, superest 3, quem pono sub 2.

Ex 30 (qui jam potest *triginta sex*, scilicet *ter duodecim*) aufer 9 quantum potest; et superest nihil: continetur enim 9 quater exactè in *triginta sex*: pono igitur 0 sub 3.

Et idèò, zero sub reliquis characteribus continget.

Undè colligo, omnes numeros, quorum ultimus character semel sumptus, penultimus verò ter (*de cæteris non curo quales sint; zero enim fortivatur*;) efficiunt numerum divisibilem per 9, dividi quoque per 9, in duodenariâ progressionè.

Sic in hac progressionè duodenariâ omnes numeri quorum singuli characteres simul sumpti efficiunt numerum divisibilem per 11, sunt et divisibiles per eundem.

In nostrâ verò progressionè denariâ, contingit omnes numeros divisibiles per 11, ita se habere, ut ultimus semel sumptus, penultimus deciès, præcedens semel, præcedens deciès, præcedens semel, præcedens deciès, et sic in infinitum, constare numerum multiplicem 11.

Hæc et alia facili studio, ex istâ methodo quisque colliget. Nos hic ea levitèr quidem tetigimus, quoniam nova et intentata placere solent; nunc verò relinquimus, ne nimia percrutatio tædium pariat.

Fin de Oeuvres Arithmétiques de Monsieur BLAISE PASCAL.

An APPENDIX to the Tract on THE SUMMATION OF INFINITE SERIES contained in the Third Volume of this Collection of Tracts, called SCRIPTORES LOGARITHMICI, pages 255, 256, 257, &c - - 312, in which an Investigation is given of the Differential Series

$$a - \frac{bx}{1+x} - \frac{b^2x^2}{1+x^2} - \frac{b^3x^3}{1+x^3} - \frac{b^{11}x^{11}}{1+x^{11}} - \frac{b^{17}x^{17}}{1+x^{17}} - \frac{b^{23}x^{23}}{1+x^{23}} - \frac{b^{29}x^{29}}{1+x^{29}} - \frac{b^{35}x^{35}}{1+x^{35}} - \frac{b^{41}x^{41}}{1+x^{41}} - \frac{b^{47}x^{47}}{1+x^{47}} - \frac{b^{53}x^{53}}{1+x^{53}} - \frac{b^{59}x^{59}}{1+x^{59}} - \frac{b^{65}x^{65}}{1+x^{65}} - \frac{b^{71}x^{71}}{1+x^{71}} - \frac{b^{77}x^{77}}{1+x^{77}} - \frac{b^{83}x^{83}}{1+x^{83}} - \frac{b^{89}x^{89}}{1+x^{89}} - \frac{b^{95}x^{95}}{1+x^{95}} - \frac{b^{101}x^{101}}{1+x^{101}} - \frac{b^{107}x^{107}}{1+x^{107}} - \frac{b^{113}x^{113}}{1+x^{113}} - \frac{b^{119}x^{119}}{1+x^{119}} - \frac{b^{125}x^{125}}{1+x^{125}} - \frac{b^{131}x^{131}}{1+x^{131}} - \frac{b^{137}x^{137}}{1+x^{137}} - \frac{b^{143}x^{143}}{1+x^{143}} - \frac{b^{149}x^{149}}{1+x^{149}} - \frac{b^{155}x^{155}}{1+x^{155}} - \frac{b^{161}x^{161}}{1+x^{161}} - \frac{b^{167}x^{167}}{1+x^{167}} - \frac{b^{173}x^{173}}{1+x^{173}} - 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\frac{b^{527}x^{527}}{1+x^{527}} - \frac{b^{533}x^{533}}{1+x^{533}} - \frac{b^{539}x^{539}}{1+x^{539}} - \frac{b^{545}x^{545}}{1+x^{545}} - \frac{b^{551}x^{551}}{1+x^{551}} - \frac{b^{557}x^{557}}{1+x^{557}} - \frac{b^{563}x^{563}}{1+x^{563}} - \frac{b^{569}x^{569}}{1+x^{569}} - \frac{b^{575}x^{575}}{1+x^{575}} - \frac{b^{581}x^{581}}{1+x^{581}} - \frac{b^{587}x^{587}}{1+x^{587}} - \frac{b^{593}x^{593}}{1+x^{593}} - \frac{b^{599}x^{599}}{1+x^{599}} - \frac{b^{605}x^{605}}{1+x^{605}} - \frac{b^{611}x^{611}}{1+x^{611}} - \frac{b^{617}x^{617}}{1+x^{617}} - \frac{b^{623}x^{623}}{1+x^{623}} - \frac{b^{629}x^{629}}{1+x^{629}} - \frac{b^{635}x^{635}}{1+x^{635}} - \frac{b^{641}x^{641}}{1+x^{641}} - \frac{b^{647}x^{647}}{1+x^{647}} - \frac{b^{653}x^{653}}{1+x^{653}} - \frac{b^{659}x^{659}}{1+x^{659}} - \frac{b^{665}x^{665}}{1+x^{665}} - \frac{b^{671}x^{671}}{1+x^{671}} - \frac{b^{677}x^{677}}{1+x^{677}} - \frac{b^{683}x^{683}}{1+x^{683}} - \frac{b^{689}x^{689}}{1+x^{689}} - \frac{b^{695}x^{695}}{1+x^{695}} - 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\frac{b^{2339}x^{2339}}{1+x^{2339}} - \frac{b^{2345}x^{2345}}{1+x^{2345}} - \frac{b^{2351}x^{2351}}{1+x^{2351}} - \frac{b^{2357}x^{2357}}{1+x^{2357}} - \frac{b^{2363}x^{2363}}{1+x^{2363}} - \frac{b^{2369}x^{2369}}{1+x^{2369}} - \frac{b^{2375}x^{2375}}{1+x^{2375}} - \frac{b^{2381}x^{2381}}{1+x^{2381}} - \frac{b^{2387}x^{2387}}{1+x^{2387}} - \frac{b^{2393}x^{2393}}{1+x^{2393}} - \frac{b^{2399}x^{2399}}{1+x^{2399}} - \frac{b^{2405}x^{2405}}{1+x^{2405}} - \frac{b^{2411}x^{2411}}{1+x^{2411}} - \frac{b^{2417}x^{2417}}{1+x^{2417}} - \frac{b^{2423}x^{2423}}{1+x^{2423}} - \frac{b^{2429}x^{2429}}{1+x^{2429}} - \frac{b^{2435}x^{2435}}{1+x^{2435}} - \frac{b^{2441}x^{2441}}{1+x^{2441}} - \frac{b^{2447}x^{2447}}{1+x^{2447}} - \frac{b^{2453}x^{2453}}{1+x^{2453}} - \frac{b^{2459}x^{2459}}{1+x^{2459}} - \frac{b^{2465}x^{2465}}{1+x^{2465}} - \frac{b^{2471}x^{2471}}{1+x^{2471}} - \frac{b^{2477}x^{2477}}{1+x^{2477}} - \frac{b^{2483}x^{2483}}{1+x^{2483}} - \frac{b^{2489}x^{2489}}{1+x^{2489}} - \frac{b^{2495}x^{2495}}{1+x^{2495}} - \frac{b^{2501}x^{2501}}{1+x^{2501}} - \frac{b^{2507}x^{2507}}{1+x^{2507}} - \frac{b^{2513}x^{2513}}{1+x^{2513}} - \frac{b^{2519}x^{2519}}{1+x^{2519}} - \frac{b^{2525}x^{2525}}{1+x^{2525}} - \frac{b^{2531}x^{2531}}{1+x^{2531}} - \frac{b^{2537}x^{2537}}{1+x^{2537}} - \frac{b^{2543}x^{2543}}{1+x^{2543}} - \frac{b^{2549}x^{2549}}{1+x^{2549}} - \frac{b^{2555}x^{2555}}{1+x^{2555}} - \frac{b^{2561}x^{2561}}{1+x^{2561}} - \frac{b^{2567}x^{2567}}{1+x^{2567}} - \frac{b^{2573}x^{2573}}{1+x^{2573}} - \frac{b^{2579}x^{2579}}{1+x^{2579}} - \frac{b^{2585}x^{2585}}{1+x^{2585}} - \frac{b^{2591}x^{2591}}{1+x^{2591}} - \frac{b^{2597}x^{2597}}{1+x^{2597}} - \frac{b^{2603}x^{2603}}{1+x^{2603}} - \frac{b^{2609}x^{2609}}{1+x^{2609}} - \frac{b^{2615}x^{2615}}{1+x^{2615}} - \frac{b^{2621}x^{2621}}{1+x^{2621}} - \frac{b^{2627}x^{2627}}{1+x^{2627}} - \frac{b^{2633}x^{2633}}{1+x^{2633}} - \frac{b^{2639}x^{2639}}{1+x^{2639}} - \frac{b^{2645}x^{2645}}{1+x^{2645}} - \frac{b^{2651}x^{2651}}{1+x^{2651}} - \frac{b^{2657}x^{2657}}{1+x^{2657}} - \frac{b^{2663}x^{2663}}{1+x^{2663}} - \frac{b^{2669}x^{2669}}{1+x^{2669}} - \frac{b^{2675}x^{2675}}{1+x^{2675}} - \frac{b^{2681}x^{2681}}{1+x^{2681}} - \frac{b^{2687}x^{2687}}{1+x^{2687}} - \frac{b^{2693}x^{2693}}{1+x^{2693}} - \frac{b^{2699}x^{2699}}{1+x^{2699}} - \frac{b^{2705}x^{2705}}{1+x^{2705}} - \frac{b^{2711}x^{2711}}{1+x^{2711}} - \frac{b^{2717}x^{2717}}{1+x^{2717}} - \frac{b^{2723}x^{2723}}{1+x^{2723}} - \frac{b^{2729}x^{2729}}{1+x^{2729}} - \frac{b^{2735}x^{2735}}{1+x^{2735}} - \frac{b^{2741}x^{2741}}{1+x^{2741}} - \frac{b^{2747}x^{2747}}{1+x^{2747}} - \frac{b^{2753}x^{2753}}{1+x^{2753}} - \frac{b^{2759}x^{2759}}{1+x^{2759}} - \frac{b^{2765}x^{2765}}{1+x^{2765}} - \frac{b^{2771}x^{2771}}{1+x^{2771}} - \frac{b^{2777}x^{2777}}{1+x^{2777}} - \frac{b^{2783}x^{2783}}{1+x^{2783}} - \frac{b^{2789}x^{2789}}{1+x^{2789}} - \frac{b^{2795}x^{2795}}{1+x^{2795}} - \frac{b^{2801}x^{2801}}{1+x^{2801}} - \frac{b^{2807}x^{2807}}{1+x^{2807}} - \frac{b^{2813}x^{2813}}{1+x^{2813}} - \frac{b^{2819}x^{2819}}{1+x^{2819}} - \frac{b^{2825}x^{2825}}{1+x^{2825}} - \frac{b^{2831}x^{2831}}{1+x^{2831}} - \frac{b^{2837}x^{2$$

&c, or $b - 2c + d$, $c - 2d + e$, $d - 2e + f$, $e - 2f + g$, $f - 2g + h$, &c, or the second differences of the original numbers b, c, d, e, f, g, h , &c, shall also form a decreasing progression; and that the differences of these second differences, or the quantities $b - 2c + d - \{c - 2d + e, c - 2d + e - \{d - 2e + f, d - 2e + f - \{e - 2f + g, e - 2f + g - \{f - 2g + h, \&c,$ or $b - 3c + 3d - e, c - 3d + 3e - f, d - 3e + 3f - g, e - 3f + 3g - h$, &c, or the third differences of the original numbers b, c, d, e, f, g, h , &c, shall also form a decreasing progression; and that the fourth, and fifth, and sixth, and every following, set of differences of the same original numbers b, c, d, e, f, g, h , &c, shall also form decreasing progressions;—and, if the first difference, $b - c$, of the first order be called v^1 , and the first difference, $b - 2c + d$, of the second order be called v^2 , and the first difference, $b - 3c + 3d - e$, of the third order be called v^3 , and the first difference of the fourth order be called v^4 , and the first differences of the fifth, sixth, seventh, eighth, and other following orders be called v^5, v^6, v^7, v^8 , &c;—I say, it was shewn in the said Tract preceeding the Tract here referred-to, that, when all these circumstances concurred, the infinite series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$ *ad infinitum*, would be equal to the infinite series $a - \frac{bx}{1+x} - \frac{v^1 x^2}{1+x^2} -$

$$- \frac{v^2 x^3}{1+x^3} - \frac{v^3 x^4}{1+x^4} - \frac{v^4 x^5}{1+x^5} - \frac{v^5 x^6}{1+x^6} - \frac{v^6 x^7}{1+x^7} - \&c$$
 ad infinitum, which

converges with a very considerable degree of swiftness. And many curious examples are given in that Tract, of the great usefulness of this latter series (which, from its involving in its terms the several differences, $v^1, v^2, v^3, v^4, v^5, v^6, v^7$, &c, of the co-efficients b, c, d, e, f, g, h , &c of the several powers of x in the original series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$, is called *The Differential Series*;) in finding the value of the original series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$ *ad infinitum*, when its terms decrease with such excessive slowness as to make it incapable of being summed in the common way, or by the mere computation of a moderate number of its first terms, and a subtraction of the sum of those which are marked with the sign — from the sum of those which are marked with the sign +. But the method of investigating this Differential Series was but slightly touched-upon in that paper, to avoid making it too long. This omission, however, I afterwards came to think was a defect of sufficient importance to deserve to be supplied in the best manner I was able. For a learned and intelligent friend of mine, who had a very clear head and a good capacity and turn for mathematical inquiries, the late Henry Boulton Cay, Esq. a Barrister at Law of the Society of the Middle Temple, (who was afterwards made Solicitor to the Exchequer in the year 1783,) upon reading that Tract, though he was greatly pleased with the Differential Series described in it, on account of its great utility in finding the value of a slowly-converging Series, yet complained at the same time that the manner of obtaining it had not been therein sufficiently explained for him to understand it thoroughly. This made me resolve to give a further explanation of

of it in as full and clear a manner as I could, and was the occasion of my drawing-up the second Tract upon that Series, which is intitled *The Investigation of the foregoing Differential Series*, and is contained in pages 255, 256, 257, &c. - - - 312 of the Third Volume of this Collection, in which the subject is treated with great care and exactness. In this Second Tract I have, first, given a very full description and demonstration of the principles by means of which I had investigated the said Differential Series, which are set forth in four Lemmas, or preliminary Propositions to the Investigation itself of the said Series; and I have then exhibited the said Investigation in a very distinct and particular manner, so far as to determine the first five terms of the said Differential Series,

which are found to be $a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x^2} - \frac{D^2x^3}{1+x^3} - \frac{D^3x^4}{1+x^4}$, of which

all but the first term a have the sign — prefixed to them, or are subtracted from the said first term a . And I added (in art. 46, page 282,) that I had, for my own satisfaction, investigated in the same manner the sixth, seventh, and eighth terms of the same Differential Series, and found them to be equal to

$\overline{b-4c+6d-4e+f} \times \frac{x^5}{1+x^5}$, and $\overline{b-5c+10d-10e+5f-g} \times \frac{x^6}{1+x^6}$,

and $\overline{b-6c+15d-20e+15f-6g+h} \times \frac{x^7}{1+x^7}$, or to $\frac{D^4x^5}{1+x^5}$, $\frac{D^5x^6}{1+x^6}$,

and $\frac{D^6x^7}{1+x^7}$, and that they also, as well as the four preceding terms, were to

be subtracted from the first term a , and consequently to have the sign — prefixed to them. And I then observed that it seemed reasonable to conjecture, from the analogy of those several terms to each other, that the same law of continuation would take place in the following terms of the Series, after the eighth

term, and that these would consequently be $\frac{D^7x^8}{1+x^8}$, $\frac{D^8x^9}{1+x^9}$, $\frac{D^9x^{10}}{1+x^{10}}$, $\frac{D^{10}x^{11}}{1+x^{11}}$,

and $\frac{D^{11}x^{12}}{1+x^{12}}$, &c. and would all have the sign — prefixed to them, or be sub-

tracted from the first term a , like the other seven terms, immediately following the first term, which I had investigated. But I confessed at the same time that I was not able to see clearly and distinctly in my own mind, and much less to demonstrate to others, that this must of necessity be so. This therefore was a defect which still remained in the Investigation, and which I was sorry to be forced to leave in it, and which the readers of it must, I doubt not, have wished to see supplied, in order to make their knowledge of the said very useful Differential Series quite complete, and to enable them to use with perfect confidence as many terms of the said Series as they might at any time have occasion for. Now this defect I have lately been enabled to supply by the assistance of the learned and ingenious Mr. John Hellins, Vicar of Potter's Pury in Northamptonshire, who has communicated to me a different method of investigating the values of the second, third, fourth, fifth, and other following terms of the assumed

assumed series $\mathbf{r}, \frac{ax}{1+x}, \frac{ax^2}{1+x^2}, \frac{ax^3}{1+x^3}, \frac{ax^4}{1+x^4}, \frac{ax^5}{1+x^5}, \frac{ax^6}{1+x^6}, \frac{ax^7}{1+x^7},$
 &c. *ad infinitum*, (which is equal to the original series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$ *ad infinitum*,) and of shewing that the same sign — must be prefixed to all of them, from that set forth in the said Second Tract, and which is of such a nature as to enable us to perceive that the law of continuation, or generation, of the terms of the said assumed series one from another which is found to take place in the terms that are actually investigated, must likewise take place in all the following terms, which have not been investigated, to whatever number those terms may be continued. This method of investigating the values of the terms of the said assumed series, and the signs that are to be prefixed to them, I shall now endeavour to explain, and, for that purpose, shall, first, lay down and demonstrate three Lemmas, or preliminary Propositions, upon which the said Investigation will be grounded. These Lemmas are as follows:

L E M M A I.

ART. 2. If there be an Infinite Series of decreasing numbers denoted by the capital letters $\mathbf{A, B, C, D, E, F, G, H, \&c}$ *ad infinitum*; and y be a variable quantity which may be of any magnitude not greater than 1, or than some quantity less than 1, how small soever that magnitude be taken; and the several terms of the series $\mathbf{A, B, C, D, E, F, G, H, \&c}$ *ad infinitum*, be multiplied into the corresponding terms of the geometrical progression $\mathbf{1, y, y^2, y^3, y^4, y^5, y^6, y^7, \&c}$ *ad infinitum*, so as to form a third series which will be $\mathbf{Ay, Ey, Cy^2, Dy^3, Ey^4, Fy^5, Gy^6, Hy^7, \&c}$ *ad infinitum*;—and, if it is at first uncertain which of the two signs + and — is to be prefixed to each of the terms $\mathbf{Ay, Ey^2, Dy^3, Fy^4, Ey^5, Cy^6, Hy^7, \&c}$, which come after the first term $\mathbf{A}$ in which the variable quantity y is not involved, or which of the said terms $\mathbf{Ey, Cy^2, Dy^3}$, &c is to be added to the said first term $\mathbf{A}$, and which of them is to be subtracted from it; and, if it is afterwards proved that the whole series $\mathbf{A, Ay, Cy^2, Dy^3, Ey^4, Fy^5, Gy^6, Hy^7, \&c}$ is always less than its first term $\mathbf{A}$, however small the variable quantity y may be taken: it will follow that the second term $\mathbf{Ay}$ of the said series must be subtracted from its first term $\mathbf{A}$, and not added to it, and consequently that it must have the sign — prefixed to it.

DEMON-

DEMONSTRATION.

For, if ay be not subtracted from the first term A , but added to it, and consequently marked with the sign $+$, it will not be possible for the whole series $A, ay, cy^2, dy^3, ey^4, fy^5, gy^6, hy^7, \&c.$ (which in this case will be $A + ay, cy^2, dy^3, ey^4, fy^5, gy^6, hy^7, \&c.$) to be always less than the first term A , of how small a magnitude soever the variable quantity y may be taken. For let us even suppose that all the other terms except ay , to wit, the terms $cy^2, dy^3, ey^4, fy^5, gy^6, hy^7, \&c.$ are subtracted from the first term A , and consequently have the sign $-$ prefixed to them; which is the supposition that tends most to make the series $A + ay, cy^2, dy^3, ey^4, fy^5, gy^6, hy^7, \&c.$ be less than its first term A : and we shall find that, even on this supposition, it will not be possible for the said series to be always less than its first term A , when the variable quantity y is taken extremely small. For, on this supposition, the said series will be $A + ay - cy^2 - dy^3 - ey^4 - fy^5 - gy^6 - hy^7 - \&c.$ Now the single term ay is to the whole series $cy^2 + dy^3 + ey^4 + fy^5 + gy^6 + hy^7 + \&c.$ in the same proportion as the single term B is to the whole series $cy + dy^2 + ey^3 + fy^4 + gy^5 + hy^6 + \&c.$ But, because the co-efficients $c, d, e, f, g, h, \&c.$ are a set of decreasing numbers, it is evident that y may be taken of so small a magnitude that the whole series $cy + dy^2 + ey^3 + fy^4 + gy^5 + hy^6 + \&c.$ shall be less than any proposed quantity how small soever. Let it then be taken of so small a magnitude that the said series shall be less than B . Then will the whole series $cy^2 + dy^3 + ey^4 + fy^5 + gy^6 + hy^7 + \&c.$ be less than the single quantity ay . And consequently the whole series $A + ay - cy^2 - dy^3 - ey^4 - fy^5 - gy^6 - hy^7 - \&c.$ *ad infinitum* will be greater than the first term A ; which is contrary to what is supposed to have been proved concerning it, to wit, that it is always less than the said first term. Therefore the supposition from which this contradiction follows, to wit, the supposition that ay is to be added to the first term A , cannot be true. Therefore the said second term ay must be subtracted from the said first term A , and must have the sign $-$ prefixed to it. Q. E. D.

L E M M A II.

Art. 3. If there be a series of decreasing numbers denoted by the capital letters $A, B, C, D, E, F, G, H, \&c.$ *ad infinitum*, as in the foregoing Lemma; and y be a variable quantity that may be of any magnitude not greater than 1, or than some quantity less than 1, how small soever that magnitude may be taken, as was supposed also in the foregoing Lemma; and the several terms of the decreasing series $A, B, C, D, E, F, G, H, \&c.$ *ad infinitum* be multiplied into the corresponding terms of the decreasing geometrical progression $1, y, y^2, y^3, y^4, y^5, y^6, y^7, \&c.$ *ad infinitum*, so as to form a third series $A, ay, cy^2, dy^3, ey^4, fy^5, gy^6, hy^7,$

ny^2 , &c. *ad infinitum*, as in the foregoing Lemme;—and, if it is at first uncertain which of the two signs + and — is to be prefixed to each of the terms ny , cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c. which come after the first term A of the said series (in which term the variable quantity y is not involved,) or which of the said terms ny , cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c. is to be added to the said first term A , and which of them is to be subtracted from it; and, if it is afterwards proved that the whole series A , ny , cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c. is always greater than its first term A , however small the variable quantity y may be taken: it will follow that the second term ny of the said series must be added to its first term A , and not subtracted from it, and consequently that it must have the sign + prefixed to it.

DEMONSTRATION.

For, if ny be not added to the first term A , but subtracted from it, and consequently marked with the sign —, it will not be possible for the whole series A , ny , cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c. (which in this case will be $A - ny$, cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c.) to be always greater than the first term A , of how small a magnitude soever the variable quantity y may be taken. For let us even suppose that all the other terms except ny , to wit, the terms cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c. are added to the first term A , and consequently have the sign + prefixed to them; which is the supposition that tends most to make the series $A - ny$, cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c. be greater than its first term A : and we shall find that, even on this supposition, it will not be possible for the whole series A , ny , cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c. (which in this case will be $A - ny$, cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c.) to be always greater than the first term A , of how small a magnitude soever the variable quantity y may be taken. For, on this supposition, the said series will be $A - ny + cy^2 + ny^2 + ey^2 + fy^2 + gy^2 + hy^2 + \&c.$ Now the single term ny is to the whole series $cy^2 + ny^2 + ey^2 + fy^2 + gy^2 + hy^2 + \&c.$ as the single quantity n is to the whole series $cy + ny^2 + ey^2 + fy^2 + gy^2 + hy^2 + \&c.$ *ad infinitum*, in the same proportion as the single quantity n is to the whole series $cy + ny^2 + ey^2 + fy^2 + gy^2 + hy^2 + \&c.$ *ad infinitum*. But, because the coefficients c , n , e , f , g , h , &c. are a set of decreasing numbers, it is evident that y may be taken of so small a magnitude that the whole series $cy + ny^2 + ey^2 + fy^2 + gy^2 + hy^2 + \&c.$ shall be less than any proposed quantity, how small soever. Let it then be taken of so small a magnitude that the said series shall be less than n . Then will the whole series $cy^2 + ny^2 + ey^2 + fy^2 + gy^2 + hy^2 + \&c.$ be less than the single quantity ny . And consequently the whole series $A - ny + cy^2 + ny^2 + ey^2 + fy^2 + gy^2 + hy^2 + \&c.$ *ad infinitum* will be less than the first term A ; which is contrary to what is supposed to have been proved concerning it, to wit, that it is always greater than the said first term. Therefore the supposition from which this contradiction follows, to wit, the supposition that ny is to be subtracted from the first term A , cannot be true. Therefore the said second term ny of the series A , ny , cy^2 , ny^2 , ey^2 , fy^2 , gy^2 , hy^2 , &c. *ad infinitum* must be added to the said first term A , and must have the sign + prefixed to it.

Q. E. D.

LEMMA

L E M M A III.

Art. 4. If the several numbers $a, b, c, d, e, f, g, h, \&c$ in the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$ *ad infinitum* form a decreasing progression of the kind mentioned above in art. 1; in which the differences of $b, c, d, e, f, g, h, \&c$, to wit, the quantities $b - c, c - d, d - e, e - f, f - g, g - h, \&c$, also form a decreasing progression; and the differences of these differences, or the second differences of the numbers $b, c, d, e, f, g, h, \&c$, also form a decreasing progression; and the differences of these second differences, or the third differences of $b, c, d, e, f, g, h, \&c$, also form a decreasing progression; and, in like manner, the fourth differences, and the fifth differences, and the sixth differences, and the differences of every following order, of the same numbers $b, c, d, e, f, g, h, \&c$ also form decreasing progressions;—and if the terms of any one of these sets of differences be multiplied into the corresponding terms of the series $1, x, x^2, x^3, x^4, x^5, x^6, x^7, \&c$, so as to produce a third series of which the first term shall be the first term of the said series of differences, and all the following terms shall involve in them the several successive powers of x , to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7, \&c$;—and, if the third, fifth, seventh, and other following odd terms of the said third series be added to its first term, and consequently marked with the sign $+$, and the second, fourth, sixth, eighth, and other following even terms of the said third series be subtracted from its first term, and consequently marked with the sign $-$; and, lastly, if the said third series be multiplied into the binomial quantity $1 + x$; the product of the said multiplication will be a fourth infinite series, of which the first term will be the same as the first term of the said third series that is multiplied into $1 + x$, or the first term of the above-mentioned series of differences, and the following terms will involve in them the several successive powers of x , to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7, \&c$; and the co-efficients of these powers of x in the terms of the said fourth series will be the several differences of the numbers $b, c, d, e, f, g, h, \&c$, of the next higher order to that of the differences contained in the foregoing, or third, series. And in the said fourth series, the second, fourth, sixth, and eighth, and other following even terms, will be marked with the sign $+$; and the third, fifth, and seventh, and other following odd terms, will be marked with the sign $-$.

DEMONSTRATION.

THE FIRST CASE.

In the first place we will suppose the differences of the co-efficients b, c, d, e, f, g, h , &c to be those of the first order, to wit, the differences $b - c, c - d, d - e, e - f, f - g, g - h$, &c. Now, if these differences are multiplied into the corresponding terms of the series $1, x, x^2, x^3, x^4, x^5, x^6$, &c, they will produce the series $\overline{b-c} \times 1$, or $b - c, \overline{c-d} \times x, \overline{d-e} \times x^2, \overline{e-f} \times x^3, \overline{f-g} \times x^4, \overline{g-h} \times x^5$, &c; and, if we prefix the sign + to the third, fifth, seventh, and other following odd terms of this series, and the sign - to the second, fourth, sixth, eighth, and other following even terms of the said series, agreeably to the directions of this Lemma, it will become $b - c - \overline{c-d} \times x + \overline{d-e} \times x^2 - \overline{e-f} \times x^3 + \overline{f-g} \times x^4 - \overline{g-h} \times x^5 + \&c$ *ad infinitum*. We are therefore to prove, that, if this last, or third, series $b - c - \overline{c-d} \times x + \overline{d-e} \times x^2 - \overline{e-f} \times x^3 + \overline{f-g} \times x^4 - \overline{g-h} \times x^5 + \&c$ be multiplied into the binomial quantity $1 + x$, the product of such multiplication will be the series $b - c + \overline{b-2c+d} \times x - \overline{c-2d+e} \times x^2 + \overline{d-2e+f} \times x^3 - \overline{e-2f+g} \times x^4 + \overline{f-2g+h} \times x^5 - \&c$ *ad infinitum*. Now this will appear from the operation itself of multiplying the former, or third, series $b - c - \overline{c-d} \times x + \overline{d-e} \times x^2 - \overline{e-f} \times x^3 + \overline{f-g} \times x^4 - \overline{g-h} \times x^5 + \&c$ into the binomial quantity $1 + x$; which will be as follows:

$$\begin{array}{r}
 b - c - \overline{c-d} \times x + \overline{d-e} \times x^2 - \overline{e-f} \times x^3 + \overline{f-g} \times x^4 - \overline{g-h} \times x^5 + \&c \\
 1 + x \\
 \hline
 b - c - \overline{c-d} \times x + \overline{d-e} \times x^2 - \overline{e-f} \times x^3 + \overline{f-g} \times x^4 - \overline{g-h} \times x^5 + \&c \\
 + b - c - \overline{c-d} \times x + \overline{d-e} \times x^2 - \overline{e-f} \times x^3 + \overline{f-g} \times x^4 - \overline{g-h} \times x^5 + \&c \\
 \hline
 b - c + \overline{b-2c+d} \times x - \overline{c-2d+e} \times x^2 + \overline{d-2e+f} \times x^3 - \overline{e-2f+g} \times x^4 + \overline{f-2g+h} \times x^5 - \&c
 \end{array}$$

By this operation it appears that the product of this multiplication is the series $b - c + \overline{b-2c+d} \times x - \overline{c-2d+e} \times x^2 + \overline{d-2e+f} \times x^3 - \overline{e-2f+g} \times x^4 + \overline{f-2g+h} \times x^5 - \&c$, of which the first term is $b - c$, or the first term of the first differences, $b - c, c - d, d - e, e - f, f - g, g - h$, &c, of the co-efficients b, c, d, e, f, g, h , &c, and the following terms involve the several successive powers of x , to wit, x, x^2, x^3, x^4, x^5 , &c, and have for their co-efficients the several quantities $b - 2c + d, c - 2d + e, d - 2e + f, e - 2f + g$, and $f - 2g + h$, &c, or the several second differ-

ences

ences of the said co-efficients $b, c, d, e, f, g, h, \&c.$ And in this series it is evident that the second, fourth, and sixth, terms, $b - 2c + d \times x$, $d - 2e + f \times x^3$, $f - 2g + h \times x^5$, and the other following even terms of the said series, will have the sign $+$ prefixed to them; and that the third and fifth terms, $c - 2d + e \times x^2$ and $e - 2f + g \times x^4$, and the other following odd terms of the said series, will have the sign $-$ prefixed to them.

Q. E. D.

THE SECOND CASE.

Secondly, Let the differences of the co-efficients $b, c, d, e, f, g, h, \&c$ be those of the second order, to wit, the differences $b - 2c + d$, $c - 2d + e$, $d - 2e + f$, $e - 2f + g$, $f - 2g + h$, $\&c.$

Now, if these differences are multiplied into the corresponding terms of the series $1, x, x^2, x^3, x^4, x^5, x^6, x^7, \&c.$ the series thereby produced will be $b - 2c + d$, $c - 2d + e \times x$, $d - 2e + f \times x^2$, $e - 2f + g \times x^3$, $f - 2g + h \times x^4$, $\&c$; and, if we prefix the sign $+$ to the third, fifth, seventh, and other following odd terms of this series, and the sign $-$ to the second, fourth, sixth, eighth, and other following even terms of the said series, agreeably to the directions of this Lemma, it will become $b - 2c + d - [c - 2d + e] \times x + [d - 2e + f] \times x^2 - [e - 2f + g] \times x^3 + [f - 2g + h] \times x^4 - \&c$ *ad infinitum*. We are therefore to prove that, if this last series $b - 2c + d - [c - 2d + e] \times x + [d - 2e + f] \times x^2 - [e - 2f + g] \times x^3 + [f - 2g + h] \times x^4 - \&c$ be multiplied into the binomial quantity $1 + x$, the product of such multiplication will be the series $b - 2c + d + b - 3c + 3d - e \times x - [c - 3d + 3e - f] \times x^2 + [d - 3e + 3f - g] \times x^3 - [e - 3f + 3g - h] \times x^4 + \&c$ *ad infinitum*. Now this will appear by making the said multiplication of the said series $b - 2c + d - [c - 2d + e] \times x + [d - 2e + f] \times x^2 - [e - 2f + g] \times x^3 + [f - 2g + h] \times x^4 - \&c$ into the binomial quantity $1 + x$; which may be performed as follows:

$$\begin{array}{r}
 b - 2c + d - [c - 2d + e] \times x + [d - 2e + f] \times x^2 - [e - 2f + g] \times x^3 + [f - 2g + h] \times x^4 - \&c \\
 1 + x \\
 \hline
 b - 2c + d - [c - 2d + e] \times x + [d - 2e + f] \times x^2 - [e - 2f + g] \times x^3 + [f - 2g + h] \times x^4 - \&c \\
 + b - 2c + d \times x - [c - 2d + e] \times x^2 + [d - 2e + f] \times x^3 - [e - 2f + g] \times x^4 + \&c \\
 \hline
 b - 2c + d + b - 3c + 3d - e \times x - [c - 3d + 3e - f] \times x^2 + [d - 3e + 3f - g] \times x^3 - [e - 3f + 3g - h] \times x^4 + \&c
 \end{array}$$

4 E 2

By

By this operation it appears that the product of this multiplication is the series $b - 2c + d + \overline{b - 3c + 3d - e} \times x - \overline{c - 3d + 3e - f} \times x^2 + \overline{d - 3e + 3f - g} \times x^3 - \overline{e - 3f + 3g - b} \times x^4 + \&c$, of which the first term is $b - 2c + d$, or the first term of the series of second differences, $b - 2c + d$, $c - 2d + e$, $d - 2e + f$, $e - 2f + g$, $f - 2g + b$, &c of the co-efficients $b, c, d, e, f, g, b, \&c$, and the following terms involve the several successive powers of x , to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7$, &c, and have for their co-efficients the several quantities $b - 3c + 3d - e$, $c - 3d + 3e - f$, $d - 3e + 3f - g$, and $e - 3f + 3g - b$, &c, or the several third differences of the said co-efficients $b, c, d, e, f, g, b, \&c$. And in this series it is evident that the second and fourth terms $\overline{b - 3c + 3d - e} \times x$ and $\overline{d - 3e + 3f - g} \times x^3$, and the other following even terms of the said series will have the sign + prefixed to them; and that the third and fifth terms $\overline{c - 3d + 3e - f} \times x^2$ and $\overline{e - 3f + 3g - b} \times x^4$, and the other following odd terms of the said series will have the sign - prefixed to them. Q. E. D.

THE THIRD CASE.

Thirdly, Let the differences of the co-efficients $b, c, d, e, f, g, b, \&c$ be those of the third order, to wit, the differences $b - 3c + 3d - e$, $c - 3d + 3e - f$, $d - 3e + 3f - g$, $e - 3f + 3g - b$, &c.

Now, if these differences are multiplied into the corresponding terms of the series $1, x, x^2, x^3, x^4, x^5, x^6, \&c$, the series thereby produced will be $b - 3c + 3d - e$, $\overline{c - 3d + 3e - f} \times x$, $\overline{d - 3e + 3f - g} \times x^2$, and $\overline{e - 3f + 3g - b} \times x^3$ &c; and, if we prefix the sign + to the third, fifth, seventh, and other following odd terms of this series, and the sign - to the second, fourth, sixth, eighth, and other following even terms of the said series, agreeably to the directions of this Lemma, it will become $b - 3c + 3d - e - \overline{c - 3d + 3e - f} \times x + \overline{d - 3e + 3f - g} \times x^2 + \overline{e - 3f + 3g - b} \times x^3 - \&c$ *ad infinitum*. We are therefore to prove, that, if this last series $b - 3c + 3d - e - \overline{c - 3d + 3e - f} \times x + \overline{d - 3e + 3f - g} \times x^2 - \overline{e - 3f + 3g - b} \times x^3 + \&c$ be multiplied into the binomial quantity $1 + x$, the product of such multiplication will be the series $b - 3c + 3d - e + \overline{b - 4c + 6d - 4e + f} \times x - \overline{c - 4d + 6e - 4f + g} \times x^2 + \overline{d - 4e + 6f - 4g + b} \times x^3 - \&c$ *ad infinitum*.

Now

Now this will appear by performing the said multiplication; which may be done as follows:

$$b-3c+3d-e-(c-3d+3e-f) \times x + d-3e+3f-g \times x^2 - e-3f+3g-b \times x^3 + \&c$$

$$b-3c+3d-e-(c-3d+3e-f) \times x + d-3e+3f-g \times x^2 - e-3f+3g-b \times x^3 + \&c$$

$$+ b-3c+3d-e-(c-3d+3e-f) \times x^2 + d-3e+3f-g \times x^3 - \&c$$

$$b-3c+3d-e+b-4c+6d-4e+f \times x - (c-4d+6e-4f+g) \times x^2 + d-4e+6f-4g+b \times x^3 - \&c$$

By this operation it appears that the product of this multiplication is the series $b-3c+3d-e+b-4c+6d-4e+f \times x - (c-4d+6e-4f+g) \times x^2 + d-4e+6f-4g+b \times x^3 - \&c$ *ad infinitum*, of which the first term is $b-3c+3d-e$, or the first term of the series of third differences, $b-3c+3d-e$, $c-3d+3e-f$, $d-3e+3f-g$, and $e-3f+3g-b$, &c, of the co-efficients $b, c, d, e, f, g, b, \&c$, and the following terms involve the several successive powers of x , to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7$, &c, and have for their co-efficients the several quantities $b-4c+6d-4e+f, c-4d+6e-4f+g$, and $d-4e+6f-4g+b$, &c, or the several fourth differences of the said co-efficients $b, c, d, e, f, g, b, \&c$. And in this series it is evident that the second and fourth terms $b-4c+6d-4e+f \times x$ and $d-4e+6f-4g+b \times x^3$, and the other following even terms of the said series, will have the sign + prefixed to them; and that the third term $c-4d+6e-4f+g \times x^2$, and the other following odd terms of the said series, will have the sign - prefixed to them.

Q. E. D.

A General Conclusion from the foregoing Cases to all other Cases whatsoever.

Art. 5. From the three foregoing multiplications of the three serieses $b-c-(c-d) \times x + d-e \times x^2 - (e-f) \times x^3 + f-g \times x^4 - (g-b) \times x^5 + \&c$ *ad infinitum*, and $b-2c+d-(c-2d+e) \times x + (d-2e+f) \times x^2 - (e-2f+g) \times x^3 + f-2g+b \times x^4 - \&c$ *ad infinitum*, and $b-3c+3d-e-(c-3d+3e-f) \times x + d-3e+3f-g \times x^2 - e-3f+3g-b \times x^3 + \&c$ *ad infinitum*, into the binomial quantity $1+x$, it is easy to see that, if any other similar series, of which the terms involved any higher set of differences of the co-efficients $b, c, d, e, f, g, b, \&c$, whatsoever, and which were marked with the signs - and + alternately, were to be multiplied into the said binomial

mial quantity $1 + x$, the first term of the series produced by such multiplication would be the same with the first term of the series multiplied, and the co-efficients of all the following terms of it would be the differences of the co-efficients of the terms of the former, or multiplied, series, by reason of the alternate succession of the signs — and + which are prefixed to the terms of the said multiplied series; and consequently, as the co-efficients of the terms of the former series are a series of the differences, of some particular order, of the original co-efficients $b, c, d, e, f, g, h, \&c$, the co-efficients of all the terms, after the first term, of the new series, produced by the said multiplication, must be the terms of the next order of differences of the said original co-efficients $b, c, d, e, f, g, h, \&c$ above those which form the co-efficients of the former, or multiplied, series. And therefore the Lemma will be universally true, of however high an order the differences which form the co-efficients of the multiplied series may happen to be.

Q. E. D.

The same Conclusion demonstrated by means of a more General Notation.

Art. 6. But this may also be shewn in a somewhat different manner by means of the following general notation.

Let the whole number n denote any order whatsoever of the differences of the original co-efficients $b, c, d, e, f, g, h, \&c$; and let the first, second, third, fourth, and other following terms of the said series of differences of the n th order be denoted by the several expressions $D_1^n, D_2^n, D_3^n, D_4^n, D_5^n, D_6^n, D_7^n, D_8^n, \&c$, *ad infinitum*: Then will the differences of the said original co-efficients $b, c, d, e, f, g, h, \&c$, of the next higher, or the $n + 1$ th, order, be $D_1^n - D_2^n, D_2^n - D_3^n, D_3^n - D_4^n, D_4^n - D_5^n, D_5^n - D_6^n, D_6^n - D_7^n, D_7^n - D_8^n, \&c$. Now, according to this notation, the Proposition laid down in the foregoing Lemma is this; to wit, that, if the series $D_1^n - D_2^n \times x + D_3^n \times x^2 - D_4^n \times x^3 + D_5^n \times x^4 - D_6^n \times x^5 + D_7^n \times x^6 - D_8^n \times x^7 + \&c$ be multiplied into the binomial quantity $1 + x$, the product of the said multiplication will be the series $D_1^n + D_1^n - D_2^n \times x - [D_2^n - D_3^n] \times x^2 + D_3^n - D_4^n \times x^3 - [D_4^n - D_5^n] \times x^4 + D_5^n - D_6^n \times x^5$

$$-\overline{D_6^n - D_7^n} \times x^6 + \overline{D_7^n - D_8^n} \times x^7 - \&c \text{ ad infinitum, or } D^n + \\ D^{\frac{n+1}{1}} \times x - D^{\frac{n+1}{2}} \times x^2 + D^{\frac{n+1}{3}} \times x^3 - D^{\frac{n+1}{4}} \times x^4 + D^{\frac{n+1}{5}} \\ \times x^5 - D^{\frac{n+1}{6}} \times x^6 + D^{\frac{n+1}{7}} \times x^7 - \&c \text{ ad infinitum.}$$

Now this will readily appear by performing the said multiplication; which may be done as follows:

$$D_1^n - D_2^n \times x + D_3^n \times x^2 - D_4^n \times x^3 + D_5^n \times x^4 - D_6^n \times x^5 + \&c \\ 1 + x$$

$$D_1^n - D_2^n \times x + D_3^n \times x^2 - D_4^n \times x^3 + D_5^n \times x^4 - D_6^n \times x^5 + \&c \\ + D_1^n \times x - D_2^n \times x^2 + D_3^n \times x^3 - D_4^n \times x^4 + D_5^n \times x^5 - \&c$$

$$D_1^n + \overline{D_1^n - D_2^n} \times x - \overline{D_2^n - D_3^n} \times x^2 + \overline{D_3^n - D_4^n} \times x^3 - \overline{D_4^n - D_5^n} \times x^4 + \overline{D_5^n - D_6^n} \times x^5 - \&c$$

By this operation it appears that the product of the said multiplication is the the series $D_1^n + \overline{D_1^n - D_2^n} \times x - \overline{D_2^n - D_3^n} \times x^2 + \overline{D_3^n - D_4^n} \times x^3 - \overline{D_4^n - D_5^n} \times x^4 + \overline{D_5^n - D_6^n} \times x^5 - \&c \text{ ad infinitum, or } D_1^n + D^{\frac{n+1}{1}} \times x - D^{\frac{n+1}{2}} \times x^2 + D^{\frac{n+1}{3}} \times x^3 - D^{\frac{n+1}{4}} \times x^4 + D^{\frac{n+1}{5}} \times x^5 - D^{\frac{n+1}{6}} \times x^6 + D^{\frac{n+1}{7}} \times x^7 - \&c \text{ ad infinitum. } Q_2 \text{ E. D.}$

Mr. Hellins's Investigation of the foregoing Differential Series.

Art. 7. These three Lemmas being premised, Mr. Hellins's method of investigating the values of the several numeral co-efficients $P, Q, R, S, T, V, W, X, \&c$ of the terms of the assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Sx^3}{1+x^3}, \frac{Tx^4}{1+x^4}, \frac{Vx^5}{1+x^5}, \frac{Wx^6}{1+x^6}, \frac{Xx^7}{1+x^7}, \&c$, and the signs that are to be prefixed to the second and other following terms of the said series, may be explained in the following manner:

An

An Investigation of the Values of the numeral Co-efficients P, Q, R, S, T, V, W, X , &c of the Terms of the assumed Series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \frac{Wx^6}{1+x}, \frac{Xx^7}{1+x}$, &c, and of the Signs + or - that are to be prefixed to the Second and other following Terms of the said Series, upon a supposition that the said Series is always equal to the original Series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ ad infinitum, of how small a magnitude soever the variable Quantity x may be supposed to be taken.

Of the First Term P .

Art. 8. Since the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ ad infinitum, is supposed to be always equal to the series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \frac{Wx^6}{1+x}, \frac{Xx^7}{1+x}$, &c ad infinitum, while x is of any magnitude not greater than 1, how small soever that magnitude be taken, it will also be equal to it when $x = 0$. But, when $x = 0$, all the terms in each of the said two serieses that involve any of the powers of x will become equal to 0 likewise, and each of the said serieses will be reduced to it's first term. Therefore P , or the first term of the second, or assumed, series, will be equal to a , or the first term of the first, or original, series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ ad infinitum. Q. E. D.

Of the Second Term $\frac{Qx}{1+x}$.

Art. 9. Secondly, since P is equal to a , and the whole series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ is less than it's first term a , (because the second and fourth and sixth and eighth, and other following even terms of the said series, which are marked with the sign -, or subtracted from the first term a , are greater, respectively, than the third and fifth and seventh, and other following odd terms of the said series which are marked with the sign +, or added to the said first term,) it follows that the whole assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \frac{Wx^6}{1+x}, \frac{Xx^7}{1+x}$, &c (which is equal to the original

ginal series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$) must be less than its first term P , which is equal to a . And this will be true, of how small a magnitude soever we suppose the quantity x , and consequently the quantity $\frac{x}{1+x}$, (which decreases when x decreases, and becomes equal to $\frac{0}{1+0}$, or to $\frac{0}{1}$, or to 0, when x becomes equal to 0,) to be taken. Therefore, by

Lemma 1, the second term $\frac{Qx}{1+x}$ of the assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \frac{Wx^6}{1+x}, \frac{Xx^7}{1+x}, \&c$ (which answers to the series $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z, \&c$ in the said Lemma,) must be subtracted from the first term P , and consequently must be marked with the sign $-$. And consequently the said equation $a - bx + cx^2 - dx^3 + \&c = P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \frac{Wx^6}{1+x}, \frac{Xx^7}{1+x}, \&c$ will be $a - bx + cx^2 - dx^3 + \&c = P - \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \frac{Wx^6}{1+x}, \frac{Xx^7}{1+x}, \&c$.

Art. 10. Having thus determined the sign that is to be prefixed to the second term, $\frac{Qx}{1+x}$, of the said assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \&c$, we must next proceed to investigate its magnitude.

Now, since the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + \&c$ is equal to the series $P - \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \&c$, and the first term P of the latter series is equal to the first term a of the former series, and each of these serieses is less than its own first term, it follows that, if each of these serieses be subtracted from its own first term, the remainders will be equal to each other. But, if the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + \&c$ be subtracted from its own first term a , the remainder will be the series $bx - cx^2 + dx^3 - ex^4 + fx^5 - \&c$ *ad infinitum*, in which the signs of the several terms will be contrary to what they were before; and, if the series $P - \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \&c$, *ad infinitum* be subtracted from its first term P , the remainder will be the series $\frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \&c$, *ad infinitum*, in which the signs to be prefixed to the second and other following terms (which are hitherto unknown,) will, when they come to be known, be contrary to what they ought to be in the foregoing, or original, equation $a - bx + cx^2 - dx^3 + ex^4 - fx^5 - \&c$.

+ &c ad infinitum = $p - \frac{qx}{1+x}, \frac{rx^2}{1+x}, \frac{sx^3}{1+x}, \frac{tx^4}{1+x}, \frac{ux^5}{1+x}, \&c \text{ ad infinitum}$. Therefore we shall, by this subtraction, obtain a second equation, which will be as follows, to wit, $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c \text{ ad infinitum} = \frac{qx}{1+x}, \frac{rx^2}{1+x}, \frac{sx^3}{1+x}, \frac{tx^4}{1+x}, \frac{ux^5}{1+x}, \frac{wx^6}{1+x}, \frac{xx^7}{1+x}, \&c \text{ ad infinitum}$.

Now let all the terms of this 2d equation be divided by x .

And we shall thereby obtain a third equation, to wit, the equation $b - cx + dx^2 - ex^3 + fx^4 - gx^5 + hx^6 - \&c \text{ ad infinitum} = \frac{q}{1+x}, \frac{rx}{1+x}, \frac{sx^2}{1+x}, \frac{tx^3}{1+x}, \frac{ux^4}{1+x}, \frac{wx^5}{1+x}, \frac{xx^6}{1+x}, \&c \text{ ad infinitum}$.

And, in the third place, let both sides of this last equation be multiplied into the binomial quantity $1 + x$, which forms the denominator of the term $\frac{q}{1+x}$; and we shall thereby obtain a fourth equation, which will be as follows; to wit,

$$\left\{ \begin{array}{l} b - cx + dx^2 - ex^3 + fx^4 - gx^5 + hx^6 - \&c \\ + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + \&c \end{array} \right\} =$$

 $Q, \frac{rx}{1+x}, \frac{sx^2}{1+x}, \frac{tx^3}{1+x}, \frac{ux^4}{1+x}, \frac{wx^5}{1+x}, \frac{xx^6}{1+x}, \&c, \text{ or } b + \overbrace{b}^{\text{---}} - c \times x$
 $- \overbrace{c-d}^{\text{---}} \times x^2 + \overbrace{d-e}^{\text{---}} \times x^3 - \overbrace{e-f}^{\text{---}} \times x^4 + \overbrace{f-g}^{\text{---}} \times x^5 - \overbrace{g-h}^{\text{---}} \times x^6 + \&c \text{ ad infinitum} = Q, \frac{rx}{1+x}, \frac{sx^2}{1+x}, \frac{tx^3}{1+x}, \frac{ux^4}{1+x}, \frac{wx^5}{1+x}, \frac{xx^6}{1+x}, \&c \text{ ad infinitum}$, in which the co-efficient Q is freed from it's multiplication into the quantity x and it's division by the binomial quantity $1 + x$.

But this equation is always true, of however small a magnitude the variable quantity x be taken. And therefore it will also be true when x is = 0. But, when x is = 0, all the terms on both sides of the equation that involve any of the powers of x will become equal to 0 likewise, and consequently the only terms remaining in it will be b and Q . Therefore Q will be = b , and consequently the second term $\frac{qx}{1+x}$ of the assumed series $p - \frac{qx}{1+x}, \frac{rx^2}{1+x}, \frac{sx^3}{1+x}, \frac{tx^4}{1+x}, \frac{ux^5}{1+x}, \frac{wx^6}{1+x}, \frac{xx^7}{1+x}, \&c$ will be = $\frac{bx}{1+x}$. Therefore the two first terms of the said assumed series will be $a - \frac{bx}{1+x}$ $Q, E, I.$

The

The Investigation of the Third Term, $\frac{xx^3}{1+x^4}$, of the said assumed Series.

Art. 11. We are next to determine what sign is to be prefixed to the third term, $\frac{xx^3}{1+x^4}$, of the said assumed series $P = \frac{Qx}{1+x} + \frac{xx^3}{1+x^4} + \frac{xx^3}{1+x^4} + \frac{xx^3}{1+x^4} + \frac{xx^3}{1+x^4}$, $\frac{xx^3}{1+x^4}$, $\frac{xx^3}{1+x^4}$, $\frac{xx^3}{1+x^4}$, &c, and what will be the magnitude of the co-efficient R .

Now it was shewn in the last article, that the series $b + b - c \times x - (c - d \times x^2 + d - e) \times x^3 - (e - f) \times x^4 + (f - g) \times x^5 - (g - h) \times x^6 + \&c$ is equal to the series $Q, \frac{xx}{1+x}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}$, &c, and also that Q was equal to b . But b is less than the whole series $b + b - c \times x - (c - d \times x^2 + d - e) \times x^3 - (e - f) \times x^4 + (f - g) \times x^5 - (g - h) \times x^6 + \&c$, which forms the left-hand side of this equation. Therefore Q (which is equal to b) must be less than the whole series $Q, \frac{xx}{1+x}$,

$\frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}$, &c, which forms the right-hand side of the same equation; or the said whole series will be greater than it's first term Q . And this will always be true, of how small a magnitude soever the quantity x , and consequently the fraction $\frac{x}{1+x}$, be taken. Therefore, by the

second of the foregoing Lemmas, the second term, $\frac{xx}{1+x}$, of the said series $Q, \frac{xx}{1+x}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}, \frac{xx^3}{1+x^4}$, &c (in which the fraction $\frac{x}{1+x}$ answers to the variable quantity y in the series $A, Ay, Cy^2, Dy^3, Ey^4, Fy^5, Gy^6, Hy^7$, &c described in that Lemma,) must be added to the first term Q , and consequently marked with the sign $+$. Therefore the last equation will be as follows; $b + b - c \times x - (c - d \times x^2 + d - e) \times x^3 - (e - f) \times x^4 + (f - g) \times x^5 - (g - h) \times x^6 + \&c$ *ad infinitum* $= Q + \frac{xx}{1+x} + \frac{xx^3}{1+x^4} + \frac{xx^3}{1+x^4} + \frac{xx^3}{1+x^4} + \frac{xx^3}{1+x^4} + \frac{xx^3}{1+x^4}$, &c *ad infinitum*. But the sign that is to be prefixed to the term $\frac{xx}{1+x}$ in this equation is contrary to the sign that is to be

prefixed to the corresponding term $\frac{rx^3}{1+x}$ in the original equation $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c = P - \frac{Qx}{1+x} + \frac{Rx^3}{1+x} - \frac{Sx^5}{1+x} + \frac{Tx^7}{1+x} - \frac{Vx^9}{1+x} + \frac{Wx^{11}}{1+x} - \frac{Xx^{13}}{1+x} + \&c$, because the last equation was derived from the said original equation by means of a subtraction of the two sides of the said original equation from their first terms a and P , which caused a change of the signs of all its terms, and likewise by means of a division of the remaining terms by x , and a multiplication of the quotients of the said division into the binomial quantity $1 + x$, neither of which two latter operations made any further changes in the signs of the terms of the equation. Therefore the sign that is to be prefixed to the term $\frac{Rx^3}{1+x}$ in the original equation $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c = P - \frac{Qx}{1+x} + \frac{Rx^3}{1+x} - \frac{Sx^5}{1+x} + \frac{Tx^7}{1+x} - \frac{Vx^9}{1+x} + \frac{Wx^{11}}{1+x} - \frac{Xx^{13}}{1+x} + \&c$ must be the contrary sign to the sign $+$ which is prefixed to the term $\frac{Rx}{1+x}$ in the last equation $b + b - c \times x - c - d \times x^2 + d - e \times x^3 + e - f \times x^4 + f - g \times x^5 + g - h \times x^6 + h - i \times x^7 + i - j \times x^8 + j - k \times x^9 + k - l \times x^{10} + l - m \times x^{11} + m - n \times x^{12} + n - o \times x^{13} + o - p \times x^{14} + p - q \times x^{15} + q - r \times x^{16} + r - s \times x^{17} + s - t \times x^{18} + t - u \times x^{19} + u - v \times x^{20} + v - w \times x^{21} + w - x \times x^{22} + x - y \times x^{23} + y - z \times x^{24} + z - \&c = Q + \frac{Rx}{1+x} - \frac{Sx^3}{1+x} + \frac{Tx^5}{1+x} - \frac{Vx^7}{1+x} + \frac{Wx^9}{1+x} - \frac{Xx^{11}}{1+x} + \frac{Yx^{13}}{1+x} - \frac{Zx^{15}}{1+x} + \&c$; that is, it must be the sign $-$. And consequently the three first terms of the said assumed series $P, \frac{Qx}{1+x}, \frac{Rx^3}{1+x}$, &c in the said original equation will be $P - \frac{Qx}{1+x} - \frac{Rx^3}{1+x}$, or $a - \frac{bx}{1+x} - \frac{cx^3}{1+x}$. Q. E. I.

Art. 12. It remains that we determine the magnitude of the co-efficient R ; which may be done as follows:

It has been shewn that the series $b + b - c \times x - c - d \times x^2 + d - e \times x^3 + e - f \times x^4 + f - g \times x^5 + g - h \times x^6 + h - i \times x^7 + i - j \times x^8 + j - k \times x^9 + k - l \times x^{10} + l - m \times x^{11} + m - n \times x^{12} + n - o \times x^{13} + o - p \times x^{14} + p - q \times x^{15} + q - r \times x^{16} + r - s \times x^{17} + s - t \times x^{18} + t - u \times x^{19} + u - v \times x^{20} + v - w \times x^{21} + w - x \times x^{22} + x - y \times x^{23} + y - z \times x^{24} + z - \&c$ is equal to the series $Q + \frac{Rx}{1+x} - \frac{Sx^3}{1+x} + \frac{Tx^5}{1+x} - \frac{Vx^7}{1+x} + \frac{Wx^9}{1+x} - \frac{Xx^{11}}{1+x} + \frac{Yx^{13}}{1+x} - \frac{Zx^{15}}{1+x} + \&c$, and likewise that Q is b . Therefore, if we subtract b and Q from the opposite sides of this equation, the remainders will be equal to each other; that is, the series $b - c \times x - c - d \times x^2 + d - e \times x^3 + e - f \times x^4 + f - g \times x^5 + g - h \times x^6 + h - i \times x^7 + i - j \times x^8 + j - k \times x^9 + k - l \times x^{10} + l - m \times x^{11} + m - n \times x^{12} + n - o \times x^{13} + o - p \times x^{14} + p - q \times x^{15} + q - r \times x^{16} + r - s \times x^{17} + s - t \times x^{18} + t - u \times x^{19} + u - v \times x^{20} + v - w \times x^{21} + w - x \times x^{22} + x - y \times x^{23} + y - z \times x^{24} + z - \&c$ will be equal to the series $\frac{Rx}{1+x} - \frac{Sx^3}{1+x} + \frac{Tx^5}{1+x} - \frac{Vx^7}{1+x} + \frac{Wx^9}{1+x} - \frac{Xx^{11}}{1+x} + \frac{Yx^{13}}{1+x} - \frac{Zx^{15}}{1+x} + \&c$.

$\frac{bx^4}{1+x^4}, \frac{wx^3}{1+x^3}, \frac{xx^2}{1+x^2}, \&c.$ Therefore, if we divide all the terms on both sides by x , we shall have $b - c - (c - d) \times x + (d - e) \times x^2 - (e - f) \times x^3 + (f - g) \times x^4 - (g - h) \times x^5 + \&c = \frac{R}{1+x}, \frac{Tx^2}{1+x^2}, \frac{Vx^3}{1+x^3}, \frac{Wx^4}{1+x^4}, \&c$; and, if we multiply all the terms of this equation into the binomial quantity $1 + x$, so as to disengage the co-efficient R from its denominator, or divisor, $1 + x$, we shall obtain the equation

$$\left\{ \begin{aligned} &b - c - (c - d) \times x + (d - e) \times x^2 - (e - f) \times x^3 + (f - g) \times x^4 - (g - h) \times x^5 + \&c \\ &+ (b - c) \times x - (c - d) \times x^2 + (d - e) \times x^3 - (e - f) \times x^4 + (f - g) \times x^5 - \&c \end{aligned} \right\} \\ = R, \frac{Tx}{1+x}, \frac{Tx^2}{1+x^2}, \frac{Vx^3}{1+x^3}, \frac{Wx^4}{1+x^4}, \frac{xx^5}{1+x^5}, \&c, \text{ or } b - c + b - 2c + d \\ \times x - (c - 2d + e) \times x^2 + (d - 2e + f) \times x^3 - (e - 2f + g) \times x^4 \\ + (f - 2g + h) \times x^5 - \&c = R, \frac{Tx}{1+x}, \frac{Tx^2}{1+x^2}, \frac{Vx^3}{1+x^3}, \frac{Wx^4}{1+x^4}, \frac{xx^5}{1+x^5}, \\ \&c.$$

Now let x become $= 0$.

Then will all the terms in this last equation that involve any of the powers of x become equal to 0 likewise, and consequently the said equation will have only the quantity $b - c$ left on one side, and only the co-efficient R left on the other. Therefore R will be $= b - c$; and consequently the three first terms of the assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Tx^3}{1+x^3}, \frac{Vx^4}{1+x^4}, \frac{Wx^5}{1+x^5}, \frac{xx^6}{1+x^6}, \&c$ will be $a - \frac{bx}{1+x} - (b - c) \times \frac{x^2}{1+x^2}$, or (because b^2 is $= b - c$)
 $a - \frac{bx}{1+x} - \frac{b^2x^2}{1+x^2}, \quad Q, E. I.$

The Investigation of the Fourth Term, $\frac{bx^3}{1+x^3}$, of the said assumed Series.

Art. 13. We are next to determine what sign is to be prefixed to the fourth term, $\frac{bx^3}{1+x^3}$, of the assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Tx^3}{1+x^3}, \frac{Vx^4}{1+x^4}, \frac{Wx^5}{1+x^5}, \&c$, and what will be the magnitude of the co-efficient S .

Now

Now it has been shewn in the last article, that the series $b - c + \overline{b - 2c + d} \times x - \overline{c - 2d + e} \times x^2 + \overline{d - 2e + f} \times x^3 - \overline{e - 2f + g} \times x^4 + \overline{f - 2g + h} \times x^5 - \&c$ is equal to the series $\mathbf{r}, \frac{bx}{1+x}, \frac{Tx^2}{1+x}, \frac{Vx^3}{1+x}, \frac{Wx^4}{1+x}, \frac{Xx^5}{1+x}, \&c$, and likewise that $\mathbf{r}$ is equal to $b - c$. Therefore, if the former of these series is greater than $b - c$, the latter of them must be greater than $\mathbf{r}$. But the former of these series is greater than $b - c$; because the next term $\overline{b - 2c + d} \times x$, which is added to $b - c$, is greater than the following term $\overline{c - 2d + e} \times x^2$, which is subtracted from $b - c$; and, in like manner, every following term in the said series that is marked with the sign $+$, or is to be added to $b - c$, will be greater than the term immediately following it, which is marked with the sign $-$, or is to be subtracted from $b - c$. Therefore the other series $\mathbf{r}, \frac{bx}{1+x}, \frac{Tx^2}{1+x}, \frac{Vx^3}{1+x}, \frac{Wx^4}{1+x}, \frac{Xx^5}{1+x}, \&c$ will be greater than its first term $\mathbf{r}$. And this will be always true, of how small a magnitude soever the quantity x , and consequently the fraction $\frac{x}{1+x}$, be supposed to be taken. Therefore, by the second of the foregoing Lemmas, the second term $\frac{bx}{1+x}$ of this series must be added to its first term $\mathbf{r}$, and consequently must have the sign $+$ prefixed to it; and therefore the last equation will be $b - c + \overline{b - 2c + d} \times x - \overline{c - 2d + e} \times x^2 + \overline{d - 2e + f} \times x^3 - \overline{e - 2f + g} \times x^4 + \overline{f - 2g + h} \times x^5 - \&c = \mathbf{r} + \frac{bx}{1+x}, \frac{Tx^2}{1+x}, \frac{Vx^3}{1+x}, \frac{Wx^4}{1+x}, \frac{Xx^5}{1+x}, \&c$. Therefore in the assumed series $\mathbf{p}, \frac{Qx}{1+x}, \frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Vx^5}{1+x}, \frac{Wx^6}{1+x}, \frac{Xx^7}{1+x}, \&c$ (from which the present series $\mathbf{r} + \frac{bx}{1+x}, \frac{Tx^2}{1+x}, \frac{Vx^3}{1+x}, \frac{Wx^4}{1+x}, \frac{Xx^5}{1+x}, \&c$ is derived by means of the subtraction mentioned in art. 10, which causes all the signs of the terms of the assumed series to be changed, and by means of certain other operations by which the signs of the terms are not affected,) the corresponding term $\frac{Rx^2}{1+x}$ must have the sign $-$ prefixed to it; And consequently the four first terms of the said assumed series will be $\mathbf{p} - \frac{Qx}{1+x} - \frac{Rx^2}{1+x} - \frac{Sx^3}{1+x}$, or $\mathbf{a} - \frac{bx}{1+x} - \frac{Tx^2}{1+x} - \frac{Vx^3}{1+x}$.

Q. E. I.

Art. 14.

Art. 14. It remains that we determine the magnitude of the co-efficient S ; which may be done as follows:

It has been shewn in the last article, that the series $b - c + \overline{b - 2c + d} \times x - \overline{c - 2d + e} \times x^2 + \overline{d - 2e + f} \times x^3 - \overline{e - 2f + g} \times x^4 + \overline{f - 2g + h} \times x^5 - \&c$ is = the series $R + \frac{bx}{1+x}, \frac{Tx^2}{1+x^2}, \frac{Vx^3}{1+x^3}, \frac{Wx^4}{1+x^4}, \frac{Xx^5}{1+x^5}, \&c$, and likewise that R is = $b - c$. Therefore, if we subtract $b - c$ and R from the opposite sides of this last equation, the remainders will be equal to each other; that is, the series $\overline{b - 2c + d} \times x - \overline{c - 2d + e} \times x^2 + \overline{d - 2e + f} \times x^3 - \overline{e - 2f + g} \times x^4 + \overline{f - 2g + h} \times x^5 - \&c$ will be equal to the series $\frac{bx}{1+x}, \frac{Tx^2}{1+x^2}, \frac{Vx^3}{1+x^3}, \frac{Wx^4}{1+x^4}, \frac{Xx^5}{1+x^5}, \&c$. And consequently, if we divide all the terms of this last equation by x , we shall have the series $\overline{b - 2c + d} - \overline{c - 2d + e} \times x + \overline{d - 2e + f} \times x^2 - \overline{e - 2f + g} \times x^3 + \overline{f - 2g + h} \times x^4 - \&c$ = the series $\frac{b}{1+x}, \frac{Tx}{1+x^2}, \frac{Vx^2}{1+x^3}, \frac{Wx^3}{1+x^4}, \frac{Xx^4}{1+x^5}, \&c$; and, if we multiply all the terms of this equation into the binomial quantity $1 + x$, in order to disengage the co-efficient S from it's divisor $1 + x$, we shall obtain the following equation, to wit,

$$\left\{ \begin{array}{l} b - 2c + d - \overline{c - 2d + e} \times x + \overline{d - 2e + f} \times x^2 - \overline{e - 2f + g} \times x^3 + \overline{f - 2g + h} \times x^4 - \&c \\ + \overline{b - 2c + d} \times x - \overline{c - 2d + e} \times x^2 + \overline{d - 2e + f} \times x^3 - \overline{e - 2f + g} \times x^4 + \&c \end{array} \right\} \\ = S, \frac{Tx}{1+x}, \frac{Vx^2}{1+x^3}, \frac{Wx^3}{1+x^4}, \frac{Xx^4}{1+x^5}, \&c, \text{ or } b - 2c + d + \\ b - 3c + 3d - e \times x - \overline{c - 3d + 3e - f} \times x^2 + \overline{d - 3e + 3f - g} \times x^3 - \\ - \overline{e - 3f + 3g - h} \times x^4 + \&c = S, \frac{Tx}{1+x}, \frac{Vx^2}{1+x^3}, \frac{Wx^3}{1+x^4}, \frac{Xx^4}{1+x^5}, \&c.$$

Now let x become = 0.

Then will all the terms in this last equation that involve any of the powers of x become equal to 0 likewise, and consequently the only quantity left on the left-hand side of the equation will be $b - 2c + d$, and the only quantity left on the right hand side of it will be S . Therefore S will be = $b - 2c + d$;

and consequently the four first terms of the assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Sx^3}{1+x^3}, \frac{Tx^4}{1+x^4}, \frac{Vx^5}{1+x^5}, \frac{Wx^6}{1+x^6}, \frac{Xx^7}{1+x^7}, \&c$ will be $a - \frac{bx}{1+x} - \overline{b - c} \times \frac{x^2}{1+x^2} - \overline{b - 2c + d} \times \frac{x^3}{1+x^3}$, or $a - \frac{bx}{1+x} - \frac{D^2x^2}{1+x^2} - \frac{D^3x^3}{1+x^3}$.

Q. E. I.

Tb

The Investigation of the Fifth Term, $\frac{Tx^4}{1+x}$, of the said assumed Series.

ART. 15. We are next to determine what sign is to be prefixed to the fifth term, $\frac{Tx^4}{1+x}$, of the assumed series P , $\frac{Qx^2}{1+x}$, $\frac{Rx^3}{1+x}$, $\frac{Sx^4}{1+x}$, $\frac{Tx^5}{1+x}$, $\frac{Ux^6}{1+x}$, $\frac{Vx^7}{1+x}$, &c, and what will be the magnitude of the co-efficient T .

Now it has been shewn in the last article, that the series $b - 2c + d + \frac{b-3c+3d-e}{1+x} \times x - \frac{c-3d+3e-f}{1+x} \times x^2 + \frac{d-3e+3f-g}{1+x} \times x^3 - \frac{e-3f+3g-h}{1+x} \times x^4 + \&c$ ad infinitum, is equal to the series S , $\frac{Tx}{1+x}$, $\frac{Vx^2}{1+x}$, $\frac{Wx^3}{1+x}$, $\frac{Xx^4}{1+x}$, &c, and likewise that S is equal to $b - 2c + d$. Therefore, if the former of these serieses is greater than $b - 2c + d$, the latter of them must be greater than S . But the former of these serieses is greater than $b - 2c + d$; because the next term $\frac{b-3c+3d-e}{1+x} \times x$, which is added to $b - 2c + d$, is greater than the following term $\frac{c-3d+3e-f}{1+x} \times x^2$, which is subtracted from $b - 2c + d$; and, in like manner, every following term in the said series that is marked with the sign $+$, or is to be added to $b - 2c + d$, will be greater than the term immediately following it, which is marked with the sign $-$, or is to be subtracted from $b - 2c + d$. Therefore the other series S , $\frac{Tx}{1+x}$, $\frac{Vx^2}{1+x}$, $\frac{Wx^3}{1+x}$, $\frac{Xx^4}{1+x}$, &c will be greater than it's first term S . And this will be always true, of how small a magnitude soever the quantity x , and consequently the fraction $\frac{x}{1+x}$, be supposed to be taken. Therefore, by the second of the foregoing Lemmas, the second term, $\frac{Tx}{1+x}$, of this series must be added to it's first term S , and consequently must have the sign $+$ prefixed to it; and therefore the last equation will be $b - 2c + d + \frac{b-3c+3d-e}{1+x} \times x - \frac{c-3d+3e-f}{1+x} \times x^2 + \frac{d-3e+3f-g}{1+x} \times x^3 - \frac{e-3f+3g-h}{1+x} \times x^4 + \&c = S + \frac{Tx}{1+x} + \frac{Vx^2}{1+x} + \frac{Wx^3}{1+x} + \frac{Xx^4}{1+x}$, &c. Therefore in the assumed series P , $\frac{Qx^2}{1+x}$, $\frac{Rx^3}{1+x}$, $\frac{Sx^4}{1+x}$, $\frac{Tx^5}{1+x}$, $\frac{Ux^6}{1+x}$, $\frac{Vx^7}{1+x}$, &c (from which the pre-

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sent series $S + \frac{Tx}{1+x}, \frac{Vx^2}{1+x}, \frac{Wx^3}{1+x}, \frac{Xx^4}{1+x}, \&c$ is derived by means of the subtraction mentioned in art. 10, which causes all the signs of that series to be changed, and by means of certain other arithmetical operations by which the signs of the terms are not affected,) the corresponding term $\frac{Tx^4}{1+x}$ must have the contrary sign — prefixed to it. And consequently the five first terms of the said assumed series will be $P - \frac{Qx}{1+x} - \frac{Rx^2}{1+x} - \frac{Sx^3}{1+x} - \frac{Tx^4}{1+x}$, or $a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x} - \frac{D^2x^3}{1+x} - \frac{Tx^4}{1+x}$. Q. E. I.

Art. 16. It remains that we determine the magnitude of the co-efficient T ; which may be done as follows:

It has been shewn in the last article, that the series $b - 2c + d + b - 3c + 3d - e \times x - \{c - 3d + 3e - f\} \times x^2 + \{d - 3e + 3f - g\} \times x^3 - \{e - 3f + 3g - h\} \times x^4 + \&c$ is = the series $S + \frac{Tx}{1+x}, \frac{Vx^2}{1+x}, \frac{Wx^3}{1+x}, \frac{Xx^4}{1+x}, \&c$, and likewise that S is = $b - 2c + d$. Therefore, if we subtract $b - 2c + d$ and S from the opposite sides of the former of these equations, the remainders will be equal to each other; that is, the series $b - 3c + 3d - e \times x - \{c - 3d + 3e - f\} \times x^2 + \{d - 3e + 3f - g\} \times x^3 - \{e - 3f + 3g - h\} \times x^4 + \&c$ *ad infinitum* will be = the series $\frac{Tx}{1+x}, \frac{Vx^2}{1+x}, \frac{Wx^3}{1+x}, \frac{Xx^4}{1+x}, \&c$. Therefore, if we divide all the terms of this equation by x , we shall have the series $b - 3c + 3d - e - \{c - 3d + 3e - f\} \times x + \{d - 3e + 3f - g\} \times x^2 - \{e - 3f + 3g - h\} \times x^3 + \&c$ *ad infinitum* = the series $\frac{T}{1+x}, \frac{Vx}{1+x}, \frac{Wx^2}{1+x}, \frac{Xx^3}{1+x}, \&c$ *ad infinitum*; and, if we multiply all the terms of this equation into the binomial quantity $1 + x$, in order to disengage the co-efficient T from it's divisor $1 + x$, we shall obtain the following equation, to wit,

$$\left\{ \begin{aligned} &b - 3c + 3d - e - \{c - 3d + 3e - f\} \times x + \{d - 3e + 3f - g\} \times x^2 - \{e - 3f + 3g - h\} \times x^3 + \&c \\ &+ b - 3c + 3d - e \times x - \{c - 3d + 3e - f\} \times x^2 + \{d - 3e + 3f - g\} \times x^3 - \&c \end{aligned} \right\} \\ = T, \frac{Vx}{1+x}, \frac{Wx^2}{1+x}, \frac{Xx^3}{1+x}, \&c, \text{ or the series } b - 3c + 3d - e$$

$$+ \frac{b - 4c + 6d - 4e + f}{1 + x} \times x - \frac{c - 4d + 6e - 4f + g}{1 + x} \times x^2 + \frac{d - 4e + 6f - 4g + h}{1 + x} \times x^3 - \&c \text{ ad infinitum} = \text{the series } \tau, \frac{vx}{1+x}, \frac{wx^2}{1+x^2}, \frac{xx^3}{1+x^3}, \&c \text{ ad infinitum}.$$

Now let x become $= 0$.

Then will all the terms in this last equation that involve any of the powers of x become equal to 0 likewise, and consequently the only quantity left on the left-hand side of the equation will be $b - 3c + 3d - e$, and the only quantity left on the right-hand side of the equation will be τ . Therefore τ will be $= b - 3c + 3d - e$; and consequently the five first terms of the assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Sx^3}{1+x^3}, \frac{Tx^4}{1+x^4}, \frac{Vx^5}{1+x^5}, \frac{Wx^6}{1+x^6}, \frac{Xx^7}{1+x^7}, \&c$ will be $a - \frac{bx}{1+x} - \frac{(b-c) \times x^2}{1+x^2} - \frac{(b-2c+d) \times x^3}{1+x^3} - \frac{(b-3c+3d-e) \times x^4}{1+x^4}$, or $a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x^2} - \frac{D^{11}x^3}{1+x^3} - \frac{D^{111}x^4}{1+x^4}$, Q. E. D.

The Investigation of the Sixth Term, $\frac{Vx^5}{1+x^5}$, of the said assumed Series.

Art. 17. We are next to determine what sign is to be prefixed to the sixth term, $\frac{Vx^5}{1+x^5}$, of the assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Sx^3}{1+x^3}, \frac{Tx^4}{1+x^4}, \frac{Vx^5}{1+x^5}, \frac{Wx^6}{1+x^6}, \frac{Xx^7}{1+x^7}, \&c$, and what will be the magnitude of the co-efficient V .

Now it has been shewn in the last article, that the series $b - 3c + 3d - e + \frac{b - 4c + 6d - 4e + f}{1 + x} \times x - \frac{c - 4d + 6e - 4f + g}{1 + x} \times x^2 + \frac{d - 4e + 6f - 4g + h}{1 + x} \times x^3 - \&c \text{ ad infinitum}$, is equal the series $\tau, \frac{vx}{1+x}, \frac{wx^2}{1+x^2}, \frac{xx^3}{1+x^3}, \&c \text{ ad infinitum}$ and likewise that τ is $= b - 3c + 3d - e$. Therefore, if the former of these series is greater than $b - 3c + 3d - e$, the latter of them must be greater than τ . But the former of these series is greater than $b - 3c + 3d - e$; because the next term $\frac{b - 4c + 6d - 4e + f}{1 + x} \times x$, which is added to $b - 3c + 3d - e$, is greater than the following term $\frac{c - 4d + 6e - 4f + g}{1 + x} \times x^2$, which is subtracted from $b - 3c + 3d - e$; and, in

in like manner, every following term in the said series that is marked with the sign +, or is to be added to $b - 3c + 3d - e$, will be greater than the term immediately following it, which is marked with the sign -, or is to be subtracted

from $b - 3c + 3d - e$. Therefore the other series $\tau, \frac{vx}{1+x}, \frac{wx^2}{1+x}, \frac{xx^3}{1+x},$ &c will be greater than it's first term τ . And this will be always true, of how small a magnitude soever the quantity x , and consequently the fraction $\frac{x}{1+x}$, be supposed to be taken. Therefore, by the second of the foregoing Lemmas,

the second term, $\frac{vx}{1+x}$, of this series must be added to its first term τ , and consequently must have the sign + prefixed to it; and therefore the last equation will be $b - 3c + 3d - e + b - 4c + 6d - 4e + f \times x - [c - 4d + 6e - 4f + g] \times x^2 + d - 4e + 6f - 4g - b \times x^3 - \&c = \tau + \frac{vx}{1+x}, \frac{wx^2}{1+x}, \frac{xx^3}{1+x}, \&c$. Therefore in the assumed series $\tau, \frac{Qx}{1+x},$

$\frac{Rx^2}{1+x}, \frac{Sx^3}{1+x}, \frac{Tx^4}{1+x}, \frac{Ux^5}{1+x}, \frac{Vx^6}{1+x}, \frac{Wx^7}{1+x}, \&c$, (from which the present series $\tau + \frac{vx}{1+x}, \frac{wx^2}{1+x}, \frac{xx^3}{1+x}, \&c$ is derived by means of the subtraction mentioned in art. 10, which causes all the signs of the terms of the said assumed series to be changed, and by means of certain other arithmetical operations by which the signs of the terms are not affected,) the corresponding term

$\frac{Vx^5}{1+x}$ must have the contrary sign - prefixed to it. And consequently the six first terms of the said assumed series will be $\tau - \frac{Qx}{1+x} - \frac{Rx^2}{1+x} - \frac{Sx^3}{1+x} - \frac{Tx^4}{1+x} - \frac{Ux^5}{1+x}$, or $a - \frac{bx}{1+x} - [b - c] \times \frac{x^2}{1+x} - [b - 2c + d] \times \frac{x^3}{1+x} - [b - 3c + 3d - e] \times \frac{x^4}{1+x} - \frac{Vx^5}{1+x}$, or $a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x} - \frac{D^{11}x^3}{1+x} - \frac{D^{111}x^4}{1+x} - \frac{Vx^5}{1+x}$.

Q. E. I.

Art. 18. It remains that we determine the magnitude of the co-efficient V ; which may be done as follows:

It has been shewn in the last article, that the series $b - 3c + 3d - e + b - 4c + 6d - 4e + f \times x - [c - 4d + 6e - 4f + g] \times x^2 + d - 4e + 6f - 4g + b \times x^3 - \&c$ *ad infinitum* is = the series $\tau + \frac{vx}{1+x},$

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$\frac{wx^2}{1+x},$

$\frac{wx^2}{1+x^2}, \frac{x^3}{1+x^2}, \&c.$, and likewise that $\tau = b - 3c + 3d - e$. Therefore, if we subtract $b - 3c + 3d - e$ and τ from the opposite sides of the former of these equations, the remainders will be equal to each other; that is, the series $\overline{b - 4c + 6d - 4e + f} \times x - \overline{c - 4d + 6e - 4f + g} \times x^2 + \overline{d - 4e + 6f - 4g + h} \times x^3 - \&c$ will be equal to the series $\frac{vx}{1+x}, \frac{wx^2}{1+x^2}, \frac{x^3}{1+x^2}, \&c.$ Therefore, if we divide all the terms of this equation by x , we shall have the series $b - 4c + 6d - 4e + f - \overline{c - 4d + 6e - 4f + g} \times x + \overline{d - 4e + 6f - 4g + h} \times x^2 - \&c$ *ad infinitum* $= \frac{v}{1+x}, \frac{wx}{1+x^2}, \frac{x^2}{1+x^2}, \&c$ *ad infinitum*; and, if we multiply all the terms of this equation into the binomial quantity $1 + x$, in order to disengage the co-efficient V from the divisor $1 + x$, we shall obtain the following equation, to wit,

$$\left\{ \begin{array}{l} b - 4c + 6d - 4e + f - \overline{c - 4d + 6e - 4f + g} \times x + \overline{d - 4e + 6f - 4g + h} \times x^2 - \&c \\ + b - 4c + 6d - 4e + f \end{array} \right\} \times x - \overline{c - 4d + 6e - 4f + g} \times x^2 + \&c \\ = v, \frac{wx}{1+x}, \frac{x^2}{1+x^2}, \&c, \text{ or } b - 4c + 6d - 4e + f + \\ \overline{b - 5c + 10d - 10e + 5f - g} \times x - \overline{c - 5d + 10e - 10f + 5g - h} \times x^2 \\ + \&c = v, \frac{wx}{1+x}, \frac{x^2}{1+x^2}, \&c.$$

Now let x become $= 0$.

Then will all the terms in this last equation that involve any of the powers of x become equal to 0 likewise; and consequently the only quantities left on the opposite sides of the equation will be $b - 4c + 6d - 4e + f$ and V . Therefore V will be $= b - 4c + 6d - 4e + f$; and consequently the six first terms of the assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Sx^3}{1+x^3}, \frac{Tx^4}{1+x^4}, \frac{Vx^5}{1+x^5}, \frac{Wx^6}{1+x^6}, \frac{x^2}{1+x^2}, \&c$ will be $a - \frac{bx}{1+x} - \overline{b - c} \times \frac{x^2}{1+x^2} - \overline{b - 2c + d} \times \frac{x^3}{1+x^3} - \overline{b - 3c + 3d - e} \times \frac{x^4}{1+x^4} - \overline{b - 4c + 6d - 4e + f} \times \frac{x^5}{1+x^5}$, or $a - \frac{bx}{1+x} - \frac{D^2x^2}{1+x^2} - \frac{D^3x^3}{1+x^3} - \frac{D^4x^4}{1+x^4} - \frac{D^5x^5}{1+x^5};$ Qc E. I.

The Investigation of the Seventh Term, $\frac{wx^6}{1+x^7}$, of the said assumed Series.

Art. 19. We are next to determine what sign is to be prefixed to the seventh term, $\frac{wx^6}{1+x^7}$, of the said assumed series $P = \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Sx^3}{1+x^3}, \frac{Tx^4}{1+x^4}, \frac{Vx^5}{1+x^5}, \frac{Wx^6}{1+x^7}$, &c, and what will be the magnitude of the co-efficient w.

Now it has been shewn in the last article, that the series $b - 4c + 6d - 4e + f + b - 5c + 10d - 10e + 5f - g \times x - c - 5d + 10e - 10f + 5g - b \times x^2 + \&c$ *ad infinitum* is = the series $v, \frac{wx}{1+x}, \frac{xa^2}{1+x^2}$, &c *ad infinitum*, and likewise that v is $= b - 4c + 6d - 4e + f$. But here, as the compound quantities on the left-hand side of the first of these equations are so long as to make it troublesome to repeat them as often as may be necessary, it will be convenient to express them in an abridged notation. I shall therefore denote the quantity $b - 4c + 6d - 4e + f$, (which is the first difference of the co-efficients $b, c, d, e, f, g, b, i, k, l, m$, &c of the fourth order,) by the expression v'' , and the several co-efficients of the powers of x on the same side of that equation, to wit, $b - 5c + 10d - 10e + 5f - g, c - 5d + 10e - 10f + 5g - b, d - 5e + 10f - 10g + 5b - i, e - 5f + 10g - 10b + 5i - k, f - 5g + 10b - 10i + 5k - l$, and $g - 5b + 10i - 10k + 5l - m$, &c, (which are the several fifth differences, or differences of the fifth order, of the same co-efficients $b, c, d, e, f, g, b, i, k, l, m$, &c,) by the expressions $D_1', D_2', D_3', D_4', D_5'$, and D_6' , &c respectively. And then the said equation will be as follows, to wit, $D_1'' + D_2' \times x - D_2' \times x^2 + D_3' \times x^3 - D_4' \times x^4 + D_5' \times x^5 - D_6' \times x^6 + \&c$ *ad infinitum* = the series $v, \frac{wx}{1+x},$

$\frac{xa^2}{1+x^2}, \frac{va^3}{1+x^3}, \frac{2x^4}{1+x^4}, \frac{p_2 \times x^5}{1+x^5}, \frac{q_3 \times x^6}{1+x^6}$, &c *ad infinitum*, in which p_2, q_3, r_2, s_2 , &c are used to denote the co-efficients of the following terms of the assumed series after the co-efficient z , which is the last letter of the alphabet. With this notation, the investigation of the sign that is to be prefixed to the seventh term, $\frac{wx^6}{1+x^7}$, of the assumed series $P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Sx^3}{1+x^3}, \frac{Tx^4}{1+x^4}, \frac{Vx^5}{1+x^5}, \frac{Wx^6}{1+x^7}$, &c will proceed as follows :

Since

Since the series $D^{1'} + D_1^* \times x - D_2^* \times x^2 + D_3^* \times x^3 - D_4^* \times x^4 + D_5^* \times x^5 - D_6^* \times x^6 + \&c$ *ad infinitum* is = the series $v, \frac{wx}{1+x}, \frac{xa^3}{1+x}, \frac{va^3}{1+x}, \frac{za^4}{1+x}, \frac{p_2 \times x^5}{1+x}, \frac{q_3 \times x^6}{1+x}, \&c$ *ad infinitum*, and V is = $D^{1'}$, it follows that, if the former series $D^{1'} + D_1^* \times x - D_2^* \times x^2 + D_3^* \times x^3 - D_4^* \times x^4 + D_5^* \times x^5 - D_6^* \times x^6 + \&c$ is greater than it's first term $D^{1'}$, the latter series $v, \frac{wx}{1+x}, \frac{xa^3}{1+x}, \frac{va^3}{1+x}, \frac{za^4}{1+x}, \frac{p_2 \times x^5}{1+x}, \frac{q_3 \times x^6}{1+x}, \&c$ will also be greater than it's first term V . But the former of these series is greater than it's first term $D^{1'}$; because the terms D_1^* , D_3^* , and D_5^* , and every following term in the series that is marked with the sign $+$, or added to the first term $D^{1'}$, are greater, respectively, than D_2^* , D_4^* , and D_6^* , and every following term in the same series that is marked with the sign $-$, or subtracted from the first term $D^{1'}$. Therefore the other series $v, \frac{wx}{1+x}, \frac{xa^3}{1+x}, \frac{va^3}{1+x}, \frac{za^4}{1+x}, \frac{p_2 \times x^5}{1+x}, \frac{q_3 \times x^6}{1+x}, \&c$ will be greater than it's first term V . And this will be always true, of how small a magnitude soever the quantity x , and consequently the fraction $\frac{x}{1+x}$, be supposed to be taken. Therefore, by the second of the foregoing Lemmas, the second term, $\frac{wx}{1+x}$, of this series must be added to it's first term V , and consequently must have the sign $+$ prefixed to it; and therefore the last equation will be $D^{1'} + D_1^* \times x - D_2^* \times x^2 + D_3^* \times x^3 - D_4^* \times x^4 + D_5^* \times x^5 - D_6^* \times x^6 + \&c$ *ad infinitum* = the series $v + \frac{wx}{1+x}, \frac{xa^3}{1+x}, \frac{va^3}{1+x}, \frac{za^4}{1+x}, \frac{p_2 \times x^5}{1+x}, \frac{q_3 \times x^6}{1+x}, \&c$ *ad infinitum*. Therefore in the assumed series $P, \frac{va^3}{1+x}, \frac{za^4}{1+x}, \frac{p_2 \times x^5}{1+x}, \frac{q_3 \times x^6}{1+x}, \frac{va^3}{1+x}, \frac{za^4}{1+x}, \frac{p_2 \times x^5}{1+x}, \frac{q_3 \times x^6}{1+x}, \&c$ (from which the present series $v, \frac{wx}{1+x}, \frac{xa^3}{1+x}, \frac{va^3}{1+x}, \frac{za^4}{1+x}, \frac{p_2 \times x^5}{1+x}, \frac{q_3 \times x^6}{1+x}, \&c$, is derived by means of the subtraction mentioned in art. 10, which causes all the terms of the said assumed series to be changed, and by means of certain other arithmetical operations by which the signs of the terms are not affected,) the corresponding term, $\frac{va^3}{1+x}$, must have the contrary sign $-$ prefixed to it.

And

And consequently the seven first terms of the said assumed series will be $p - \frac{qx}{1+x} - \frac{2x^2}{1+x^2} - \frac{3x^3}{1+x^3} - \frac{4x^4}{1+x^4} - \frac{5x^5}{1+x^5} - \frac{6x^6}{1+x^6}$, or $a - \frac{bx}{1+x} - \frac{b^2x^2}{1+x^2} - \frac{b^3x^3}{1+x^3} - \frac{b^4x^4}{1+x^4} - \frac{b^5x^5}{1+x^5} - \frac{b^6x^6}{1+x^6}$. Q. E. I.

Art. 20. It remains that we determine the magnitude of the co-efficient W ; which may be done as follows:

It has been shewn in the last article, that the series $D^v + D_1^v \times x - D_2^v \times x^2 + D_3^v \times x^3 - D_4^v \times x^4 + D_5^v \times x^5 - D_6^v \times x^6 + \&c$ *ad infinitum* is = the series $v + \frac{wx}{1+x} - \frac{2x^2}{1+x^2} + \frac{3x^3}{1+x^3} - \frac{4x^4}{1+x^4} + \frac{5x^5}{1+x^5} - \frac{6x^6}{1+x^6} + \&c$ *ad infinitum*, and likewise that V is = D^v . Therefore, if we subtract V and D^v from the opposite sides of this equation, the remainders will be equal to each other; that is, the series $D_1^v \times x - D_2^v \times x^2 + D_3^v \times x^3 - D_4^v \times x^4 + D_5^v \times x^5 - D_6^v \times x^6 + \&c$ *ad infinitum* will be = the series $\frac{wx}{1+x} - \frac{2x^2}{1+x^2} + \frac{3x^3}{1+x^3} - \frac{4x^4}{1+x^4} + \frac{5x^5}{1+x^5} - \frac{6x^6}{1+x^6} + \&c$ *ad infinitum*. Therefore, if we divide all the terms of this equation by x , we shall have the series $D_1^v - D_2^v \times x + D_3^v \times x^2 - D_4^v \times x^3 + D_5^v \times x^4 - D_6^v \times x^5 + \&c$ = the series $\frac{w}{1+x} - \frac{x}{1+x^2} + \frac{x^2}{1+x^3} - \frac{x^3}{1+x^4} + \frac{x^4}{1+x^5} - \frac{x^5}{1+x^6} + \&c$; and, if we multiply all the terms of this equation into the binomial quantity $1+x$, in order to disengage the co-efficient W from it's divisor $1+x$, we shall obtain the following equation, to wit,

$$\left\{ \begin{array}{l} D_1^v - D_2^v \times x + D_3^v \times x^2 - D_4^v \times x^3 + D_5^v \times x^4 - D_6^v \times x^5 + \&c \\ + D_1^v \times x - D_2^v \times x^2 + D_3^v \times x^3 - D_4^v \times x^4 + D_5^v \times x^5 - \&c \end{array} \right\} \\ = w, \frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \frac{x^5}{1+x^5} - \frac{x^6}{1+x^6} + \&c, \text{ or } D_1^v + D_1^v - D_2^v \times x \\ - [D_2^v - D_3^v \times x^2 + D_3^v - D_4^v \times x^3 - D_4^v + D_5^v \times x^4 + D_5^v - D_6^v \times x^5] \times x^2 \\ - \&c \text{ ad infinitum.} = w, \frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \frac{x^5}{1+x^5} - \frac{x^6}{1+x^6} + \&c.$$

Now let x become = 0.

Then

Then will all the terms in this last equation that involve any of the powers of x become equal to 0 likewise; and consequently the only quantities left on the opposite sides of the equation will be D_1^v and V . Therefore V will be $= D_1^v$, or D^v ; and consequently the seven first terms of the assumed series

$$P, \frac{Qx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Sx^3}{1+x^3}, \frac{Tx^4}{1+x^4}, \frac{Vx^5}{1+x^5}, \frac{Wx^6}{1+x^6}, \frac{Xx^7}{1+x^7}, \&c. \text{ will be}$$

$$a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x^2} - \frac{D^{12}x^3}{1+x^3} - \frac{D^{112}x^4}{1+x^4} - \frac{D^{122}x^5}{1+x^5} - \frac{D^{1222}x^6}{1+x^6} - \frac{D^{12222}x^7}{1+x^7} \quad Q_2 \text{ E. r.}$$

General Conclusions from the Five last Investigations concerning the following Terms of the said assumed Series.

ART. 21. From an attentive observation of the five preceding investigations, by which we have found that the third term, $\frac{Rx^2}{1+x^2}$, of the assumed series is to have the sign — prefixed to it, and that it is equal to $\overline{b-d} \times \frac{x^2}{1+x^2}$, or $\frac{D^1x^2}{1+x^2}$, and that the fourth term, $\frac{Sx^3}{1+x^3}$, of the said series is likewise to have the sign — prefixed to it, and is equal to $\overline{b-2c+d} \times \frac{x^3}{1+x^3}$, or $\frac{D^{12}x^3}{1+x^3}$, and that the fifth, sixth, and seventh terms, $\frac{Tx^4}{1+x^4}$, $\frac{Vx^5}{1+x^5}$, and $\frac{Wx^6}{1+x^6}$, of the said series, are, all of them, in like manner, to have the sign — prefixed to them, and that they are equal to $\frac{D^{112}x^4}{1+x^4}$, $\frac{D^{122}x^5}{1+x^5}$, and $\frac{D^{1222}x^6}{1+x^6}$, respectively, it seems evident that, if we were to make use of the like investigations to discover the signs that are to be prefixed to the following terms, $\frac{Xx^7}{1+x^7}$, $\frac{Yx^8}{1+x^8}$, $\frac{Zx^9}{1+x^9}$, $\frac{P_2x^{10}}{1+x^{10}}$, $\frac{Q_2x^{11}}{1+x^{11}}$, &c of the said assumed series, and the values of their several co-efficients X , Y , Z , P_2 , Q_2 , &c, we should find that all these terms would have the sign — prefixed to them, and that the said co-efficients would be, respectively, equal to D_1^v , D_1^{vi} , D_1^{vii} , D_1^{viii} , and D_1^i , &c, or D^v , D^{vi} , D^{vii} , D^{viii} , D^i .

D^{viii} , D^ix , and D^x , &c, or to the first differences of the sixth, seventh, eighth, ninth, and tenth, and other following orders of differences, of the original co-efficients $b, c, d, e, f, g, h, i, k, l, m$, &c, to whatever number of terms the said assumed series might be continued. For in every new investigation the processes and the reasonings upon them would be exactly similar to those in the next preceeding investigation, and consequently must produce similar results, which are, 1st, that the new term of the assumed series is to have the same sign — prefixed to it which was prefixed to the next preceeding term; and, 2dly, that the numeral co-efficient involved in it will be the first difference of the next higher order of differences of the original co-efficients $b, c, d, e, f, g, h, i, k, l, m$, &c, to that of which the numeral co-efficient of the next preceeding term of the assumed series was the first difference. And therefore we may conclude, without an actual investigation of any more than the foregoing seven terms which have been here investigated, that the following terms of the said series will, all, have the sign — prefixed to them, and will be equal to $\frac{D^{vii}x^7}{1+x}^7$,

$\frac{D^{vi}x^6}{1+x}^6, \frac{D^{v}x^5}{1+x}^5, \frac{D^{iv}x^4}{1+x}^4, \frac{D^{iii}x^3}{1+x}^3, \frac{D^{ii}x^2}{1+x}^2, \frac{D^ix^1}{1+x}^1$, &c *ad infinitum*, or to whatever number of terms the said assumed series may be continued. And thus, by this method of investigating the terms of the said assumed series suggested by Mr. Heilins, we have discovered the law of the continuation of the terms of the said assumed series.

Q. E. I.

ART. 22. But, perhaps, the resemblance between the processes and the reasonings employed in these successive investigations of new terms of the assumed series, may be made to appear more clearly by expressing them in a closer and conciser manner than has been used in the foregoing articles. And therefore I shall now repeat, or recapitulate, the five last investigations in as short and compressed a manner as I can, and with the same abridged notation as is used in the last of those investigations, in which the several successive differences of the fifth order of the original co-efficients $b, c, d, e, f, g, h, i, k, l, m$, &c are denoted by the expressions $D_1^v, D_2^v, D_3^v, D_4^v, D_5^v, D_6^v$, &c. And therefore I shall now denote the first order of those differences, to wit, the differences $b - c, c - d, d - e, e - f, f - g$, and $g - h$, &c, by the terms $D_1^i, D_2^i, D_3^i, D_4^i, D_5^i$, and D_6^i , &c, and the second order of those differences, to wit, the differences $b - 2c + d, c - 2d + e, d - 2e + f, e - 2f + g, f - 2g + h$, and $g - 2h + i$, &c, by the terms $D_1^{ii}, D_2^{ii}, D_3^{ii}, D_4^{ii}, D_5^{ii}$, and D_6^{ii} , &c, and the third order of those differences, to wit, the differences $b - 3c + 3d - e, c - 3d + 3e - f, d - 3e + 3f - g, e - 3f + 3g - h, f - 3g + 3h - i$, and

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and

and $g - 3b + 3i - k$, &c by the terms D_1''' , D_2''' , D_3''' , D_4''' , D_5''' , and D_6''' , &c, and the fourth and fifth orders of those differences by the terms D_1'' , D_2'' , D_3'' , D_4'' , D_5'' , &c, and D_1' , D_2' , D_3' , D_4' , D_5' , D_6' , &c. With this notation the five foregoing investigations contained in art. 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, may be expressed in the manner following.

A Recapitulation of the foregoing Investigations of the 3d, 4th, 5th, 6th, and 7th Terms of the aforesaid assumed Series, in a more concise Notation.

The Investigation of the Third Term, $\frac{bx^3}{1+x^2}$, of the said assumed Series.

Art. 23. It is shewn in art. 10, that the series $b + D_1' \times x - D_2' \times x^2 + D_3' \times x^3 - D_4' \times x^4 + D_5' \times x^5 - D_6' \times x^6 + \&c$ ad infinitum is = the series $Q, \frac{bx}{1+x}, \frac{bx^3}{1+x^2}, \frac{bx^5}{1+x^3}, \frac{bx^7}{1+x^4}, \frac{bx^9}{1+x^5}, \frac{bx^{11}}{1+x^6}, \&c$ ad infinitum, and also that Q is = b . And it is evident that the former series is greater than its first term b ; because those terms of it that are marked with the sign $+$ are greater than those that immediately follow them and are marked with the sign $-$. Therefore the latter series must be also greater than its first term Q . And this is always true, of how small a magnitude soever we may suppose the quantity x , and consequently the fraction $\frac{x}{1+x}$, to be taken. Therefore, by Lemma 2d, the second term, $\frac{bx}{1+x}$, of that series must be marked with the sign $+$. And consequently, by reason of the subtraction mentioned in art. 10, the corresponding term, $\frac{bx^3}{1+x^2}$, of the assumed series must be marked with the contrary sign $-$. Q. N. 1.

Secondly, by subtracting the equal quantities b and Q from the opposite sides of the equation $b + D_1' \times x - D_2' \times x^2 + D_3' \times x^3 - D_4' \times x^4 + D_5' \times x^5 -$

$-D_6' \times x^2 + \&c = R + \frac{Rx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Tx^3}{1+x^3}, \frac{Vx^4}{1+x^4}, \frac{Wx^5}{1+x^5}, \frac{Xx^6}{1+x^6},$
 $\&c$, we shall have the equation $D_1' \times x - D_2' \times x^2 + D_3' \times x^3 - D_4' \times x^4$
 $+ D_5' \times x^5 - D_6' \times x^6 + \&c = \frac{Rx}{1+x}, \frac{Rx^2}{1+x^2}, \frac{Tx^3}{1+x^3}, \frac{Vx^4}{1+x^4}, \frac{Wx^5}{1+x^5},$
 $\frac{Xx^6}{1+x^6}, \&c$; and, by dividing all these terms by x , we shall have the equation
 $D_1' - D_2' \times x + D_3' \times x^2 - D_4' \times x^3 + D_5' \times x^4 - D_6' \times x^5 + \&c =$
 $\frac{R}{1+x}, \frac{Tx}{1+x^2}, \frac{Tx^2}{1+x^3}, \frac{Vx^3}{1+x^4}, \frac{Wx^4}{1+x^5}, \frac{Xx^5}{1+x^6}, \&c$; and, lastly, by multi-
 plying all the terms of this equation into the binomial quantity $1+x$, we
 shall have the equation

$$\left\{ \begin{array}{l} D_1' - D_2' \times x + D_3' \times x^2 - D_4' \times x^3 + D_5' \times x^4 - D_6' \times x^5 + \&c \\ + D_1' \times x - D_2' \times x^2 + D_3' \times x^3 - D_4' \times x^4 + D_5' \times x^5 - \&c \end{array} \right\} \\
 = R, \frac{Rx}{1+x}, \frac{Tx^2}{1+x^2}, \frac{Vx^3}{1+x^3}, \frac{Wx^4}{1+x^4}, \frac{Xx^5}{1+x^5}, \&c, \text{ or } D_1' + \overbrace{D_1' - D_2'}^{(1)} \times x \\
 - \overbrace{(D_2' - D_3')}^{(2)} \times x^2 + \overbrace{(D_3' - D_4')}^{(3)} \times x^3 - \overbrace{(D_4' - D_5')}^{(4)} \times x^4 + \overbrace{(D_5' - D_6')}^{(5)} \\
 \times x^5 - \&c = R, \frac{Rx}{1+x}, \frac{Tx^2}{1+x^2}, \frac{Vx^3}{1+x^3}, \frac{Wx^4}{1+x^4}, \frac{Xx^5}{1+x^5}, \&c, \text{ or } D_1'' + \\
 D_1'' \times x - D_2'' \times x^2 + D_3'' \times x^3 - D_4'' \times x^4 + D_5'' \times x^5 - \&c = R, \\
 \frac{Rx}{1+x}, \frac{Tx^2}{1+x^2}, \frac{Vx^3}{1+x^3}, \frac{Wx^4}{1+x^4}, \frac{Xx^5}{1+x^5}, \frac{Yx^6}{1+x^6}, \&c. \text{ Now let } x \text{ become} \\
 = 0. \text{ And we shall have } D_1' = R, \text{ or } R = D_1', \text{ or } D'. \quad Q.E.D.$$

The Investigation of the Fourth Term, $\frac{Rx^3}{1+x^3}$, of the said assumed Series.

Art. 24. It has been shewn in the last article, that the series $D_1'' + D_1'' \times x$
 $- D_2'' \times x^2 + D_3'' \times x^3 - D_4'' \times x^4 + D_5'' \times x^5 - D_6'' \times x^6 + \&c$ is =
 the series $R, \frac{Rx}{1+x}, \frac{Tx^2}{1+x^2}, \frac{Vx^3}{1+x^3}, \frac{Wx^4}{1+x^4}, \frac{Xx^5}{1+x^5}, \frac{Yx^6}{1+x^6}, \&c$, and like-

wife that $x = D_1'$. But the former series is greater than it's first term D_1' ; because the following terms $D_1'' \times x$, $D_3'' \times x^3$, $D_5'' \times x^5$, &c, which are marked with the sign +, are greater, respectively, than the terms $D_2'' \times x^2$, $D_4'' \times x^4$, $D_6'' \times x^6$, &c, which immediately follow them and are marked with the sign -. Therefore the latter series must also be greater than it's first term x . And this is always true, of how small a magnitude soever the quantity x , and consequently the fraction $\frac{x}{1+x}$, be supposed to be taken. Therefore, by Lemma 2, the second term, $\frac{x^2}{1+x}$, of that series must have the sign + prefixed to it. And consequently, by reason of the subtraction mentioned in art. 10, the corresponding term, $\frac{x^2}{1+x}$, of the assumed series must have the contrary sign — prefixed to it. Q. E. I.

Secondly, by subtracting the equal quantities D_1' and x from the opposite sides of the equation $D_1' + D_1'' \times x - D_2'' \times x^2 + D_3'' \times x^3 - D_4'' \times x^4 + D_5'' \times x^5 - D_6'' \times x^6 + \&c = x + \frac{x^2}{1+x} - \frac{x^3}{1+x} + \frac{x^4}{1+x} - \frac{x^5}{1+x} + \frac{x^6}{1+x} - \frac{x^7}{1+x} + \&c$, we shall have the equation $D_1'' \times x - D_2'' \times x^2 + D_3'' \times x^3 - D_4'' \times x^4 + D_5'' \times x^5 - D_6'' \times x^6 + \&c = \frac{x^2}{1+x} - \frac{x^3}{1+x} + \frac{x^4}{1+x} - \frac{x^5}{1+x} + \frac{x^6}{1+x} - \frac{x^7}{1+x} + \&c$; and, by dividing all the terms by x , we shall have the equation $D_1'' - D_2'' \times x + D_3'' \times x^2 - D_4'' \times x^3 + D_5'' \times x^4 - D_6'' \times x^5 + \&c = \frac{x}{1+x} - \frac{x^2}{1+x} + \frac{x^3}{1+x} - \frac{x^4}{1+x} + \frac{x^5}{1+x} - \frac{x^6}{1+x} + \&c$; and, by multiplying all these terms into $1+x$, we shall have the equation

$$\left\{ \begin{array}{l} D_1'' - D_2'' \times x + D_3'' \times x^2 - D_4'' \times x^3 + D_5'' \times x^4 - D_6'' \times x^5 + \&c \\ + D_1'' \times x - D_2'' \times x^2 + D_3'' \times x^3 - D_4'' \times x^4 + D_5'' \times x^5 - \&c \end{array} \right\} \\ = S, \frac{x}{1+x} - \frac{x^2}{1+x} + \frac{x^3}{1+x} - \frac{x^4}{1+x} + \frac{x^5}{1+x} - \frac{x^6}{1+x} + \&c; \text{ or } D_1'' + D_1'' - D_2'' \times x \\ - (D_2'' - D_3'') \times x^2 + (D_3'' - D_4'') \times x^3 - (D_4'' - D_5'') \times x^4 + (D_5'' - D_6'') \times x^5$$

$\times x^3 - \&c = S, \frac{Tx}{1+x}, \frac{Vx^2}{1+x}, \frac{Wx^3}{1+x}, \frac{Xx^4}{1+x}, \frac{Yx^5}{1+x}, \&c$, or $D_1'' + D_1''' \times x - D_2''' \times x^2 + D_3''' \times x^3 - D_4''' \times x^4 + D_5''' \times x^5 - \&c = S, \frac{Tx}{1+x}, \frac{Vx^2}{1+x}, \frac{Wx^3}{1+x}, \frac{Xx^4}{1+x}, \frac{Yx^5}{1+x}, \&c$. Now let x become $= 0$. And we shall then have $S = D_1''$.

Q. E. I.

The Investigation of the Fifth Term, $\frac{Tx^4}{1+x}$, of the said assumed Series.

Art. 25. Since the series $D_1'' + D_1''' \times x - D_2''' \times x^2 + D_3''' \times x^3 - D_4''' \times x^4 + D_5''' \times x^5 - D_6''' \times x^6 + \&c$ is $=$ the series $S, \frac{Tx}{1+x}, \frac{Vx^2}{1+x}, \frac{Wx^3}{1+x}, \frac{Xx^4}{1+x}, \frac{Yx^5}{1+x}, \frac{Zx^6}{1+x}, \&c$, and $D_1'' = S$, and the former series is evidently greater than it's first term D_1'' , it follows that the latter series must be greater than it's first term S . And this will be always true, of however small a magnitude we may suppose x , and consequently the fraction $\frac{x}{1+x}$ to be taken. Therefore, by Lemma 2d, the second term, $\frac{Tx}{1+x}$, of the latter series must have the sign $+$ prefixed to it. And consequently, by reason of the subtraction mentioned in art. 10, the corresponding term, $\frac{Tx^4}{1+x}$, of the assumed series must have the contrary sign $-$ prefixed to it.

Q. E. I.

Secondly, by subtracting the equal quantities D_1'' and S from the opposite sides of the equation $D_1'' + D_1''' \times x - D_2''' \times x^2 + D_3''' \times x^3 - D_4''' \times x^4 + D_5''' \times x^5 - D_6''' \times x^6 + \&c = S, \frac{Tx}{1+x}, \frac{Vx^2}{1+x}, \frac{Wx^3}{1+x}, \frac{Xx^4}{1+x}, \frac{Yx^5}{1+x}, \frac{Zx^6}{1+x}, \&c$, we shall obtain the equation $D_1''' \times x - D_2''' \times x^2 + D_3''' \times x^3 - D_4''' \times x^4 + D_5''' \times x^5 - D_6''' \times x^6 + \&c = \frac{Tx}{1+x}, \frac{Vx^2}{1+x}, \frac{Wx^3}{1+x}, \frac{Xx^4}{1+x}, \frac{Yx^5}{1+x}, \frac{Zx^6}{1+x}, \&c$.

$\frac{wx^3}{1+x}, \frac{xx^4}{1+x}, \frac{yx^5}{1+x}, \frac{zx^6}{1+x}, \&c$; and, by dividing all these terms by x , we shall have $D_1''' - D_2''' \times x + D_3''' \times x^2 - D_4''' \times x^3 + D_5''' \times x^4 - D_6''' \times x^5 + \&c = \frac{\tau}{1+x}, \frac{yx}{1+x}, \frac{wx^2}{1+x}, \frac{xx^3}{1+x}, \frac{yx^4}{1+x}, \frac{zx^5}{1+x}, \&c$; and, lastly, by multiplying all these terms into $1+x$, we shall obtain the equation

$$\left\{ \begin{aligned} &D_1''' - D_2''' \times x + D_3''' \times x^2 - D_4''' \times x^3 + D_5''' \times x^4 - D_6''' \times x^5 + \&c \\ &+ D_1''' \times x - D_2''' \times x^2 + D_3''' \times x^3 - D_4''' \times x^4 + D_5''' \times x^5 - \&c \end{aligned} \right\} =$$

$$\tau, \frac{yx}{1+x}, \frac{wx^2}{1+x}, \frac{xx^3}{1+x}, \frac{yx^4}{1+x}, \frac{zx^5}{1+x}, \&c, \text{ or } D_1''' + D_1''' - D_2''' \times x$$

$$- [D_2''' - D_3'''] \times x^2 + [D_3''' - D_4'''] \times x^3 - [D_4''' - D_5'''] \times x^4 +$$

$$[D_5''' - D_6'''] \times x^5 - \&c = \tau, \frac{yx}{1+x}, \frac{wx^2}{1+x}, \frac{xx^3}{1+x}, \frac{yx^4}{1+x}, \frac{zx^5}{1+x}, \&c,$$

$$\text{or } D_1''' + D_5''' \times x - D_2''' \times x^2 + D_3''' \times x^3 - D_4''' \times x^4 + D_5''' \times x^5 -$$

$$\&c = \tau, \frac{yx}{1+x}, \frac{wx^2}{1+x}, \frac{xx^3}{1+x}, \frac{yx^4}{1+x}, \frac{zx^5}{1+x}, \&c. \text{ Now let } x \text{ become } = 0.$$

And we shall then have $\tau = D_1'''$, or D_1''' . Q. E. D.

The Investigation of the Sixth Term, $\frac{yx^5}{1+x}$, of the said assumed Series.

Art. 26. Since the series $D_1''' + D_1''' \times x - D_2''' \times x^2 + D_3''' \times x^3 - D_4''' \times x^4 + D_5''' \times x^5 - D_6''' \times x^6 + \&c$ is = the series $\tau, \frac{yx}{1+x}, \frac{wx^2}{1+x}, \frac{xx^3}{1+x}, \frac{yx^4}{1+x}, \frac{zx^5}{1+x}, \frac{yx^6}{1+x}, \&c$, and D_1''' is = τ , and the former series is evidently greater than it's first term D_1''' , it follows that the latter series must be greater than it's first term τ . And this will always be true, of however small a magnitude we may suppose x , and consequently the fraction, $\frac{x}{1+x}$, to be taken.

taken. Therefore, by Lemma 2, the second term, $\frac{vx}{1+x}$, of the latter series must have the sign + prefixed to it. And consequently, by reason of the subtraction mentioned in art. 10, the corresponding term, $\frac{vx^3}{1+x}$, of the assumed series must have the contrary sign - prefixed to it.

Q. E. I.

Secondly, by subtracting the equal quantities D_1^{iii} and τ from the opposite sides of the equation $D_1^{iii} + D_1^{iv} \times x - D_2^{iv} \times x^2 + D_3^{iv} \times x^3 - D_4^{iv} \times x^4 + D_5^{iv} \times x^5 - D_6^{iv} \times x^6 + \&c = \tau$, $\frac{vx}{1+x}$, $\frac{wx^3}{1+x}$, $\frac{xx^3}{1+x}$, $\frac{yx^4}{1+x}$, $\frac{zx^5}{1+x}$, $\frac{p^2 \times x^6}{1+x}$, &c, we shall obtain the equation $D_1^{iv} \times x - D_2^{iv} \times x^2 + D_3^{iv} \times x^3 - D_4^{iv} \times x^4 + D_5^{iv} \times x^5 - D_6^{iv} \times x^6 + \&c = \frac{vx}{1+x}$, $\frac{wx^3}{1+x}$, $\frac{xx^3}{1+x}$, $\frac{yx^4}{1+x}$, $\frac{zx^5}{1+x}$, $\frac{p^2 \times x^6}{1+x}$, &c; and, by dividing all these terms by x , we shall have $D_1^{iv} - D_2^{iv} \times x + D_3^{iv} \times x^2 - D_4^{iv} \times x^3 + D_5^{iv} \times x^4 - D_6^{iv} \times x^5 + \&c = \frac{v}{1+x}$, $\frac{wx}{1+x}$, $\frac{xx^2}{1+x}$, $\frac{yx^3}{1+x}$, $\frac{zx^4}{1+x}$, $\frac{p^2 \times x^5}{1+x}$, &c; and, lastly, by multiplying all these terms into $1+x$, we shall obtain the equation

$$\left\{ \begin{array}{l} D_1^{iv} - D_2^{iv} \times x + D_3^{iv} \times x^2 - D_4^{iv} \times x^3 + D_5^{iv} \times x^4 - D_6^{iv} \times x^5 + \&c \\ D_1^{iv} \times x - D_2^{iv} \times x^2 + D_3^{iv} \times x^3 - D_4^{iv} \times x^4 + D_5^{iv} \times x^5 - \&c \end{array} \right\} \\ = v, \frac{wx}{1+x}, \frac{xx^2}{1+x}, \frac{yx^3}{1+x}, \frac{zx^4}{1+x}, \frac{p^2 \times x^5}{1+x}, \&c, \text{ or } D_1^{iv} + D_1^{iv} - D_2^{iv} \times x \\ - [D_2^{iv} - D_3^{iv}] \times x^2 + [D_3^{iv} - D_4^{iv}] \times x^3 - [D_4^{iv} - D_5^{iv}] \times x^4 + [D_5^{iv} - D_6^{iv}] \times x^5 \\ - \&c = v, \frac{wx}{1+x}, \frac{xx^2}{1+x}, \frac{yx^3}{1+x}, \frac{zx^4}{1+x}, \frac{p^2 \times x^5}{1+x}, \&c, \text{ or } D_1^{iv} + D_1^{iv} \times x \\ - D_2^{iv} \times x^2 + D_3^{iv} \times x^3 - D_4^{iv} \times x^4 + D_5^{iv} \times x^5 - \&c = v, \frac{wx}{1+x}, \\ \frac{xx^2}{1+x}, \frac{yx^3}{1+x}, \frac{zx^4}{1+x}, \frac{p^2 \times x^5}{1+x}, \&c.$$

Now let x become = 0. And we shall then have $V = D_1^{iv}$, or D^{iv} .

Q. E. I.

The.

The Investigation of the Seventh Term, $\frac{wx^6}{1+x}^6$, of the said assumed Series.

Art. 27. Since the series $D_1^{iv} + D_1^v \times x - D_2^v \times x^2 + D_3^v \times x^3 - D_4^v \times x^4 + D_5^v \times x^5 - D_6^v \times x^6 + \&c$ *ad infinitum* is = the series $v, \frac{wx}{1+x}, \frac{x^2}{1+x}, \frac{yx^3}{1+x}, \frac{zx^4}{1+x}, \frac{px^5}{1+x}, \frac{qx^6}{1+x}, \&c$ *ad infinitum*, and D_1^{iv} is = V , and the former of these serieses is evidently greater than it's first term D_1^{iv} , it follows that the latter series must also be greater than it's first term V . And this will always be true, of how small a magnitude soever the quantity x , and consequently the fraction $\frac{x}{1+x}$, be supposed to be taken. Therefore, by Lemma 2, the second term, $\frac{wx}{1+x}$, of the latter series must have the sign + prefixed to it. And consequently, by reason of the subtraction mentioned in art. 10, the corresponding term, $\frac{wx^6}{1+x}^6$, of the assumed series must have the contrary sign - prefixed to it. Q. E. I.

Secondly, by subtracting the equal quantities D_1^{iv} and V from the opposite sides of the equation $D_1^{iv} + D_1^v \times x - D_2^v \times x^2 + D_3^v \times x^3 - D_4^v \times x^4 + D_5^v \times x^5 - D_6^v \times x^6 + \&c = v, \frac{wx}{1+x}, \frac{x^2}{1+x}, \frac{yx^3}{1+x}, \frac{zx^4}{1+x}, \frac{px^5}{1+x}, \frac{qx^6}{1+x}, \&c$, we shall obtain the equation $D_1^v \times x - D_2^v \times x^2 + D_3^v \times x^3 - D_4^v \times x^4 + D_5^v \times x^5 - D_6^v \times x^6 + \&c = \frac{wx}{1+x}, \frac{x^2}{1+x}, \frac{yx^3}{1+x}, \frac{zx^4}{1+x}, \frac{px^5}{1+x}, \frac{qx^6}{1+x}, \&c$; and, by dividing all these terms by x , we shall have $D_1^v - D_2^v \times x + D_3^v \times x^2 - D_4^v \times x^3 + D_5^v \times x^4 - D_6^v \times x^5 + \&c = \frac{w}{1+x}, \frac{x}{1+x}, \frac{yx^2}{1+x}, \frac{zx^3}{1+x}, \frac{px^4}{1+x}, \frac{qx^5}{1+x}, \&c$; and, lastly, by multiplying all these terms into $1+x$, we shall obtain the equation

$$D_1^v$$

$$\left\{ \begin{aligned} & D_1^* - D_2^* \times x + D_3^* \times x^2 - D_4^* \times x^3 + D_5^* \times x^4 - D_6^* \times x^5 + \&c \\ & + D_1^* \times x - D_2^* \times x^2 + D_3^* \times x^3 - D_4^* \times x^4 + D_5^* \times x^5 - \&c \end{aligned} \right\}$$

$$= w, \frac{x^0}{1+x}, \frac{y_1^2}{1+x^2}, \frac{z_1^3}{1+x^3}, \frac{p_2 \times x^4}{1+x^4}, \frac{q_2^2 \times x^5}{1+x^5}, \&c, \text{ or } D_1^* + D_1^* - D_2^* \times x$$

$$- [D_2^* - D_3^*] \times x^2 + D_3^* - D_4^* \times x^3 - [D_4^* - D_5^*] \times x^4 + D_5^* - D_6^* \times x^5$$

$$- \&c = w, \frac{x^0}{1+x}, \frac{y_1^2}{1+x^2}, \frac{z_1^3}{1+x^3}, \frac{p_2 \times x^4}{1+x^4}, \frac{q_2^2 \times x^5}{1+x^5}, \&c, \text{ or } D_1^* + D_1^* \times x$$

$$- D_2^* \times x^2 + D_3^* \times x^3 - D_4^* \times x^4 + D_5^* \times x^5 - \&c = w, \frac{x^0}{1+x},$$

$$\frac{y_1^2}{1+x^2}, \frac{z_1^3}{1+x^3}, \frac{p_2 \times x^4}{1+x^4}, \frac{q_2^2 \times x^5}{1+x^5}, \&c.$$

Now let x become $= 0$. And we shall then have $D_1^* = w$, or $w = D_1^*$,
or D^* . Q. E. I.

Remarks on the last Five Investigations of New Terms in the assumed Series

$$P, \frac{Q^2}{1+x}, \frac{R^3}{1+x^2}, \frac{S^4}{1+x^3}, \frac{T^5}{1+x^4}, \frac{V^6}{1+x^5}, \frac{W^7}{1+x^6}, \frac{X^8}{1+x^7}, \&c \text{ ad infinitum.}$$

Art. 23. These investigations are all exactly similar to each other, and consist of the same arithmetical operations and reasonings, repeated over and over, and applied to new infinite series involving new orders of differences of the original co-efficients $b, c, d, e, f, g, h, i, k, l, m$, &c *ad infinitum*. For in each of them we set out with considering an equation between two infinite series that have the following properties. The first, or left-hand, series in this equation is a decreasing progression of terms, of which all the terms, except the first, involve in them the several powers of x in their natural order, to wit, $x, x^2, x^3, x^4, x^5, x^6$, &c, combined with, or multiplied into, the several successive terms of some order of differences of the said original co-efficients $b, c, d, e, f, g, h, i, k, l, m$, &c, respectively; and the said terms, so involving the powers of x , are marked with the sign $+$ and the sign $-$ alternately, or are added to the first term (which is a known quantity not combined with x) and subtracted from it alternately: and consequently the whole series is necessarily greater than it's first term. These are the properties of the first series. And the second series in this equation is a series of quantities of which all the terms, except the first

term, involve in them the several powers of the fraction $\frac{x}{1+x}$ in their natural order, to wit, $\frac{x}{1+x}$, $\frac{x^2}{1+x^2}$, $\frac{x^3}{1+x^3}$, $\frac{x^4}{1+x^4}$, $\frac{x^5}{1+x^5}$, $\frac{x^6}{1+x^6}$, &c, combined with, or multiplied into, certain numeral co-efficients denoted by the capital letters R, S, T, V, W, X, &c, that are hitherto unknown, and are to be determined gradually, one after another, by means of the following investigations; and the first term of this series is a known quantity, not involved with the fraction $\frac{x}{1+x}$, or with x , to wit, the co-efficient of the term of the assumed series that was determined by the last preceding investigation; and this term has been shewn to be equal to the first term of the former series. And hence it follows that, since the whole of the said former series is greater than it's first term, the whole of this second series must likewise be greater than it's first term. And this will be true, of how small a magnitude soever the quantity x , and consequently the fraction $\frac{x}{1+x}$, may be taken. And therefore, by Lemma 2d, the second term of the said second series must be added to it's first term, and consequently must have the sign + prefixed to it. And consequently, by reason of the subtraction mentioned in art. 10, (which causes the signs of all the terms of the assumed series to be changed,) the correspondent term of the assumed series must have the contrary sign - prefixed to it. These are the reasonings contained in the first parts of each of the foregoing investigations, and by which it is discovered that the sign to be prefixed to the new term of the assumed series will always be the sign -.

Art. 29. The remaining part of each of the said investigations relates to the determination of the magnitude of the co-efficient of the new term of the assumed series; which co-efficient is also the co-efficient of the fraction $\frac{x}{1+x}$ in the second term of the second of the two infinite serieses above-described, which are equal to each other. And to determine the magnitude of this co-efficient, we perform, in each of the said investigations, the three following operations on the said two equal serieses.

In the first place, we subtract from them their two first terms, which are equal to each other: and by this operation we obtain a second equation between the two remainders; which are infinite serieses that have the following properties. The first of them is a series of which all the terms involve the several powers of x in their natural order, to wit, the powers x , x^2 , x^3 , x^4 , x^5 , x^6 , &c, combined with, or multiplied into, the several successive terms of the aforesaid order of differences of the original co-efficients b , c , d , e , f , g , h , i , l , m , &c, and connected with each other by the sign - and the sign + alternately; the second, fourth, sixth, eighth, and other following even terms having the sign - prefixed to them; and the third, fifth, seventh, and other following odd terms having the sign + prefixed to them. And the other series is a series of which all

all the terms involve the several powers of the fraction $\frac{x}{1+x}$ in their natural order, to wit, the powers $\frac{x}{1+x}$, $\frac{x^2}{1+x^2}$, $\frac{x^3}{1+x^3}$, $\frac{x^4}{1+x^4}$, $\frac{x^5}{1+x^5}$, $\frac{x^6}{1+x^6}$, &c, combined with, or multiplied into, certain numeral co-efficients that are hitherto unknown, and of which the first is now to be determined.

We then, in the 2d place, divide all the terms of this second equation by x ; and we thereby obtain a third equation, of which the first, or left-hand, side contains an infinite series of which the first term does not involve any power of x , but all the following terms involve the several powers of x in their natural order; and the first term, and the numeral co-efficients of the powers of x in the following terms are the several successive terms of the aforesaid order of differences of the original co-efficients $b, c, d, e, f, g, b, i, k, l, m$, &c; and the second and other following terms of the said series are connected with it's first term by the sign — and the sign + alternately: and the second, or right-hand, side of the said third equation contains an infinite series, of which the first term is a fraction that has for it's numerator the new co-efficient which we are now endeavouring to determine, and for it's denominator the binomial quantity $1+x$; and the second and other following terms of the said series involve in their numerators the several powers of x in their natural order, beginning with x , to wit, $x, x^2, x^3, x^4, x^5, x^6$, &c, and involve in their denominators the several powers of the binomial quantity $1+x$ in their natural order, beginning from $1+x^1$, to wit, $1+x^1, 1+x^2, 1+x^3, 1+x^4, 1+x^5, 1+x^6$, &c, and involve also in their numerators, besides the said powers of x , the several numeral co-efficients of the terms of the assumed series that are hitherto undetermined.

And we then, in the 3d place, in order to disengage the new co-efficient, (of which we are now endeavouring to determine the magnitude,) from it's denominator, or divisor, $1+x$, multiply all the terms of this last, or 3d, equation into the said binomial quantity $1+x$; which produces a new, or 4th, equation between two infinite serieses, which may be described as follows.

The first term of the first series, which forms the left-hand side of this 4th equation, is the same with the first term of the first series contained in the former, or 3d, equation, or is the first difference of the above-mentioned order of differences of the original co-efficients $b, c, d, e, f, g, b, i, k, l, m$, &c; and the second and other following terms of the said series involve in them the several powers of x in their natural order, (to wit, x, x^2, x^3, x^4, x^5 , &c,) combined with, or multiplied into, the first, second, third, and other following differences of the original co-efficients $b, c, d, e, f, g, b, i, k, l, m$, &c, of the next higher order to that of the differences of the said co-efficients contained in the former, or 3d, equation, and they are connected with the first term of the said series by addition and subtraction alternately, or have the sign + and the sign — prefixed to them alternately: and the first term of the second series (which forms the right-hand side of this 4th equation,) is the new co-efficient, (of which we are endeavouring to determine the magnitude,) not combined with, or multiplied into,

any other quantity, and the second and other following terms of the said second series involve in them the several powers of the fraction $\frac{x}{1+x}$, in their natural order, to wit, the powers $\frac{x}{1+x}$, $\frac{x^2}{1+x^2}$, $\frac{x^3}{1+x^3}$, $\frac{x^4}{1+x^4}$, $\frac{x^5}{1+x^5}$, $\frac{x^6}{1+x^6}$,

&c, combined with, or multiplied into, the several co-efficients of the terms of the assumed series that are hitherto undetermined.

And then, in the 4th place, we suppose x to become $= 0$; which causes all the terms on both sides of this last, or 4th, equation, except the first term on each side, to become equal to 0 likewise: and we thence conclude that the said two terms must be equal to each other, or that the new co-efficient, of which we have been endeavouring to determine the magnitude, is equal to the first term of the former series which forms the first, or left-hand, side of the last, or 4th, equation, or to the first difference of the above-mentioned order of differences of the original co-efficients $b, c, d, e, f, g, h, i, k, l, m$, &c, contained in the 3d equation before the multiplication of it's terms into the binomial quantity $1+x$.

Art. 30. These are the processes and reasonings contained in the second part of each of the foregoing investigations; by which it has been discovered that every new co-efficient of the assumed series thereby determined, is the first difference of the next higher order of differences of the said original co-efficients $b, c, d, e, f, g, h, i, k, l, m$, &c, to that of which the last preceding co-efficient in the said assumed series was the first difference. And, upon an attentive examination of these investigations, and a comparison of them with each other, it will appear that, in each of these investigations, the series which forms the first, or left-hand, side of the last, or 4th, equation mentioned in the preceding, or 29th, article, (which 4th equation was obtained by the multiplication of the next preceding, or the 3d, equation into the binomial quantity $1+x$;) is always a series of the same kind with that which forms the first, or left-hand, side of the equation with which we set out in the beginning of the investigation; that is, it's first term is the first difference of the order of differences contained in that first equation, and it's following terms contain the several successive differences of the next higher order to the said former order, combined with, or multiplied into, the several powers of x in their natural order, to wit, x, x^2, x^3, x^4, x^5 , &c, and connected with the first term, or the first difference of the preceeding order, by the sign $+$ and the sign $-$ alternately.

The foregoing Conclusion proved in a more General Manner.

Art. 31. And that this must universally be the effect of such a multiplication of a series of quantities involving the terms of any order of differences of the said original

original co-efficients $b, c, d, e, f, g, h, i, k, l, m, \&c$, in which series the first term does not involve any power of x , but the second and other following terms involve in them the several powers of x in their natural order, to wit, the powers $x, x^2, x^3, x^4, x^5, x^6, \&c$, and in which the second and other following terms are connected with the first term by the sign $-$ and the sign $+$ alternately;—I say, that this must universally be the effect of a multiplication of a series of this kind into the binomial quantity $1 + x$, will appear by denoting the number which expresses the order of these differences contained in the series that is to be multiplied, in a general manner by the letter n , so that the series that is to be multiplied shall be the series $D_1^n - D_2^n \times x + D_3^n \times x^2 - D_4^n \times x^3 +$

$D_5^n \times x^4 - D_6^n \times x^5 + \&c$, and multiplying the said series into $1 + x$.

This may be done as follows :

$$\begin{array}{r} D_1^n - D_2^n \times x + D_3^n \times x^2 - D_4^n \times x^3 + D_5^n \times x^4 - D_6^n \times x^5 + \&c \\ 1 + x \end{array}$$

$$\begin{array}{r} D_1^n - D_2^n \times x + D_3^n \times x^2 - D_4^n \times x^3 + D_5^n \times x^4 - D_6^n \times x^5 + \&c \\ + D_1^n \times x - D_2^n \times x^2 + D_3^n \times x^3 - D_4^n \times x^4 + D_5^n \times x^5 - \&c \end{array}$$

$$D_1^n + \left[\begin{array}{c} D_1^n \\ -D_2^n \end{array} \right] \times x - \left[\begin{array}{c} D_2^n \\ -D_3^n \end{array} \right] \times x^2 + \left[\begin{array}{c} D_3^n \\ -D_4^n \end{array} \right] \times x^3 - \left[\begin{array}{c} D_4^n \\ -D_5^n \end{array} \right] \times x^4 + \left[\begin{array}{c} D_5^n \\ -D_6^n \end{array} \right] \times x^5 - \&c$$

which is $= D_1^n + D_1^{n+1} \times x - D_2^{n+1} \times x^2 + D_3^{n+1} \times x^3 - D_4^{n+1} \times x^4 + D_5^{n+1} \times x^5 - \&c$.

This series $D_1^n + D_1^{n+1} \times x - D_2^{n+1} \times x^2 + D_3^{n+1} \times x^3 - D_4^{n+1} \times x^4 + D_5^{n+1} \times x^5 - \&c$, obtained by the foregoing multiplication, is of the same kind with the series $D^{n-1} + D_1^n \times x - D_2^n \times x^2 + D_3^n \times x^3 - D_4^n \times x^4 + D_5^n \times x^5 - \&c$ with which we must have begun the investigation which ends with the said series obtained by the said multiplication. For the series with which we must have begun the said investigation which ends with that last series, is the series $D^{n-1} + D_1^n \times x - D_2^n \times x^2 + D_3^n \times x^3 - D_4^n \times x^4 + D_5^n \times x^5 - \&c$; this being the series upon which, if we perform the three operations of subtraction, division, and multiplication, mentioned in

art. 29, we shall thereby obtain the last series $D_1^n + D_1^{n+1} \times x - D_2^{n+1} \times x^2 - D_3^{n+1} \times x^3 + D_4^{n+1} \times x^4 - D_5^{n+1} \times x^5 + \&c$. For, by subtracting from it its first term D_1^{n-1} , we shall obtain the series $D_1^n \times x - D_2^n \times x^2 + D_3^n \times x^3 - D_4^n \times x^4 + D_5^n \times x^5 - \&c$; and, by dividing all the terms of this series by x , we shall obtain the series $D_1^n - D_2^n \times x + D_3^n \times x^2 - D_4^n \times x^3 + D_5^n \times x^4 - D_6^n \times x^5 + \&c$; and, by multiplying this last series into $1 + x$, we shall obtain (as we have just now seen,) the equation $D_1^n + D_1^{n+1} \times x - D_2^{n+1} \times x^2 + D_3^{n+1} \times x^3 - D_4^{n+1} \times x^4 + D_5^{n+1} \times x^5 - D_6^{n+1} \times x^6 + \&c$ *ad infinitum*. And therefore, since these three operations, which are performed in every new investigation, always produce a similar series, at the end of the investigation, to that with which the investigation begins, it follows that, whatever order of the said differences the letter n be supposed to express, or denote, in any one investigation, (as, for example, in the last investigation, by which we obtained the value of the co-efficient w), all the following co-efficients of the terms of the assumed series p , $\frac{Qx}{1+x}$, $\frac{Rx^2}{1+x}$, $\frac{Tx^3}{1+x}$, $\frac{Ux^4}{1+x}$, $\frac{Vx^5}{1+x}$, $\frac{Wx^6}{1+x}$, $\frac{Xx^7}{1+x}$, $\frac{Yx^8}{1+x}$, $\frac{Zx^9}{1+x}$, $\frac{Px \times x^{10}}{1+x^{10}}$, $\frac{Qx \times x^{11}}{1+x^{11}}$, $\frac{Rx \times x^{12}}{1+x^{12}}$, &c will be respectively equal to D^{n+1} , D^{n+2} , D^{n+3} , D^{n+4} , D^{n+5} , &c *ad infinitum*. And thus we may safely conclude that the following terms of the assumed series, which have not been investigated, will be analogous to those that have been investigated, or will be such as the above-mentioned law of generation, or continuation, will produce. Q. E. D.

End of the Appendix to the Treatise in Vol. III. of this Collection, pages 255, 256, 257, &c — — — 312, concerning the Investigation of the Differential Series

$$a - \frac{bx}{1+x} - \frac{D^2xx}{(1+x)^2} - \frac{D^3x^2}{(1+x)^3} - \frac{D^{11}x^4}{(1+x)^4} - \frac{D^{25}x^5}{(1+x)^5} - \frac{D^7x^6}{(1+x)^6} - \frac{D^9x^7}{(1+x)^7} - \frac{D^{11}x^8}{(1+x)^8} - \&c \text{ ad infinitum.}$$

A METHOD

A METHOD of SUMMING an INFINITE SERIES of DECREASING QUANTITIES of the following Form, to wit, $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ ad infinitum, with Exactness, when the several Terms of the said Series decrease so slowly as to make the Summation of it in the common Way, or by the mere Computation and Addition of a moderate Number of it's initial Terms, impracticable.

Invented by the Reverend Mr. JOHN HELLINS, Vicar of Potter's Pury, in Northamptonshire.

Article 1. **I**N this series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ ad infinitum, it is supposed, in the first place, that x denotes a quantity that is less than 1, but falls short of it by a very small difference, (as, for example, that it denotes a quantity that is equal to the fraction $\frac{9}{10}$, or the fraction $\frac{99}{100}$, of which the former differs from 1 by only the small fraction $\frac{1}{10}$,

and the latter differs from it by the still smaller fraction $\frac{1}{100}$;) and, consequently, that the series 1, or $x^0, x^1, x^2, x^3, x^4, x^5, x^6, x^7, \&c$, of the several successive powers of x , (which are involved in the terms of the said series,) will be a decreasing geometrical progression, of which the several terms will decrease exceeding slowly: and it is supposed, in the second place, that the letters $a, b, c, d, e, f, g, h, \&c$ (which are the co-efficients of x^0 , or 1, and of $x^1, x^2, x^3, x^4, x^5, x^6, x^7, \&c$ in the said series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$, or $a \times 1 + bx^1 + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$, respectively,) do likewise form a decreasing progression of terms, so that b shall be less than a , and c than b , d than c , e than d , f than e , g than f , h than g , and every following co-efficient than the co-efficient immediately preceding it: and it is supposed, in the 3d place, that the decrease of these co-efficients $a, b, c, d, e, f, g, h, \&c$ is very slow, and consequently that the multiplication of these co-efficients into the several corresponding terms of the geometrical series 1, $x, x^2, x^3, x^4, x^5, x^6, x^7, \&c$, (by which the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ is produced,) will very little accelerate the decrease of the terms of the said geometrical series, and therefore that the terms of the series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ (which
arises

arises from such multiplication,) will also decrease very slowly: and it is supposed, in the 4th place, that, if we subtract the third term c of the series $a, b, c, d, e, f, g, h, \&c$ from the second term b , and the fourth term d from the third term c ; and every following term from the term next before it, the several differences thence arising, to wit, the quantities $b - c, c - d, d - e, e - f, f - g, g - h, \&c$, will be a series of decreasing quantities; and, in like manner, that the differences of these differences, or the second differences of the co-efficients $b, c, d, e, f, g, h, \&c$, to wit, the quantities $b - c - (c - d), c - d - (d - e), d - e - (e - f), e - f - (f - g), \&c$, and $f - g - (g - h), \&c$, or $b - 2c + d, c - 2d + e, d - 2e + f, e - 2f + g, \&c$, will also be a series of decreasing quantities; and that the third differences of the said co-efficients $b, c, d, e, f, g, h, \&c$, or the differences of their second differences, to wit, the quantities $b - 3c + 3d - e, c - 3d + 3e - f, d - 3e + 3f - g, \&c$, will also be a series of decreasing quantities; and, in like manner, that the fourth differences, and the fifth differences, and the sixth differences, and all the following differences, of the said co-efficients $b, c, d, e, f, g, h, \&c$, will be series of decreasing quantities. The logarithmick series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8}$

+ &c *ad infinitum*, (which is equal to the logarithm of the ratio of 1 to $1 - x$ in Napier's System of Logarithms,) will afford us an example of such a series,

if we suppose x to be $= \frac{9}{10}$. For then the said series will become equal to

$$\frac{9}{10} + \frac{1}{2} \times \left[\frac{9}{10}\right]^2 + \frac{1}{3} \times \left[\frac{9}{10}\right]^3 + \frac{1}{4} \times \left[\frac{9}{10}\right]^4 + \frac{1}{5} \times \left[\frac{9}{10}\right]^5 + \frac{1}{6} \times \left[\frac{9}{10}\right]^6 + \frac{1}{7} \times \left[\frac{9}{10}\right]^7 + \frac{1}{8} \times \left[\frac{9}{10}\right]^8 + \&c \text{ ad infinitum, or to } \frac{9}{10} \times \text{the series } 1 + \frac{1}{2} \times \left[\frac{9}{10}\right] + \frac{1}{3} \times \left[\frac{9}{10}\right]^2 + \frac{1}{4} \times \left[\frac{9}{10}\right]^3 + \frac{1}{5} \times \left[\frac{9}{10}\right]^4 + \frac{1}{6} \times \left[\frac{9}{10}\right]^5 + \frac{1}{7} \times \left[\frac{9}{10}\right]^6 + \frac{1}{8} \times \left[\frac{9}{10}\right]^7 + \&c \text{ ad infinitum; which last series is of the same form}$$

with the general series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ *ad infinitum*, the fraction $\frac{9}{10}$ corresponding to x , and the co-efficients $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \&c$, *ad infinitum*, corresponding to the co-efficients

$a, b, c, d, e, f, g, h, \&c$ *ad infinitum*.

Art. 2. These co-efficients $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \&c$ do indeed, for a few of the first terms, decrease rather faster than might be expected in a series that converges very slowly. But their decrease soon becomes so slow

as to contribute very little to the diminution of the powers of the fraction $\frac{9}{10}$ with which they are combined; inasmuch that, if we were to endeavour to find the value of the said series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$ *ad infinitum*, or $\frac{9}{10} + \frac{1}{2} \times \left(\frac{9}{10}\right)^2 + \frac{1}{3} \times \left(\frac{9}{10}\right)^3 + \frac{1}{4} \times \left(\frac{9}{10}\right)^4 + \frac{1}{5} \times \left(\frac{9}{10}\right)^5 + \frac{1}{6} \times \left(\frac{9}{10}\right)^6 + \frac{1}{7} \times \left(\frac{9}{10}\right)^7 + \frac{1}{8} \times \left(\frac{9}{10}\right)^8 + \&c$ *ad infinitum*, in the common way, or by merely computing some of it's initial terms, and adding them together, the first thirty-two terms of it would give us the value of the whole series exact only to one figure, the said thirty-two terms being equal to the mixt number 2.294,933,820,483,84, which agrees with the true value of the said series only in the first, or highest, figure 2, the said true value being 2.302,585,092,994,045,684,017, &c, or Napier's Logarithm of the ratio of 1 to $1 - \frac{9}{10}$, or of the ratio of 1 to $\frac{1}{10}$, or of the ratio of 10 to 1, or (according to the common inaccurate expression,) Napier's Logarithm of the number 10. So little would the decrease of the co-efficients $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$, &c (though tolerably swift for the three or four first terms,) contribute to accelerate the decrease of the powers of the fraction $\frac{9}{10}$, and thereby facilitate the calculation. And, if we were to endeavour to find the value of the said series, or logarithm, exact to twenty places of decimal figures, in this way, or by the mere computation and addition of it's terms, it would, as I apprehend, be necessary to compute very nearly four hundred of it's terms for that purpose. And, if we were to suppose x in the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$ *ad infinitum* to be equal to $\frac{99}{100}$, instead of $\frac{9}{10}$, the difficulty of obtaining the value of the said series to this degree of exactness in the common way would be much greater than in the former case, and indeed so great as to be justly considered as unformountable. It is therefore very desirable, when our object is to obtain the value of the said logarithmick series to any great degree of exactness in either of these cases, to have some easier method of discovering the said value than the direct, or common, one, of merely computing and adding together a sufficient number of it's initial terms. And accordingly many eminent Mathematicians have exerted their abilities in endeavouring to find some such methods of discovering the values of infinite serieses of the foregoing form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + \&c$, when their terms decrease with this excessive slowness.

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ART. 3.

ART. 3. Now it is remarkable that a slowly-converging infinite series of the foregoing form, $a + bx + cxx + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*, in which all the terms are connected together by the sign +, or are added to each other, has been found to be much more difficult to be summed to a considerable degree of exactness, than a series consisting of the very same terms, in which the terms are marked with the signs — and + alternately, or in which the second, and the fourth, and the sixth, and the eighth, terms, and all the following even terms, are marked with the sign —, or are subtracted from the other terms, such as the series $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ *ad infinitum*. This has been long ago observed by Mr. James Stirling, in his learned Treatise on the Summation of Infinite Series, page 17, where he tells us that *Series quarum termini sunt per vices negativi et affirmativi, sunt magis tractabiles quam altera, ubi de summatione agitur*: and I have often experienced the truth of this remark. For, as to the latter series $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ *ad infinitum*, it has been shewn in a paper that I sent to the Royal Society in the year 1777, and that was printed in the Philosophical Transactions for that year, (and that has been lately re-printed, with some additions, in the Third Volume of this Collection of Tracts, called *Scriptores Logarithmici*, pages 219, 220, 221, &c - - - 252,) that, if D^1 be put for $b - e$, or the first difference of the first order, of the co-efficients $b, c, d, e, f, g, \&c$, (that involve the several powers of x), and D'' be put for $b - 2c + d$, or the first difference of the second order, of the same co-efficients, and D''' be put for $b - 3c + 3d - e$, or the first difference of the third order, of the same co-efficients, and $D^4, D^5, D^6, \&c$ for the first differences of the fourth, fifth, sixth, and other following orders of the same co-efficients respectively, the said slowly-converging series $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ *ad infinitum*, will be equal to the differential series $a - \frac{bx}{1+x} - \frac{D^1x^2}{1+x^2} - \frac{D^2x^3}{1+x^3} - \frac{D^3x^4}{1+x^4} - \frac{D^4x^5}{1+x^5} - \frac{D^5x^6}{1+x^6} - \frac{D^6x^7}{1+x^7} - \&c$ *ad infinitum*; which series will always, even in the most difficult cases, converge with a sufficient degree of swiftness to give us the value of the whole of itself, and consequently of it's equal, the series $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ *ad infinitum*, exact to eight, or nine, places of decimal figures by computing only eight of it's terms. But, when I have endeavoured to find a similar series for the summation of the former series $a + bx + cxx + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*, I have always found my efforts to be unsuccessful, though I have attempted it in a great variety of different ways. Nor have I met with any general and convenient method of doing this, for an infinite series of this form, in any of the learned writers who have treated of this subject; though many of them have given very practicable methods of summing some particular series, in which the terms were all connected by the sign +, or added to each other, when the co-efficients of the powers of x , in the terms of the said series, have observed some particular law of decrease that was favourable to their summation.

ART. 4.

ART. 4. But now at last this difficulty seems to be overcome. For my learned friend, the Reverend Mr. John Hellins, Vicar of Potter's Pury in Northamptonshire, (who furnished me with some ingenious quadratures of the circle, which have been infered in the preceeding volume of this Collection;) has lately

found out a method of applying that very differential series $a = \frac{bx}{1+x} - \frac{dx^3}{1+x^3}$

$-\frac{d^3x^3}{1+x^3} - \frac{d^5x^5}{1+x^5} - \frac{d^7x^7}{1+x^7} - \frac{d^9x^9}{1+x^9} - \frac{d^{11}x^{11}}{1+x^{11}} - \&c$ *ad infinitum*, to the

summation of the said general series $a + bx + cx^3 + dx^5 + ex^7 + fx^9 + gx^{11} + bx^7 + \&c$ *ad infinitum* with the same success as it was before applied to the summation of the other series $a - bx + cxx - dx^3 + ex^5 - fx^7 + gx^9 - bx^7 + \&c$ *ad infinitum*, or so as to obtain the true value of the said series $a + bx + cxx + dx^3 + ex^5 + fx^7 + gx^9 + bx^7 + \&c$ *ad infinitum* to as great a degree of exactness as we before obtained, by means of it, the true value of the series $a - bx + cxx - dx^3 + ex^5 - fx^7 + gx^9 - bx^7 + \&c$: which is, surely, a very valuable discovery. This summation, however, cannot be performed so easily as the summation of the former series $a - bx + cxx - dx^3 + ex^5 - fx^7 + gx^9 - bx^7 + \&c$ *ad infinitum*; because it will usually require three, or four, and, in some cases, seven, or eight, repeated applications of the said

differential series $a = \frac{bx}{1+x} - \frac{d^3x^3}{1+x^3} - \frac{d^5x^5}{1+x^5} - \frac{d^7x^7}{1+x^7} - \frac{d^9x^9}{1+x^9} - \frac{d^{11}x^{11}}{1+x^{11}} -$

$\frac{d^{13}x^{13}}{1+x^{13}} - \frac{d^{15}x^{15}}{1+x^{15}} - \&c$ *ad infinitum* to the summation of as many serieses of

the said form $a - bx + cxx - dx^3 + ex^5 - fx^7 + gx^9 - bx^7 + \&c$ *ad infinitum*, before we can accomplish the summation of the series of the other form $a + bx + cxx + dx^3 + ex^5 + fx^7 + gx^9 + bx^7 + \&c$ *ad infinitum*, to the proposed degree of exactness. And these several operations will be found to be very laborious, though much less so than finding the value of the original series to the same degree of exactness in the common way, or by the mere computation of it's terms and addition of them to each other. This method of summing an infinite series of this form, (or of which the terms are all connected with each other by addition,) invented by Mr. Hellins, I shall now endeavour to explain.

ART. 5. But, before we enter upon the description of this ingenious method of applying the differential series $a = \frac{bx}{1+x} - \frac{d^3x^3}{1+x^3} - \frac{d^5x^5}{1+x^5} - \frac{d^7x^7}{1+x^7} -$

$\frac{d^9x^9}{1+x^9} - \frac{d^{11}x^{11}}{1+x^{11}} - \frac{d^{13}x^{13}}{1+x^{13}} - \&c$ to the summation of an infinite series of the

foregoing form, $a + bx + cxx + dx^3 + ex^5 + fx^7 + gx^9 + bx^7 + \&c$ *ad infinitum*, it will be proper to observe, that, when any such series is to be summed in the method that is now to be delivered, it will always be found most convenient to begin our operations by computing about thirty, or forty, of it's terms in the common way, and adding them together into one sum, (which we shall

here denote by the capital letter A ,) and then to proceed to compute the remainder of the series, (which we will here denote by the capital letter R ,) by the application of the differential series above-mentioned in the manner directed by Mr. Hellins: after which, by adding the value found for the series R , or the remainder of the first series, to A , or the value of the initial terms of the said series which we had before computed, we shall obtain the value of $A + R$, or of the whole series at first proposed, to the required degree of exactness. Thus, for example, if the series that is to be summed, is the above-mentioned

series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$ *ad infinitum*, and

x is supposed to be equal to the fraction $\frac{9}{10}$, (and consequently the whole series is equal to the logarithm of the ratio of x to $1 - \frac{9}{10}$, or of x to $\frac{1}{10}$, or of

10 to 1, in Napier's System of Logarithms, which logarithm we know, from other grounds, to be $= 2.302,585,092,994,045,684,017$. &c,) it will be found most convenient to compute about thirty, or forty, of the terms of this series, to wit, the several terms $\frac{9}{10}, \frac{1}{2} \times \frac{9}{10}, \frac{1}{3} \times \frac{9}{10}, \frac{1}{4} \times \frac{9}{10}, \frac{1}{5} \times \frac{9}{10},$

&c as far as the term $\frac{1}{30} \times \frac{9}{10}$, or the term $\frac{1}{40} \times \frac{9}{10}$, inclusively, in the common way, and add them all together into one sum, and then to proceed to the investigation of the value of the remaining terms of the series, that is, if we have computed only thirty terms of the original series, to the investigation

of the value of the infinite series $\frac{1}{31} \times \frac{9}{10} + \frac{1}{32} \times \frac{9}{10} + \frac{1}{33} \times \frac{9}{10} +$

$\frac{1}{34} \times \frac{9}{10} + \frac{1}{35} \times \frac{9}{10} + \frac{1}{36} \times \frac{9}{10} + \frac{1}{37} \times \frac{9}{10} + \frac{1}{38} \times \frac{9}{10} +$

&c *ad infinitum*, or, if we have computed forty terms of the original series, to the investigation of the value of the infinite series $\frac{1}{41} \times \frac{9}{10} + \frac{1}{42} \times \frac{9}{10} +$

$\frac{1}{43} \times \frac{9}{10} + \frac{1}{44} \times \frac{9}{10} + \frac{1}{45} \times \frac{9}{10} + \frac{1}{46} \times \frac{9}{10} + \frac{1}{47} \times \frac{9}{10} +$

$\frac{1}{48} \times \frac{9}{10} + \&c$ *ad infinitum*, by the application of the differential series

above-mentioned in the manner directed by Mr. Hellins: after which, by adding the value thereby found for this remaining series, to the value of the first 30, or 40, terms of the original series which we had before computed, we shall

obtain the value of the whole original series $\frac{9}{10} + \frac{1}{2} \times \frac{9}{10} + \frac{1}{3} \times \frac{9}{10} +$

$\frac{1}{4} \times \frac{9}{10} + \frac{1}{5} \times \frac{9}{10} + \frac{1}{6} \times \frac{9}{10} + \frac{1}{7} \times \frac{9}{10} + \frac{1}{8} \times \frac{9}{10} +$

&c *ad infinitum*, which was to be discovered.

Art. 6.

ART. 6. Mr. Hellins has applied his method to the summation of this series, and has, for this purpose, computed the first thirty-two terms of it, and added them all together into one sum, which he has found to be = 2.294,933,820, 483,84, &c, which agrees with the true value of the whole series, to wit, 2.302,585,092,994,045,684,017, &c, only in the first, or highest, figure 2 : and he has then proceeded to investigate the value of the remaining series $\frac{x}{33}$

$\times \left(\frac{9}{10}\right)^{31} + \frac{1}{34} \times \left(\frac{9}{10}\right)^{32} + \frac{1}{35} \times \left(\frac{9}{10}\right)^{33} + \frac{1}{36} \times \left(\frac{9}{10}\right)^{34} + \frac{1}{37} \times \left(\frac{9}{10}\right)^{35} + \frac{1}{38} \times \left(\frac{9}{10}\right)^{36} + \&c$ ad infinitum, by means of the above-mentioned differential series

$$a = \frac{bx}{1+x} - \frac{D^1 x^2}{1+x)^2} - \frac{D^2 x^3}{1+x)^3} - \frac{D^3 x^4}{1+x)^4} - \frac{D^4 x^5}{1+x)^5} - \frac{D^5 x^6}{1+x)^6} - \frac{D^6 x^7}{1+x)^7} -$$

&c ad infinitum in the method which he has lately discovered, and finds it to be = 0.007,651,272,511,2 ; which being added to 2.294,933,820,483,8, gives

2.302,585,092,995,0 for the value of the whole series $\frac{9}{10} + \frac{1}{2} \times \left(\frac{9}{10}\right)^2 + \frac{1}{3}$

$\times \left(\frac{9}{10}\right)^3 + \frac{1}{4} \times \left(\frac{9}{10}\right)^4 + \frac{1}{5} \times \left(\frac{9}{10}\right)^5 + \frac{1}{6} \times \left(\frac{9}{10}\right)^6 + \frac{1}{7} \times \left(\frac{9}{10}\right)^7 + \frac{1}{8} \times \left(\frac{9}{10}\right)^8 + \&c$ ad infinitum, or for Napier's logarithm of the ratio of 10 to 1 ; which

value agrees with the more accurate value of that logarithm, to wit, the number 2.302,585,092,994,045,684,017, &c in the first twelve figures 2.302,585,092,99. This is a very great degree of exactness, and shews the excellence and importance of Mr. Hellins's method of investigation, which I shall now proceed to describe.

A Description of Mr. Hellins's Method of applying the above-mentioned Differential

$$\text{Series } a = \frac{bx}{1+x} - \frac{D^1 x^2}{1+x)^2} - \frac{D^2 x^3}{1+x)^3} - \frac{D^3 x^4}{1+x)^4} - \frac{D^4 x^5}{1+x)^5} - \frac{D^5 x^6}{1+x)^6} -$$

$$\frac{D^6 x^7}{1+x)^7} - \frac{D^7 x^8}{1+x)^8} - \&c \text{ ad infinitum to the Summation of a slowly-converging}$$

Series of the foregoing Form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ ad infinitum.

ART. 7. If, when we are desirous of obtaining the value of a slowly converging infinite series of the foregoing form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ ad infinitum, we begin our operations by computing a certain number

number of it's first terms, and adding them together into one sum to be denoted by the letter A , before we have recourse to the application of the above-mentioned differential series, agreeably to the advice contained in the last article; —and the number of the initial terms so computed be denoted by the letter m ; —it is evident that the remaining series R , which we shall then be under a necessity of summing by the help of the said differential series, will be of the following form, $ax^m + bx^{m+1} + cx^{m+2} + dx^{m+3} + ex^{m+4} + fx^{m+5} + gx^{m+6} + bx^{m+7} + \&c$ *ad infinitum*, or (that we may distinguish these co-efficients from the co-efficients a, b, c, d, e, f, g, h , of the first eight terms of the series,) of the following form, $a'x^m + b'x^{m+1} + c'x^{m+2} + d'x^{m+3} + e'x^{m+4} + f'x^{m+5} + g'x^{m+6} + b'x^{m+7} + \&c$ *ad infinitum*, in which series the co-efficients $a', b', c', d', e', f', g', b'$, differ from the letters a, b, c, d, e, f, g, h , (which denote the co-efficients of the first eight terms of the series,) only by having a mark, or accent, ' placed over them. Now this series $a'x^m + b'x^{m+1} + c'x^{m+2} + d'x^{m+3} + e'x^{m+4} + f'x^{m+5} + g'x^{m+6} + b'x^{m+7} + \&c$ *ad infinitum* is equal to $x^m \times$ the series $a' + b'x + c'xx + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ *ad infinitum*; which last series is of the same form as the proposed series $a + bx + cxx + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*. We will therefore now endeavour to apply the differential series above-mentioned to the summation of the series $a' + b'x + c'xx + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ *ad infinitum*; in which, if we succeed, we must multiply the value of the said series thereby obtained, into x^m , and the product thence arising will be equal to the series $a'x^m + b'x^{m+1} + c'x^{m+2} + d'x^{m+3} + e'x^{m+4} + f'x^{m+5} + g'x^{m+6} + b'x^{m+7} + \&c$, *ad infinitum*, or to R ; which, being added to A , or the sum of the m initial terms of the original series $a + bx + cxx + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum* which were computed at first, will give us the value of $A + R$, or of the whole original series $a + bx + cxx + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*, which was to be summed. We will therefore now endeavour, by means of the above-mentioned differential series, to find the value of the infinite series $a' + b'x + c'xx + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ *ad infinitum*.

The First Step of this Method of Summing the Series $a' + b'x + c'x^2 + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ ad infinitum, by means of the above-mentioned Differential Series,

ART. 8. Now here we may observe that the series $a' + b'x + c'x^2 + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ ad infinitum, (of which we are endeavouring to find the value,) is equal to the sum of the two following serieses, to wit, the series $d' - b'x + c'x^2 - d'x^3 + e'x^4 - f'x^5 + g'x^6 - b'x^7 + \&c$ ad infinitum and the series $2b'x + 2d'x^3 + 2f'x^5 + 2b'x^7 + \&c$ ad infinitum, or to the sum of the series $d' - b'x + c'xx - d'x^3 + e'x^4 - f'x^5 + g'x^6 - b'x^7 + \&c$ ad infinitum and $2x \times$ the series $b' + d'x^2 + f'x^4 + b'x^6 + \&c$ ad infinitum.

Let the value of the series $d' - b'x + c'xx - d'x^3 + e'x^4 - f'x^5 + g'x^6 - b'x^7 + \&c$ ad infinitum be found by means of the differential series $a - \frac{bx}{1+x} - \frac{D^1 x^2}{1+x^2} - \frac{D^2 x^3}{1+x^3} - \frac{D^3 x^4}{1+x^4} - \frac{D^4 x^5}{1+x^5} - \frac{D^5 x^6}{1+x^6} - \frac{D^6 x^7}{1+x^7} - \&c$ ad infinitum in the manner described in the 3d Volume of the *Scriptores Logarithmici* pages 219, 220, &c — — — $\pi 52$; and let the said value be denoted by the small Greek letter ω .

Then will the series $a' + b'x + c'xx + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ ad infinitum, (of which we are now seeking the value,) be equal to $\omega + 2x \times$ the series $b' + d'xx + f'x^4 + b'x^6 + \&c$ ad infinitum, which last series is of the same form with the series $a' + b'x + c'xx + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ ad infinitum, (of which we are now seeking the value,) but converges with a greater degree of swiftness than the said series, because it's terms involve the several successive powers of xx , to wit, $xx, xx^2, xx^3, \&c$, or $xx, x^2, x^4, \&c$, instead of the like powers of x itself. And thus we shall have reduced our remaining business in this investigation to the summation of the series $b' + d'xx + f'x^4 + b'x^6 + \&c$ ad infinitum, which converges less slowly by one degree than the former series $a' + b'x + c'xx + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ ad infinitum. This is the first step of Mr. Hellins's investigation.

The Second Step of the said Method of Summing the said Infinite Series $d' + b'x + c'x^2 + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ ad infinitum.

Art. 9. Before we enter upon the next step of Mr. Hellins's investigation it will be proper to continue the series $d' + b'x + c'xx + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ to sixteen terms instead of eight: and, in doing this, it will, I think, be convenient to denote the co-efficients of the next following eight terms by the same letters a, b, c, d, e, f, g , and b , as have been used to denote the co-efficients of the former eight terms, but with two accents, ", placed over them instead of one, to distinguish them from the said former co-efficients, which were marked with only one accent. And then the said series will be as follows, to wit, $d' + b'x + c'xx + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + d''x^8 + b''x^9 + c''xx + d''x^3 + e''x^4 + f''x^5 + g''x^6 + b''x^7 + \&c$ ad infinitum, and the series $b' + d'xx + f'x^4 + b'x^6 + \&c$, (of which we are still to find the value,) will be $b' + d'xx + f'x^4 + b'x^6 + b'x^8 + d'x^{10} + f'x^{12} + b'x^{14} + \&c$ ad infinitum.

Art. 10. These things being premised, Mr. Hellins's next observation is as follows: The series $b' + d'xx + f'x^4 + b'x^6 + b'x^8 + d'x^{10} + f'x^{12} + b'x^{14} + \&c$ ad infinitum, (of which we are still to find the value,) is equal to the sum of the series $b' - d'xx + f'x^4 - b'x^6 + b''x^8 - d''xx + f''x^{10} - b'x^{12} + \&c$ ad infinitum, and the series $2d'xx + 2b'x^6 + 2d''x^{10} + 2b''x^{14} + \&c$ ad infinitum, or to the sum of the series $b' - d'xx + f'x^4 - b'x^6 + b'x^8 - d'x^{10} + f'x^{12} - b'x^{14} + \&c$ ad infinitum, and $2xx \times$ the series $d' + b'x^4 + d'x^8 + b'x^{12} + \&c$ ad infinitum; the former of which serieses may be summed by means of the above-mentioned Differential series, and the latter, to wit, $d' + b'x^4 + d'x^8 + b'x^{12} + \&c$ ad infinitum is of the same form with the series $b' + d'xx + f'x^4 + b'x^6 + \&c$ ad infinitum, from which it was derived, but converges more swiftly than the said series, because it's terms involve the several successive powers of x^4 , to wit, $x^4, x^8, x^{12}, x^{16}, \&c$, or $x^4, x^8, x^{12}, \&c$, instead of the like powers of xx . Let the series $b' - d'xx + f'x^4 - b'x^6 + b'x^8 - d'x^{10} + f'x^{12} - b'x^{14} + \&c$ ad infinitum be summed by means of the above-mentioned Differential series, and let the value of it, thereby obtained, be denoted by the small Greek letter ζ . Then will the series $b' + d'xx + f'x^4 + b'x^6 + b'x^8 + d'x^{10} + f'x^{12} + b'x^{14} + \&c$ ad infinitum, (of which we were now to find the value,) be equal to the quantity $\zeta + 2xx \times$ the series $d' + b'x^4 + d'x^8 + b'x^{12} + \&c$ ad infinitum, which converges much more swiftly than the first series $d' + b'x + c'xx + d'x^3 + e'x^4 + f'x^5 + g'x^6 + b'x^7 + \&c$ ad infinitum, of which we were to find the value.

Still, however, the terms of the series $d' + b'x^4 + d'x^8 + b'x^{12} + \&c$ ad infinitum, will, if x is equal to $\frac{1}{10}$, or any greater fraction, converge too slowly

to

to make the summation of it to any great degree of exactness in the common way, or by the mere computation and addition of it's terms, easily practicable.

For the fourth power of $\frac{9}{10}$ is $\frac{6561}{10000}$, which is greater than $\frac{1}{2}$; so that about four terms would be necessary to be computed for every new figure that would be exact in the value of it obtained in this manner, and consequently forty-eight, or fifty terms of it must be computed in order to obtain the said value exact to twelve places of figures. It therefore will be expedient to increase the convergency of the final series by another degree, or two, before we proceed to determine it's value in the common way, or by the mere computation and addition of it's terms.

The Third Step of the said Method of Summing the said Series $a' + b'x + c'x^2 + d'x^3 + e'x^4 + f'x^5 + g'x^6 + h'x^7 + \&c$ ad infinitum.

Art. 11. And here it will be expedient to continue the number of terms of the last series $a' + b'x + d'x^3 + b'x^{13} + \&c$, (of which we are still to find the value,) from four terms to eight; and for this purpose it will be necessary to continue the terms of the series $a' + b'x + c'x^2 + d'x^3 + e'x^4 + f'x^5 + g'x^6 + h'x^7 + \&c$ ad infinitum to thirty-two terms, instead of sixteen; and in doing this it will be convenient to denote the co-efficients of every set of eight terms by the same letters a, b, c, d, e, f, g, h , with one, two, three, and four, accents placed over them; so that the said series will be as follows, $a' + b'x + c'x^2 + d'x^3 + e'x^4 + f'x^5 + g'x^6 + h'x^7 + a''x^8 + b''x^9 + c''x^{10} + d''x^{11} + e''x^{12} + f''x^{13} + g''x^{14} + h''x^{15} + a'''x^{16} + b'''x^{17} + c'''x^{18} + d'''x^{19} + e'''x^{20} + f'''x^{21} + g'''x^{22} + h'''x^{23} + a''''x^{24} + b''''x^{25} + c''''x^{26} + d''''x^{27} + e''''x^{28} + f''''x^{29} + g''''x^{30} + h''''x^{31} + \&c$ ad infinitum. Then will the series $a' + b'x + d'x^3 + b'x^{13} + \&c$ (of which we are still to find the value,) be $a' + b'x + d'x^3 + b'x^{13} + a''x^{10} + b''x^{20} + a'''x^{15} + b'''x^{25} + \&c$ ad infinitum.

Art. 12. These things being premised, Mr. Hellins observes, as before, that the said series $a' + b'x + d'x^3 + b'x^{13} + a''x^{10} + b''x^{20} + a'''x^{15} + b'''x^{25} + \&c$ ad infinitum (of which we are still to find the value,) is equal to the sum of the series $a' - b'x + d'x^3 - b'x^{13} + a''x^{10} - b''x^{20} + a'''x^{15} - b'''x^{25} + \&c$ ad infinitum and the series $2b'x + 2b''x^{13} + 2b'''x^{20} + 2b''''x^{25} + \&c$, or to the sum of the series $a' - b'x + d'x^3 - b'x^{13} + d'x^{10} - b''x^{20} + d'x^{15} - b''x^{25} + \&c$ ad infinitum and $2x^4 \times$ the series $b' + b'x^3 + b'x^6 + b'x^9 + b'x^{12} + b'x^{15} + \&c$ ad infinitum; of which the former series $a' - b'x + d'x^3 - b'x^{13} + d'x^{10} - b''x^{20} + d'x^{15} - b''x^{25} + \&c$ ad infinitum may be summed by the above-mentioned differential series, and the latter series $b' + b'x^3 + b'x^6 + b'x^9 + b'x^{12} + b'x^{15} + \&c$ ad infinitum

$b'''x^{10} + b''''x^4 + \&c$ *ad infinitum* is of the same form with the series $a' + b'x^4 + a''x^8 + b''x^{12} + a'''x^{16} + b'''x^{20} + a''''x^{24} + b''''x^{28} + \&c$ *ad infinitum*, (from which it was derived,) but converges more swiftly than the said series, because its terms involve the successive powers of x^4 , to wit, $x^4, x^8, x^{12}, \&c$, or $x^4, x^{10}, x^{16}, \&c$, instead of the like powers of x^2 . Now let the said series $a' + b'x^4 + a''x^8 + b''x^{12} + a'''x^{16} + b'''x^{20} + a''''x^{24} + b''''x^{28} + \&c$ *ad infinitum* be summed by means of the above-mentioned differential series; and let the value of it, thereby obtained, be denoted by the small Greek letter γ . Then will the series $a' + b'x^4 + a''x^8 + b''x^{12} + a'''x^{16} + b'''x^{20} + a''''x^{24} + b''''x^{28} + \&c$ *ad infinitum*, (of which we were now to find the value,) be equal to the quantity $\gamma + 2x^4 \times$ the series $b' + b''x^4 + b'''x^8 + b''''x^{12} + \&c$ *ad infinitum*, which converges with a much greater degree of swiftness than the first series $a' + b'x^4 + a''x^8 + a'''x^{12} + a''''x^{16} + a'''''x^{20} + a''''''x^{24} + \&c$ *ad infinitum*, of which we were to find the value.

Art. 13. And in the same manner, if it be thought necessary, we may, by another process of the same kind as the three foregoing processes, reduce the remaining business of the investigation to the computation of an infinite series of quantities that would involve the several successive powers of x^{10} , to wit, $x^{10}, x^{15}, x^{20}, x^{25}, x^{30}, x^{35}, x^{40}, x^{45}, x^{50}, \&c$, instead of the powers of x^4 , which are those involved in the series $b' + b''x^4 + b'''x^8 + b''''x^{12} + \&c$ last-obtained; and, by a fifth process of the same kind, we might reduce it to the computation of an infinite series of quantities that would involve the several successive powers of x^{12} , to wit, $x^{12}, x^{24}, x^{36}, x^{48}, x^{60}, x^{72}, x^{84}, x^{96}, x^{108}, \&c$, which, if x be supposed $= \frac{9}{10}$, would converge with a great degree of swiftness.

And thus at last we might, by repeating these processes often enough, and making, in every new process, an application of the differential series above mentioned to the summation of a series in which the terms are alternately marked with the signs $-$ and $+$, reduce the remaining business of the investigation from the summation of a slowly-converging infinite series to the summation of a swiftly-converging infinite series, which it would be easy to sum to a great degree of exactness in the common way, by computing a moderate number of it's terms and adding their values to each other. And when this is done, the value of the original infinite series will be obtained by adding together the values of the several infinite series into which it has been resolved. Q. E. D.

Art. 14. Thus, for example, if we make use of only the three processes that have been above-described, and find the value of the last-obtained series $b' + b''x^4 + b'''x^{10} + b''''x^{16} + \&c$ *ad infinitum* in the common way, or by computing a sufficient number of it's terms and adding their values together, and call the sum of the said values L , we shall have the value of the whole original series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ *ad infinitum* $=$

A + R

$$\begin{aligned}
A + R &= A + \text{the series } d'x^m + b'x^{m+1} + c'x^{m+2} + d'x^{m+3} + e'x^{m+4} \\
&\quad + f'x^{m+5} + g'x^{m+6} + b'x^{m+7} + \&c \text{ ad infinitum} \\
&= A + x^m \times \text{the series } d' + b'x + c'x^2 + d'x^3 + e'x^4 + f'x^5 + g'x^6 \\
&\quad + b'x^7 + \&c \text{ ad infinitum} \\
&= A + x^m \times \text{the series } d' - b'x + c'x^2 - d'x^3 + e'x^4 - f'x^5 + g'x^6 \\
&\quad - b'x^7 + \&c \text{ ad infinitum} + x^m \times \text{the series } 2b'x + 2d'x^3 + 2f'x^5 \\
&\quad + 2b'x^7 + \&c \text{ ad infinitum} \\
&= (\text{by art. 8,}) A + x^m \times \alpha + x^m \times 2x \times \text{the series } b' + d'xx + f'x^4 \\
&\quad + b'x^6 + b'x^8 + d''x^{10} + f''x^{12} + b''x^{14} + \&c \text{ ad infinitum} \\
&= A + x^m \times \alpha + x^m \times 2x \times \text{the series } b' - d'xx + f'x^4 - b'x^6 + \\
&\quad b'x^8 - d''x^{10} + f''x^{12} - b''x^{14} + \&c \text{ ad infinitum} + x^m \times 2x \times \\
&\quad \text{the series } 2d'xx + 2b'x^6 + 2d''x^{10} + 2b''x^{14} + \&c \text{ ad infinitum} \\
&= (\text{by art. 10,}) A + x^m \times \alpha + x^m \times 2x \times \zeta + x^m \times 2x \times 2xx \times \\
&\quad \text{the series } d' + b'x^4 + d''x^8 + b''x^{12} + \&c \text{ ad infinitum} \\
&= (\text{by art. 11,}) A + x^m \times \alpha + x^m \times 2x \times \zeta + x^m \times 2x \times 2xx \times \\
&\quad \text{the series } d' + b'x^4 + d''x^8 + b''x^{12} + d'''x^{16} + b'''x^{20} + d''''x^{24} \\
&\quad + b''''x^{28} + \&c \text{ ad infinitum} \\
&= A + x^m \times \alpha + x^m \times 2x \times \zeta + x^m \times 2x \times 2xx \times \text{the series} \\
&\quad d' - b'x^4 + d''x^8 - b''x^{12} + d'''x^{16} - b'''x^{20} + d''''x^{24} - b''''x^{28} \\
&\quad + \&c \text{ ad infinitum} + x^m \times 2x \times 2xx \times \text{the series } 2b'x^4 + 2b''x^{12} \\
&\quad + 2b'''x^{20} + 2b''''x^{28} + \&c \text{ ad infinitum} \\
&= (\text{by art. 12,}) A + x^m \times \alpha + x^m \times 2x \times \zeta + x^m \times 2x \times 2xx \\
&\quad \times \gamma + x^m \times 2x \times 2xx \times 2x^4 \times \text{the series } b' + b''x^8 + b'''x^{16} \\
&\quad + b''''x^{24} + \&c \text{ ad infinitum} \\
&= A + x^m \times \alpha + x^m \times 2x \times \zeta + x^m \times 2x \times 2xx \times \gamma + x^m \times \\
&\quad 2x \times 2xx \times 2x^4 \times L.
\end{aligned}$$

Q. E. I.

4 L 2

Art. 15.

Art. 15. And, if we were to make use of two more of these processes, in order that the final series should converge by the powers of x^{18} , (instead of converging by the powers of x^6 , as the final series $L' + L''x^6 + L'''x^{12} + L''''x^{18} + \&c$ *ad infinitum*, or L , does in the last article,) and we were to denote the said final series (which was to be computed in the common way, or by simply computing a few of it's initial terms and adding them together into one sum,) by the capital letter N , we should find the whole original series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum* to be $= A + x^m \times a + x^m \times 2x \times 6 + x^m \times 2x \times 2xx \times \gamma + x^m \times 2x \times 2xx \times 2x^4 \times \delta + x^m \times 2x \times 2xx \times 2x^4 \times 2x^8 \times \epsilon + x^m \times 2x \times 2xx \times 2x^4 \times 2x^8 \times 2x^{16} \times N$, or $A + x^m \times a + x^m \times 2x \times 6 + x^m \times 4x^2 \times \gamma + x^m \times 8x^3 \times \delta + x^m \times 16x^{18} \times \epsilon + x^m \times 32x^{31} \times N$. Q. E. I.

Art. 16. The foregoing description of Mr. Hellins's method of proceeding to find the sum of a slowly-converging series of the above-mentioned form, $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*, by repeated applications of the differential series $a - \frac{bx}{1+x} - \frac{p^1x^2}{1+x^2} - \frac{p^2x^3}{1+x^3} - \frac{p^3x^4}{1+x^4} - \frac{p^4x^5}{1+x^5} - \frac{p^5x^6}{1+x^6} - \frac{p^6x^7}{1+x^7} - \&c$ *ad infinitum* will be better understood by applying it to a particular example. And therefore I shall now proceed to apply it to the summation of the above-mentioned slowly-converging logarithmick series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \&c$ *ad infinitum*, when x is $= \frac{9}{10}$.

An Example of the foregoing Method of finding the Value of an Infinite Series of the foregoing Form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$, when it's Terms decrease very slowly.

Art. 17. Let it be required to find by the method above-described the value of the infinite series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$, when x is equal to the fraction $\frac{9}{10}$.

To

To bring this series under the form of the general series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ *ad infinitum*, we must consider it as being the product of the multiplication of x into the series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum*, which is of the said form.

ART. 18. Now let the first thirty-two terms of this last series be computed to eighteen places of decimal figures. This may be done in the manner following :

Since x is supposed to be equal to $\frac{9}{10}$, or 0.9, we shall have

$xx (= 0.9^2)$	$= 0.810,000,000,000,000,000,$
and $x^3 (= 0.9^3 \times 0.9)$	$= 0.729,000,000,000,000,000,$
and $x^4 (= 0.9^4 \times 0.9)$	$= 0.656,100,000,000,000,000,$
and $x^5 (= 0.9^5 \times 0.9)$	$= 0.590,490,000,000,000,000,$
and $x^6 (= 0.9^6 \times 0.9)$	$= 0.531,441,000,000,000,000,$
and $x^7 (= 0.9^7 \times 0.9)$	$= 0.478,296,900,000,000,000,$
and $x^8 (= 0.9^8 \times 0.9)$	$= 0.430,467,210,000,000,000,$
and $x^9 (= 0.9^9 \times 0.9)$	$= 0.387,420,489,000,000,000,$
and $x^{10} (= 0.9^{10} \times 0.9)$	$= 0.348,678,440,100,000,000,$
and $x^{11} (= 0.9^{11} \times 0.9)$	$= 0.313,810,596,090,000,000,$
and $x^{12} (= 0.9^{12} \times 0.9)$	$= 0.282,429,536,481,000,000,$
and $x^{13} (= 0.9^{13} \times 0.9)$	$= 0.254,186,582,832,500,000,$
and $x^{14} (= 0.9^{14} \times 0.9)$	$= 0.228,767,924,549,610,000,$
and $x^{15} (= 0.9^{15} \times 0.9)$	$= 0.205,891,132,094,649,000,$
and $x^{16} (= 0.9^{16} \times 0.9)$	$= 0.185,302,018,885,184,100,$
and $x^{17} (= 0.9^{17} \times 0.9)$	$= 0.166,771,816,996,665,690,$
and $x^{18} (= 0.9^{18} \times 0.9)$	$= 0.150,094,635,296,949,121,$
and $x^{19} (= 0.9^{19} \times 0.9)$	$= 0.135,085,171,767,299,208,$
and $x^{20} (= 0.9^{20} \times 0.9)$	$= 0.121,576,254,590,569,288,$
and $x^{21} (= 0.9^{21} \times 0.9)$	$= 0.109,418,589,131,512,359,$
and $x^{22} (= 0.9^{22} \times 0.9)$	$= 0.098,477,090,218,361,123,$
and $x^{23} (= 0.9^{23} \times 0.9)$	$= 0.088,629,381,196,525,000,$
and $x^{24} (= 0.9^{24} \times 0.9)$	$= 0.079,766,443,076,872,500,$
and $x^{25} (= 0.9^{25} \times 0.9)$	$= 0.071,789,798,769,185,250,$
and $x^{26} (= 0.9^{26} \times 0.9)$	$= 0.064,610,818,892,266,725,$
and $x^{27} (= 0.9^{27} \times 0.9)$	$= 0.057,149,737,003,040,053,$
and $x^{28} (= 0.9^{28} \times 0.9)$	$= 0.051,334,763,302,736,047,$

and

$$\begin{aligned} \text{and } a^{10} &= (\overline{0.9})^{10} \times 0.9 = 0.047,101,286,972,462,443, \\ \text{and } a^{10} &= (\overline{0.1})^9 \times 0.9 = 0.042,391,158,275,216,198, \\ \text{and } a^{11} &= (\overline{0.9})^{10} \times 0.9 = 0.038,152,042,447,694,578, \\ \text{and } a^{11} &= (\overline{0.1})^9 \times 0.9 = 0.034,336,838,202,925,120. \end{aligned}$$

Therefore the first thirty-two terms of the series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5}$
 $\frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ will be as follows:

$$\begin{aligned} 1 &\text{ is } = 1.000,000,000,000,000,000, \\ \frac{x}{2} &\text{ is } (= \frac{0.999,999,999,999,999,999}{2}) = 0.499,999,999,999,999,999, \\ \frac{x^2}{3} &\text{ is } (= \frac{0.999,999,999,999,999,999}{3}) = 0.270,000,000,000,000,000, \\ \frac{x^3}{4} &\text{ is } (= \frac{0.729,999,999,999,999,999}{4}) = 0.182,250,000,000,000,000, \\ \frac{x^4}{5} &\text{ is } (= \frac{0.656,100,000,000,000,000}{5}) = 0.131,220,000,000,000,000, \\ \frac{x^5}{6} &\text{ is } (= \frac{0.590,499,999,999,999,999}{6}) = 0.098,415,000,000,000,000, \\ \frac{x^6}{7} &\text{ is } (= \frac{0.531,441,000,000,000,000}{7}) = 0.075,920,142,857,142,857, \\ \frac{x^7}{8} &\text{ is } (= \frac{0.478,196,999,999,999,999}{8}) = 0.059,787,112,500,000,000, \\ \frac{x^8}{9} &\text{ is } (= \frac{0.410,167,210,000,000,000}{9}) = 0.047,829,690,000,000,000, \\ \frac{x^9}{10} &\text{ is } (= \frac{0.187,420,489,000,000,000}{10}) = 0.038,742,048,900,000,000, \\ \frac{x^{10}}{11} &\text{ is } (= \frac{0.348,678,440,100,000,000}{11}) = 0.031,698,040,009,090,909, \\ \frac{x^{11}}{12} &\text{ is } (= \frac{0.313,805,999,999,999,999}{12}) = 0.026,150,883,007,500,000, \\ \frac{x^{12}}{13} &\text{ is } (= \frac{0.282,429,536,481,000,000}{13}) = 0.021,725,348,960,076,923, \\ \frac{x^{13}}{14} &\text{ is } (= \frac{0.254,186,582,832,000,000}{14}) = 0.018,156,184,488,064,285, \\ \frac{x^{14}}{15} &\text{ is } (= \frac{0.228,767,724,549,610,000}{15}) = 0.015,251,194,969,974,000, \\ \frac{x^{15}}{16} &\text{ is } (= \frac{0.200,871,132,094,649,000}{16}) = 0.012,868,195,755,915,562, \\ \frac{x^{16}}{17} &\text{ is } (= \frac{0.186,302,018,585,184,100}{17}) = 0.010,900,118,757,952,005, \\ \frac{x^{17}}{18} &\text{ is } (= \frac{0.166,771,816,076,665,600}{18}) = 0.009,265,100,944,259,205, \end{aligned}$$

$$\frac{x^{18}}{19}$$

$$\begin{aligned}
\frac{x^{19}}{19} & \text{ is } (= \frac{0.150,794,615,796,999,121}{19}) = 0.007,899,717,647,210,480, \\
\frac{x^{20}}{20} & \text{ is } (= \frac{0.135,086,171,767,299,208}{20}) = 0.006,754,258,588,364,900, \\
\frac{x^{21}}{21} & \text{ is } (= \frac{0.121,576,654,599,569,288}{21}) = 0.005,789,364,504,312,823, \\
\frac{x^{22}}{22} & \text{ is } (= \frac{0.109,418,987,131,612,359}{22}) = 0.004,973,590,415,068,743, \\
\frac{x^{23}}{23} & \text{ is } (= \frac{0.098,477,090,318,161,123}{23}) = 0.004,281,612,618,189,614, \\
\frac{x^{24}}{24} & \text{ is } (= \frac{0.087,619,181,196,525,000}{24}) = 0.003,692,890,883,188,541, \\
\frac{x^{25}}{25} & \text{ is } (= \frac{0.079,766,443,076,871,000}{25}) = 0.003,190,657,723,074,900, \\
\frac{x^{26}}{26} & \text{ is } (= \frac{0.071,837,68,169,186,250}{26}) = 0.002,761,146,106,507,125, \\
\frac{x^{27}}{27} & \text{ is } (= \frac{0.064,108,818,902,266,725}{27}) = 0.002,392,993,292,306,175, \\
\frac{x^{28}}{28} & \text{ is } (= \frac{0.058,149,737,003,049,053}{28}) = 0.002,076,776,321,537,144, \\
\frac{x^{29}}{29} & \text{ is } (= \frac{0.052,134,761,102,736,047}{29}) = 0.001,804,647,010,439,174, \\
\frac{x^{30}}{30} & \text{ is } (= \frac{0.047,101,286,971,462,143}{30}) = 0.001,570,042,899,082,081, \\
\frac{x^{31}}{31} & \text{ is } (= \frac{0.042,391,158,275,216,198}{31}) = 0.001,367,456,718,555,361, \\
\frac{x^{32}}{32} & \text{ is } (= \frac{0.038,152,012,447,604,578}{32}) = 0.001,192,251,326,490,455.
\end{aligned}$$

Art. 19. These thirty-two terms must now be added together into one sum; which may be done as follows:

The first eight terms are as follows, to wit,

$$\begin{aligned}
1 & \text{ is } = 1.000,000,000,000,000,000, \\
\frac{x}{2} & \text{ is } = 0.450,000,000,000,000,000, \\
\frac{x^2}{3} & \text{ is } = 0.270,000,000,000,000,000, \\
\frac{x^3}{4} & \text{ is } = 0.182,250,000,000,000,000, \\
\frac{x^4}{5} & \text{ is } = 0.131,220,000,000,000,000, \\
\frac{x^5}{6} & \text{ is } = 0.098,415,000,000,000,000, \\
\frac{x^6}{7} & \text{ is } = 0.075,920,142,857,142,857, \\
\frac{x^7}{8} & \text{ is } = 0.059,787,112,500,000,000,
\end{aligned}$$

The sum = 2.267,592,255,357,142,857.

The next eight terms are as follows, to wit,

$$\frac{x^8}{9} \text{ is } = 0.047,829,690,000,000,000,$$

$$\frac{x^9}{10} \text{ is } = 0.038,742,048,900,000,000,$$

$$\frac{x^{10}}{11} \text{ is } = 0.031,698,040,009,090,909,$$

$$\frac{x^{11}}{12} \text{ is } = 0.026,150,883,007,500,000,$$

$$\frac{x^{12}}{13} \text{ is } = 0.021,725,348,960,076,923,$$

$$\frac{x^{13}}{14} \text{ is } = 0.018,156,184,488,064,285,$$

$$\frac{x^{14}}{15} \text{ is } = 0.015,251,194,969,974,000,$$

$$\frac{x^{15}}{16} \text{ is } = 0.012,868,195,755,915,562,$$

The sum $= 0.212,421,586,090,621,679.$

The next eight terms are as follows, to wit,

$$\frac{x^{16}}{17} \text{ is } = 0.010,900,118,757,952,005,$$

$$\frac{x^{17}}{18} \text{ is } = 0.009,265,100,944,259,205,$$

$$\frac{x^{18}}{19} \text{ is } = 0.007,899,717,647,210,480,$$

$$\frac{x^{19}}{20} \text{ is } = 0.006,754,258,588,364,960,$$

$$\frac{x^{20}}{21} \text{ is } = 0.005,789,364,504,312,823,$$

$$\frac{x^{21}}{22} \text{ is } = 0.004,973,590,415,068,743,$$

$$\frac{x^{22}}{23} \text{ is } = 0.004,281,612,618,189,614,$$

$$\frac{x^{23}}{24} \text{ is } = 0.003,692,890,883,188,541,$$

The sum $= 0.053,556,654,358,546,371.$

And

And the last eight terms are as follows, to wit,

$$\frac{x^{24}}{25} \text{ is } = 0.003,190,657,723,074,900,$$

$$\frac{x^{25}}{26} \text{ is } = 0.002,761,146,105,507,125,$$

$$\frac{x^{26}}{27} \text{ is } = 0.002,392,993,292,306,175,$$

$$\frac{x^{27}}{28} \text{ is } = 0.002,076,776,321,537,144,$$

$$\frac{x^{28}}{29} \text{ is } = 0.001,804,647,010,439,174,$$

$$\frac{x^{29}}{30} \text{ is } = 0.001,570,042,899,082,081,$$

$$\frac{x^{30}}{31} \text{ is } = 0.001,367,456,718,555,361,$$

$$\frac{x^{31}}{32} \text{ is } = 0.001,192,251,326,490,455,$$

The sum $= 0.016,355,971,397,992,415.$

These four sums total must now be added together, as follows :

$$2.267,592,255,357,142,857,$$

$$0.212,421,586,050,621,679,$$

$$0.053,556,654,358,546,371,$$

$$0.016,355,971,397,992,415,$$

$$2.549,926,467,204,303,322.$$

Therefore the sum of the first thirty-two terms of the series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum*, when x is equal to $\frac{9}{10}$, is $= 2.549,926,467,204,303,322$, or, according to the notation of art. 5, A will be $= 2.549,926,467,204,303,322$.

Art. 20. Since the sum of the first thirty-two terms of the series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum*, is $= 2.549,926,467,204,303,322$, it follows that the sum of the first thirty-two terms of the original series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$ *ad infinitum* will

be = $2.549,926,467,204,303,322 \times 0.9 = 2.294,933,820,483,872,989,8$, which agrees with the true value of that whole series in only the first, or highest, figure 2, that true value being $2.302,585,092,994,045,684,017$, &c, or Napier's logarithm of the ratio of 10 to 1. This shews how slowly the said original series converges, and how difficult it consequently would be to find it's value exact to ten or twelve places of figures in the common way, or by the mere computation and addition of it's terms, and how necessary it therefore is to have recourse to some such ingenious contrivance as this of Mr. Hellins, whenever we have occasion to investigate it's value to so great a degree of exactness.

Art. 21. Having found the sum of the first thirty-two terms of the series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum*, or the value of A, to be = $2.549,926,467,204,303,322$, we must now endeavour to find the value of the remaining part of that series, after the said first thirty-two terms, or the value of the infinite series $\frac{x^{32}}{33} + \frac{x^{33}}{34} + \frac{x^{34}}{35} + \frac{x^{35}}{36} + \frac{x^{36}}{37} + \frac{x^{37}}{38} + \frac{x^{38}}{39} + \frac{x^{39}}{40} + \&c$ *ad infinitum*, which series will, according to the notation of art. 5, be = R. And, when this value is found to the required degree of exactness, we must add it to A, or the number $2.549,926,467,204,303,322$, and the sum A + R will be the value of the whole series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum* to the same degree of exactness, and consequently $A + R \times x$, or $A + R \times \frac{9}{10}$, will be the value of the whole original series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$ *ad infinitum* to the same degree of exactness.

Art. 22. Now this remainder R, or the infinite series $\frac{x^{32}}{33} + \frac{x^{33}}{34} + \frac{x^{34}}{35} + \frac{x^{35}}{36} + \frac{x^{36}}{37} + \frac{x^{37}}{38} + \frac{x^{38}}{39} + \frac{x^{39}}{40} + \&c$, is evidently equal to $x^{32} \times$ the infinite series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$, or to $0.9^{32} \times$ the said series, or to $0.034,336,838,202,925,120 \times$ the said series. We must therefore now endeavour to find the value of this series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$ *ad infinitum*.

Art. 23.

Art. 23. Now this series is equal to the sum of the series $\frac{1}{33} - \frac{x}{34} + \frac{x^2}{35}$
 $- \frac{x^3}{36} + \frac{x^4}{37} - \frac{x^5}{38} + \frac{x^6}{39} - \frac{x^7}{40} + \&c$ *ad infinitum*, and the series $\frac{2x}{34} -$
 $\frac{2x^3}{36} + \frac{2x^5}{38} - \frac{2x^7}{40} + \&c$ *ad infinitum*, or to the series $\frac{1}{33} - \frac{x}{34} + \frac{x^2}{35} - \frac{x^3}{36}$
 $+ \frac{x^4}{37} - \frac{x^5}{38} + \frac{x^6}{39} - \frac{x^7}{40} + \&c$ *ad infinitum* and $2x \times$ the series $\frac{1}{34} + \frac{x^2}{36}$
 $+ \frac{x^4}{38} + \frac{x^6}{40} + \&c$ *ad infinitum*; which last series is of the same form as the
 series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$, (of which we
 are seeking the value,) but converges one degree less slowly.

Art. 24. The former of these two serieses, to wit, $\frac{1}{33} - \frac{x}{34} + \frac{x^2}{35} - \frac{x^3}{36}$
 $+ \frac{x^4}{37} - \frac{x^5}{38} + \frac{x^6}{39} - \frac{x^7}{40} + \&c$ *ad infinitum* may be summed by means of
 the differential series $a = \frac{bx}{1+x} - \frac{D^1 x x}{(1+x)^2} - \frac{D^2 x^2}{(1+x)^3} - \frac{D^3 x^3}{(1+x)^4} - \frac{D^4 x^4}{(1+x)^5}$
 $- \frac{D^5 x^5}{(1+x)^6} - \frac{D^6 x^6}{(1+x)^7} - \&c$ *ad infinitum*, by proceeding in the manner fol-
 lowing :

$$\text{Here } a \text{ is } = \frac{1}{33} = 0.030,303,030,303,030,303,$$

$$b \text{ is } = \frac{1}{34} = 0.029,411,764,705,882,352,$$

$$c \text{ is } = \frac{1}{35} = 0.028,571,428,571,428,571,$$

$$d \text{ is } = \frac{1}{36} = 0.027,777,777,777,777,777,$$

$$e \text{ is } = \frac{1}{37} = 0.027,027,027,027,027,027,$$

$$f \text{ is } = \frac{1}{38} = 0.026,315,789,473,684,210,$$

$$g \text{ is } = \frac{1}{39} = 0.025,641,025,641,025,641,$$

$$h \text{ is } = \frac{1}{40} = 0.025,000,000,000,000,000,$$

$$i \text{ is } = \frac{1}{41} = 0.024,390,243,902,439,024,$$

$$k \text{ is } = \frac{1}{42} = 0.023,809,523,809,523,809,$$

$$l \text{ is } = \frac{1}{43} = 0.023,255,813,953,488,372,$$

$$\text{and } m \text{ is } = \frac{1}{44} = 0.022,727,272,727,272,727,$$

4 M 2

And

$$\text{And consequently } b - c = \left\{ \begin{array}{l} 0.020,411,764,705,882,352 \\ -0.028,571,428,571,428,571 \end{array} \right\} \\ = 0.000,840,336,134,453,781;$$

$$\text{And } c - d = \left\{ \begin{array}{l} 0.028,571,428,571,428,571 \\ -0.027,777,777,777,777,777 \end{array} \right\} \\ = 0.000,793,650,793,650,793;$$

$$\text{And } d - e = \left\{ \begin{array}{l} 0.027,777,777,777,777,777 \\ -0.027,027,027,027,027,027 \end{array} \right\} \\ = 0.000,750,750,750,750,750;$$

$$\text{And } e - f = \left\{ \begin{array}{l} 0.027,027,027,027,027,027 \\ -0.026,315,789,473,684,210 \end{array} \right\} \\ = 0.000,711,237,553,342,817;$$

$$\text{And } f - g = \left\{ \begin{array}{l} 0.026,315,789,473,684,210 \\ -0.025,641,025,641,025,641 \end{array} \right\} \\ = 0.000,674,763,832,658,569;$$

$$\text{And } g - b = \left\{ \begin{array}{l} 0.025,641,025,641,025,641 \\ -0.025,000,000,000,000,000 \end{array} \right\} \\ = 0.000,641,025,641,025,641;$$

$$\text{And } b - i = \left\{ \begin{array}{l} 0.025,000,000,000,000,000 \\ -0.024,390,243,902,439,024 \end{array} \right\} \\ = 0.000,609,756,097,560,976;$$

$$\text{And } i - k = \left\{ \begin{array}{l} 0.024,390,243,902,439,024 \\ -0.023,809,523,809,523,809 \end{array} \right\} \\ = 0.000,580,720,092,915,215;$$

$$\text{And } k - l = \left\{ \begin{array}{l} 0.023,809,523,809,523,809 \\ -0.023,255,813,953,488,372 \end{array} \right\} \\ = 0.000,553,709,856,035,437;$$

$$\text{And } l - m = \left\{ \begin{array}{l} 0.023,255,813,953,488,372 \\ -0.022,727,272,727,272,727 \end{array} \right\} \\ = 0.000,528,541,226,215,645;$$

that is, the first differences of the co-efficients $b, c, d, e, f, g, h, i, k, l$, and m ,

or $\frac{1}{34}, \frac{1}{35}, \frac{1}{36}, \frac{1}{37}, \frac{1}{38}, \frac{1}{39}, \frac{1}{40}, \frac{1}{41}, \frac{1}{42}, \frac{1}{43}$, and $\frac{1}{44}$, are

0.000,

0.000,840,336,134,458,781,
 0.000,793,650,793,650,793,
 0.000,750,750,750,750,750,
 0.000,711,237,553,342,817,
 0.000,674,763,832,658,569,
 0.000,641,025,641,025,641,
 0.000,609,756,097,560,976,
 0.000,580,720,092,915,215,
 0.000,553,709,856,035,437,
 and 0.000,528,541,226,215,645.

Therefore the second differences of the said co-efficients $b, c, d, e, f, g, h, i, k, l$, and m , or the differences of their first differences, will be as follows; to wit,

0.000,046,685,340,802,988,
 0.000,042,900,042,900,043,
 0.000,039,513,197,407,933,
 0.000,036,473,720,684,248,
 0.000,033,738,191,632,928,
 0.000,031,269,543,464,665,
 0.000,029,036,004,645,761,
 0.000,027,010,236,879,778,
 and 0.000,025,168,629,819,792;

And their third differences will be as follows; to wit,

0.000,003,785,297,902,945,
 0.000,003,586,845,492,110,
 0.000,003,039,476,723,685,
 0.000,002,735,529,051,320,
 0.000,002,468,648,168,263,
 0.000,002,233,538,818,904,
 0.000,002,025,767,765,983,
 and 0.000,001,841,607,059,986;

And their fourth differences will be as follows; to wit,

0.000,000,398,452,410,835,
 0.000,000,347,368,768,425,
 0.000,000,303,947,672,365,
 0.000,000,266,880,883,057,
 0.000,000,235,109,349,359,
 0.000,000,207,771,051,921,
 and 0.000,000,184,100,705,997;

And

And their fifth differences will be as follows ; to wit,

0.000,000,051,083,642,410,

0.000,000,043,421,096,060,

0.000,000,037,066,789,308,

0.000,000,031,771,533,698,

0.000,000,027,338,296,438,

and 0.000,000,023,610,346,924 ;

And their sixth differences will be as follows ; to wit,

0.000,000,007,662,546,350,

0.000,000,006,354,306,752,

0.000,000,005,295,255,610,

0.000,000,004,433,237,260,

0.000,000,003,727,949,514 ;

And their seventh differences will be as follows ; to wit,

0.000,000,001,308,239,598,

0.000,000,001,059,051,142,

0.000,000,000,862,018,350,

and 0.000,000,000,705,287,746 ;

And their eighth differences will be as follows ; to wit,

0.000,000,000,249,188,456,

0.000,000,000,197,032,792,

and 0.000,000,000,156,730,604 ;

And their ninth differences will be as follows ; to wit,

0.000,000,000,052,155,664,

0.000,000,000,040,302,188 ;

And their tenth difference will be

0.000,000,000,011,853,476.

Therefore D' , or the first difference of the co-efficients $b, c, d, e, f, g, h, i, k, l$, and m , of the first order, will be $= 0.000,840,336,134,453,781$; and D'' , or the first difference of the same co-efficients, of the second order, will be $= 0.000,046,685,340,802,988$; and D''' , or the first difference of the same co-efficients, of the third order, will be $= 0.000,003,785,297,902,945$; and D''' , or their first difference, of the fourth order, will be $= 0.000,000,398,452,410,835$; and D'''' , or their first difference, of the fifth order, will be $= 0.000,$

0.000,000,051,083,642,410; and D'' , or their first difference, of the sixth order, will be = 0.000,000,007,662,546,350; and D''' , or their first difference, of the seventh order, will be = 0.000,000,001,308,239,598; and D'''' , or their first difference, of the eighth order, will be = 0.000,000,000,249,188,456; and D''''' , or their first difference, of the ninth order, will be = 0.000,000,000,052,155,664; and D'''''' , or their first difference, of the tenth order, will be = 0.000,000,000,011,853,476.

Therefore the differential series $a = \frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^{11}x^4}{1+x^{11}x^4} - \frac{x^{12}x^5}{1+x^{12}x^5} + \frac{x^{13}x^6}{1+x^{13}x^6} - \frac{x^{14}x^7}{1+x^{14}x^7} + \frac{x^{15}x^8}{1+x^{15}x^8} - \frac{x^{16}x^9}{1+x^{16}x^9} + \frac{x^{17}x^{10}}{1+x^{17}x^{10}} - \frac{x^{18}x^{11}}{1+x^{18}x^{11}} -$
&c *ad infinitum* will be =

$$\begin{aligned} & 0.030,303,030,303,030,303 \\ & - 0.029,411,764,705,882,352 \times \frac{x}{1+x} \\ & - 0.000,840,336,134,453,781 \times \frac{x^2}{1+x^2} \\ & - 0.000,046,635,340,802,988 \times \frac{x^3}{1+x^3} \\ & - 0.000,003,785,297,902,945 \times \frac{x^4}{1+x^4} \\ & - 0.000,000,398,452,410,835 \times \frac{x^5}{1+x^5} \\ & - 0.000,000,051,083,642,410 \times \frac{x^6}{1+x^6} \\ & - 0.000,000,007,662,546,350 \times \frac{x^7}{1+x^7} \\ & - 0.000,000,001,308,239,598 \times \frac{x^8}{1+x^8} \\ & - 0.000,000,000,249,188,456 \times \frac{x^9}{1+x^9} \\ & - 0.000,000,000,052,155,664 \times \frac{x^{10}}{1+x^{10}} \\ & - 0.000,000,000,011,853,476 \times \frac{x^{11}}{1+x^{11}} \\ & - \text{\&c ad infinitum.} \end{aligned}$$

But, because x is = $\frac{9}{10}$, we shall have $1+x (= 1 + \frac{9}{10} = \frac{19}{10} + \frac{9}{10} = \frac{10+9}{10}) = \frac{19}{10}$, and consequently $\frac{x}{1+x} (= \frac{\frac{9}{10}}{\frac{19}{10}}) = \frac{9}{19}$.

Therefore

Therefore the foregoing differential series will be equal to

$$\begin{aligned}
 & 0.030,303,030,303,030,303 \\
 & - 0.029,411,764,705,882,352 \times \frac{9}{19} \\
 & - 0.000,840,336,134,453,781 \times \frac{9}{19} \\
 & - 0.000,046,685,340,802,988 \times \frac{9}{19} \\
 & - 0.000,003,785,297,902,945 \times \frac{9}{19} \\
 & - 0.000,000,398,152,410,835 \times \frac{9}{19} \\
 & - 0.000,000,051,083,642,410 \times \frac{9}{19} \\
 & - 0.000,000,007,662,546,350 \times \frac{9}{19} \\
 & - 0.000,000,001,308,239,598 \times \frac{9}{19} \\
 & - 0.000,000,000,249,188,456 \times \frac{9}{19} \\
 & - 0.000,000,000,052,155,664 \times \frac{9}{19} \\
 & - 0.000,000,000,011,853,476 \times \frac{9}{19}
 \end{aligned}$$

— &c *ad infinitum*, and consequently, (by performing the several necessary multiplications into the powers of 9, and divisions by the powers of 19,) equal to

$$\begin{aligned}
 & 0.030,303,030,303,030,303 \\
 & - 0.013,931,888,544,891,640 \\
 & - 0.000,188,551,875,043,646 \\
 & - 0.000,004,961,891,448,517 \\
 & - 0.000,000,190,570,510,824 \\
 & - 0.000,000,009,502,130,733 \\
 & - 0.000,000,000,577,052,473 \\
 & - 0.000,000,000,041,001,096 \\
 & - 0.000,000,000,003,315,878 \\
 & - 0.000,000,000,000,299,176 \\
 & - 0.000,000,000,000,029,661 \\
 & - \&c \text{ ad infinitum} \\
 & = \left\{ \begin{array}{l} 0.030,303,030,303,030,303 \\ - 0.014,125,603,005,523,644 \end{array} \right\}
 \end{aligned}$$

= 0.016,177,427,297,506,659; of which number the first eleven significant figures 0.016,177,427,297, or the first twelve figures, reckoning from the place of units, will be exact.

But the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ *ad infinitum*, or $\frac{x}{33} - \frac{x^2}{34} + \frac{x^3}{35} - \frac{x^4}{36} + \frac{x^5}{37} - \frac{x^6}{38} + \frac{x^7}{39} - \frac{x^8}{40} + \&c$ *ad infinitum*, is equal to the said differential series. Therefore the said series $\frac{1}{33} - \frac{x}{34} + \frac{x^2}{35} - \frac{x^3}{36} + \frac{x^4}{37} - \frac{x^5}{38} + \frac{x^6}{39} - \frac{x^7}{40} + \&c$ *ad infinitum* will also be equal to 0.016,177,427,297,506,659, &c. Q. E. T.

End of the Summation of the Infinite Series $\frac{1}{33} - \frac{x}{34} + \frac{x^2}{35} - \frac{x^3}{36} + \frac{x^4}{37} - \frac{x^5}{38}$
+ &c by means of the above-mentioned Differential Series.

Art. 25. But, by art. 23, the series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38}$
 $+ \frac{x^6}{39} + \frac{x^7}{40} + \&c$ *ad infinitum* is equal to the series $\frac{1}{33} - \frac{x}{34} + \frac{x^2}{35} - \frac{x^3}{36}$
 $+ \frac{x^4}{37} - \frac{x^5}{38} + \frac{x^6}{39} - \frac{x^7}{40} + \&c$ *ad infinitum* together with the product of
 the multiplication of $2x$ into the series $\frac{1}{34} + \frac{x}{36} + \frac{x^2}{38} + \frac{x^3}{40} + \&c$ *ad infinitum*. We must therefore now, in the next place, endeavour to find the value of the said series $\frac{1}{34} + \frac{x}{36} + \frac{x^2}{38} + \frac{x^3}{40} + \&c$ *ad infinitum*.

Art. 26. Now this series, if continued to twelve terms, will be as follows, to wit, $\frac{1}{34} + \frac{x}{36} + \frac{x^2}{38} + \frac{x^3}{40} + \frac{x^4}{42} + \frac{x^5}{44} + \frac{x^6}{46} + \frac{x^7}{48} + \frac{x^8}{50} + \frac{x^9}{52} + \frac{x^{10}}{54}$
 $+ \frac{x^{11}}{56} + \&c$ *ad infinitum*; or, if we substitute the single letter y instead of x in it's terms, (which will be found convenient in the processes we are now going to enter upon,) the said series will be $\frac{1}{34} + \frac{y}{36} + \frac{y^2}{38} + \frac{y^3}{40} + \frac{y^4}{42} + \frac{y^5}{44}$
 $+ \frac{y^6}{46} + \frac{y^7}{48} + \frac{y^8}{50} + \frac{y^9}{52} + \frac{y^{10}}{54} + \frac{y^{11}}{56} + \&c$ *ad infinitum*. This therefore will be the series of which we are now to find the value.

Art. 27. Now the series $\frac{1}{34} + \frac{y}{36} + \frac{y^2}{38} + \frac{y^3}{40} + \frac{y^4}{42} + \frac{y^5}{44} + \frac{y^6}{46} + \frac{y^7}{48}$
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+ $\frac{y^3}{50} + \frac{y^4}{52} + \frac{y^{10}}{54} + \frac{y^{18}}{56} + \&c$ *ad infinitum* is equal to the sum of the two following series, to wit, the series $\frac{1}{34} - \frac{y}{36} + \frac{y^2}{38} - \frac{y^3}{40} + \frac{y^4}{42} - \frac{y^5}{44} + \frac{y^6}{46} - \frac{y^7}{48} + \frac{y^8}{50} - \frac{y^9}{52} + \frac{y^{10}}{54} - \frac{y^{11}}{56} + \&c$ *ad infinitum* and the series $\frac{2y}{36} + \frac{2y^2}{40} + \frac{2y^3}{48} + \frac{2y^4}{52} + \frac{2y^{11}}{56} + \&c$ *ad infinitum*; of which two series the latter is equal to $2y \times$ the series $\frac{1}{36} + \frac{y^2}{40} + \frac{y^4}{44} + \frac{y^6}{48} + \frac{y^8}{52} + \frac{y^{10}}{56} + \&c$ *ad infinitum*. We must therefore now endeavour to find the values of the two infinite series $\frac{1}{34} - \frac{y}{36} + \frac{y^2}{38} - \frac{y^3}{40} + \frac{y^4}{42} - \frac{y^5}{44} + \frac{y^6}{46} - \frac{y^7}{48} + \frac{y^8}{50} - \frac{y^9}{52} + \frac{y^{10}}{54} - \frac{y^{11}}{56} + \&c$ and $\frac{1}{36} + \frac{y^2}{40} + \frac{y^4}{44} + \frac{y^6}{48} + \frac{y^8}{52} + \frac{y^{10}}{56} + \&c$.

The Summation of the Infinite Series $\frac{1}{34} - \frac{y}{36} + \frac{y^2}{38} - \frac{y^3}{40} + \frac{y^4}{42} - \frac{y^5}{44} + \&c$ by means of the Differential Series above-mentioned.

Art. 28. The series $\frac{1}{34} - \frac{y}{36} + \frac{y^2}{38} - \frac{y^3}{40} + \frac{y^4}{42} - \frac{y^5}{44} + \frac{y^6}{46} - \frac{y^7}{48} + \frac{y^8}{50} - \frac{y^9}{52} + \frac{y^{10}}{54} - \frac{y^{11}}{56} + \&c$ *ad infinitum* may be summed by means of the foregoing differential series $a = \frac{bx}{1+x} - \frac{D^2x^2}{1+x^2} - \frac{D^4x^3}{1+x^3} - \frac{D^{11}x^4}{1+x^4} - \frac{D^{12}x^5}{1+x^5} - \frac{D^{13}x^6}{1+x^6} - \frac{D^{14}x^7}{1+x^7} - \frac{D^{15}x^8}{1+x^8} - \frac{D^{16}x^9}{1+x^9} - \frac{D^{17}x^{10}}{1+x^{10}} - \frac{D^{18}x^{11}}{1+x^{11}} - \&c$ *ad infinitum*, by only substituting the letter y instead of the letter x in the terms of the said series, which will then be $a = \frac{by}{1+y} - \frac{D^2y^2}{1+y^2} - \frac{D^4y^3}{1+y^3} - \frac{D^{11}y^4}{1+y^4} - \frac{D^{12}y^5}{1+y^5} - \frac{D^{13}y^6}{1+y^6} - \frac{D^{14}y^7}{1+y^7} - \frac{D^{15}y^8}{1+y^8} - \frac{D^{16}y^9}{1+y^9} - \frac{D^{17}y^{10}}{1+y^{10}} - \frac{D^{18}y^{11}}{1+y^{11}} - \&c$ *ad infinitum*. We must therefore compute this last differential series, which may be done in the manner following:

Art. 29. In this series, it is evident, we shall have

$a =$

$$\begin{aligned}
 a &= \frac{1}{34} = 0.029,411,764,705,882,352, \\
 \text{and } b &= \frac{1}{36} = 0.027,777,777,777,777,777, \\
 \text{and } c &= \frac{1}{38} = 0.026,315,789,473,684,210, \\
 \text{and } d &= \frac{1}{40} = 0.025,000,000,000,000,000, \\
 \text{and } e &= \frac{1}{42} = 0.023,809,523,809,523,809, \\
 \text{and } f &= \frac{1}{44} = 0.022,727,272,727,272,727, \\
 \text{and } g &= \frac{1}{46} = 0.021,739,130,434,782,608, \\
 \text{and } h &= \frac{1}{48} = 0.020,833,333,333,333,333, \\
 \text{and } i &= \frac{1}{50} = 0.020,000,000,000,000,000, \\
 \text{and } k &= \frac{1}{52} = 0.019,230,769,230,769,230, \\
 \text{and } l &= \frac{1}{54} = 0.018,518,518,518,518,518, \\
 \text{and } m &= \frac{1}{56} = 0.017,857,142,857,142,857.
 \end{aligned}$$

$$\begin{aligned}
 \text{Therefore } b - c \text{ will be } &= \left\{ \begin{array}{l} 0.027,777,777,777,777,777 \\ - 0.026,315,789,473,684,210 \end{array} \right\} \\
 &= 0.001,461,988,304,093,567;
 \end{aligned}$$

$$\begin{aligned}
 \text{and } c - d \text{ will be } &= \left\{ \begin{array}{l} 0.026,315,789,473,684,210 \\ - 0.025,000,000,000,000,000 \end{array} \right\} \\
 &= 0.001,315,789,473,684,210;
 \end{aligned}$$

$$\begin{aligned}
 \text{and } d - e \text{ will be } &= \left\{ \begin{array}{l} 0.025,000,000,000,000,000 \\ - 0.023,809,523,809,523,809 \end{array} \right\} \\
 &= 0.001,190,476,190,476,191;
 \end{aligned}$$

$$\begin{aligned}
 \text{and } e - f \text{ will be } &= \left\{ \begin{array}{l} 0.023,809,523,809,523,809 \\ - 0.022,727,272,727,272,727 \end{array} \right\} \\
 &= 0.001,082,251,082,251,082;
 \end{aligned}$$

$$\begin{aligned}
 \text{and } f - g \text{ will be } &= \left\{ \begin{array}{l} 0.022,727,272,727,272,727 \\ - 0.021,739,130,434,782,608 \end{array} \right\} \\
 &= 0.000,988,142,292,490,119;
 \end{aligned}$$

$$\text{and } g - b \text{ will be } = \left\{ \begin{array}{l} 0.021,739,130,434,782,608 \\ - 0.020,833,333,333,333,333 \end{array} \right\} \\ = 0.000,905,797,101,449,275;$$

$$\text{and } b - i \text{ will be } = \left\{ \begin{array}{l} 0.020,833,333,333,333,333 \\ - 0.020,000,000,000,000,000 \end{array} \right\} \\ = 0.000,833,333,333,333,333;$$

$$\text{and } i - k \text{ will be } = \left\{ \begin{array}{l} 0.020,000,000,000,000,000 \\ - 0.019,230,769,230,769,230 \end{array} \right\} \\ = 0.000,769,230,769,230,770;$$

$$\text{and } k - l \text{ will be } = \left\{ \begin{array}{l} 0.019,230,769,230,769,230 \\ - 0.018,518,518,518,518,518 \end{array} \right\} \\ = 0.000,712,250,712,250,712;$$

$$\text{and } l - m \text{ will be } = \left\{ \begin{array}{l} 0.018,518,518,518,518,518 \\ - 0.017,857,142,857,142,857 \end{array} \right\} \\ = 0.000,661,375,661,375,661;$$

that is, the first differences of the co-efficients $b, c, d, e, f, g, h, i, k, l,$ and $m,$

or $\frac{1}{36}, \frac{1}{33}, \frac{1}{40}, \frac{1}{42}, \frac{1}{44}, \frac{1}{46}, \frac{1}{48}, \frac{1}{50}, \frac{1}{52}, \frac{1}{54},$ and $\frac{1}{56},$ will be

$$\begin{aligned} & 0.001,461,988,304,093,567, \\ & 0.001,315,789,473,684,210, \\ & 0.001,190,476,190,476,191, \\ & 0.001,082,251,082,251,082, \\ & 0.000,988,142,292,490,119, \\ & 0.000,905,797,101,449,275, \\ & 0.000,833,333,333,333,333, \\ & 0.000,769,230,769,230,770, \\ & 0.000,712,250,712,250,712, \\ & \text{and } 0.000,661,375,661,375,661. \end{aligned}$$

Therefore the differences of these differences, or the second differences of the said co-efficients, $\frac{1}{36}, \frac{1}{33}, \frac{1}{40}, \frac{1}{42}, \frac{1}{44}, \frac{1}{46}, \frac{1}{48}, \frac{1}{50}, \frac{1}{52}, \frac{1}{54},$ and $\frac{1}{56},$ will be as follows; to wit,

$$\begin{aligned} & 0.000,146,198,830,409,357, \\ & 0.000,125,313,283,208,019, \\ & 0.000,108,225,108,225,109, \\ & 0.000,094,108,789,760,963, \end{aligned}$$

0.000,

$0.000,082,345,191,040,844,$
 $0.000,072,463,768,115,942,$
 $0.000,064,102,564,102,563,$
 $0.000,056,980,056,980,058,$
 and $0.000,050,875,050,875,051;$

And their third differences will be

$0.000,020,885,547,201,338,$
 $0.000,017,088,174,982,910,$
 $0.000,014,116,318,464,146,$
 $0.000,011,763,598,720,119,$
 $0.000,009,881,422,924,902,$
 $0.000,008,361,204,013,379,$
 $0.000,007,122,507,122,505,$
 and $0.000,006,105,006,105,007;$

And their fourth differences will be

$0.000,003,797,372,218,428,$
 $0.000,002,971,856,518,764,$
 $0.000,002,352,719,744,027,$
 $0.000,001,882,175,795,217,$
 $0.000,001,520,218,911,523,$
 $0.000,001,238,696,890,874,$
 and $0.000,001,017,501,017,498;$

And their fifth differences will be

$0.000,000,825,515,699,664,$
 $0.000,000,619,136,774,737,$
 $0.000,000,470,543,948,810,$
 $0.000,000,361,956,883,694,$
 $0.000,000,281,522,020,649,$
 and $0.000,000,221,195,873,276;$

And their sixth differences will be

$0.000,000,206,378,924,927,$
 $0.000,000,148,592,825,927,$
 $0.000,000,108,587,065,116,$
 $0.000,000,080,434,863,045,$
 and $0.000,000,060,326,147,373;$

And

And their seventh differences will be

$$0.000,000,057,786,099,000,$$

$$0.000,000,040,005,760,811,$$

$$0.000,000,028,152,202,071,$$

$$\text{and } 0.000,000,020,108,715,672;$$

And their eighth differences will be

$$0.000,000,017,780,338,189,$$

$$0.000,000,011,853,558,740,$$

$$\text{and } 0.000,000,008,043,486,399;$$

And their ninth differences will be

$$0.000,000,005,926,779,449,$$

$$\text{and } 0.000,000,003,810,072,341;$$

And their tenth difference will be

$$0.000,000,002,116,707,108.$$

Therefore D' , or the first difference of the co-efficients $b, c, d, e, f, g, h, i, k, l$, and m , or $\frac{1}{36}, \frac{1}{38}, \frac{1}{40}, \frac{1}{42}, \frac{1}{44}, \frac{1}{46}, \frac{1}{48}, \frac{1}{50}, \frac{1}{52}, \frac{1}{54}$, and $\frac{1}{56}$, of the first order, will be $= 0.001,461,988,304,093,567$; and D'' , or the first difference of the same co-efficients, of the second order, will be $= 0.000,146,198,830,409,357$; and D''' , or their first difference, of the third order, will be $= 0.000,020,885,547,201,338$; and D^{IV} , or their first difference, of the fourth order, will be $= 0.000,003,797,372,218,428$; and D^V , or their first difference, of the fifth order, will be $= 0.000,000,825,515,699,664$; and D^{VI} , or their first difference, of the sixth order, will be $= 0.000,000,206,378,924,927$; and D^{VII} , or their first difference, of the seventh order, will be $= 0.000,000,057,786,099,000$; and D^{VIII} , or their first difference, of the eighth order, will be $= 0.000,000,017,780,338,189$; and D^X , or their first difference, of the ninth order, will be $= 0.000,000,005,926,779,449$; and D^{10} , or their difference, of the tenth order, will be $= 0.000,000,002,116,707,108$.

Therefore the differential series $a - \frac{by}{1+y} - \frac{D^1y^2}{(1+y)^2} - \frac{D^2y^3}{(1+y)^3} - \frac{D^3y^4}{(1+y)^4} - \frac{D^4y^5}{(1+y)^5} - \frac{D^5y^6}{(1+y)^6} - \frac{D^6y^7}{(1+y)^7} - \frac{D^7y^8}{(1+y)^8} - \frac{D^8y^9}{(1+y)^9} - \frac{D^9y^{10}}{(1+y)^{10}} - \frac{D^{10}y^{11}}{(1+y)^{11}} - \dots$

&c ad infinitum will be =

$$0.029,411,764,705,882,352$$

$$- 0.027,777,777,777,777 \times \frac{y}{1+y}$$

$$- 0.001,461,988,304,093,567 \times \frac{y^2}{(1+y)^2}$$

$$\begin{aligned}
& - 0.000,146,198,830,409,357 \times \frac{y^3}{(1+y)^3} \\
& - 0.000,020,885,547,201,338 \times \frac{y^4}{(1+y)^4} \\
& - 0.000,003,797,372,218,428 \times \frac{y^5}{(1+y)^5} \\
& - 0.000,000,825,515,699,664 \times \frac{y^6}{(1+y)^6} \\
& - 0.000,000,206,378,924,927 \times \frac{y^7}{(1+y)^7} \\
& - 0.000,000,057,786,099,000 \times \frac{y^8}{(1+y)^8} \\
& - 0.000,000,017,780,338,189 \times \frac{y^9}{(1+y)^9} \\
& - 0.000,000,005,926,779,449 \times \frac{y^{10}}{(1+y)^{10}} \\
& - 0.000,000,002,116,707,108 \times \frac{y^{11}}{(1+y)^{11}} \\
& - \&c \text{ ad infinitum.}
\end{aligned}$$

But, because y is $= x = \frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$, $1+y$ will be $= 1 + \frac{81}{100}$
 $= \frac{100}{100} + \frac{81}{100} = \frac{181}{100}$, and consequently $\frac{y}{1+y}$ will be $= \frac{\frac{81}{100}}{\frac{181}{100}} = \frac{81}{181}$.

Therefore the foregoing differential series $a - \frac{by}{1+y} - \frac{b^2y^2}{(1+y)^2} - \frac{b^3y^3}{(1+y)^3} - \frac{b^4y^4}{(1+y)^4} - \frac{b^5y^5}{(1+y)^5} - \frac{b^6y^6}{(1+y)^6} - \frac{b^7y^7}{(1+y)^7} - \frac{b^8y^8}{(1+y)^8} - \frac{b^9y^9}{(1+y)^9} - \frac{b^{10}y^{10}}{(1+y)^{10}} - \frac{b^{11}y^{11}}{(1+y)^{11}} -$
 $\&c \text{ ad infinitum}$ will be $=$

$$\begin{aligned}
& 0.029,411,764,705,882,352 \\
& - 0.027,777,777,777,777 \times \frac{81}{181} \\
& - 0.001,461,988,304,093,567 \times \frac{81^2}{181^2} \\
& - 0.000,146,198,830,409,357 \times \frac{81^3}{181^3} \\
& - 0.000,020,885,547,201,338 \times \frac{81^4}{181^4} \\
& - 0.000,003,797,372,218,428 \times \frac{81^5}{181^5}
\end{aligned}$$

$\frac{D^{11}y^7}{1+y^7} - \frac{D^{11}y^8}{1+y^8} - \frac{D^{11}y^9}{1+y^9} - \frac{D^{11}y^{10}}{1+y^{10}} - \frac{D^{11}y^{11}}{1+y^{11}} - \&c \text{ ad infinitum}$, will also be equal to $0.016,674,011,463,729,39\dots$ Q. E. I.

End of the Summation of the Infinite Series $\frac{1}{34} - \frac{y}{36} + \frac{y^3}{38} - \frac{y^5}{40} + \frac{y^7}{42} - \frac{y^9}{44} + \&c$ by means of the Differential Series above-mentioned.

Art. 30. We must now endeavour to find the value of the infinite series $\frac{1}{36} + \frac{y^2}{40} + \frac{y^4}{44} + \frac{y^6}{48} + \&c$, which, if it be continued to twelve terms, will be $\frac{1}{36} + \frac{y^2}{40} + \frac{y^4}{44} + \frac{y^6}{48} + \frac{y^8}{52} + \frac{y^{10}}{56} + \frac{y^{12}}{60} + \frac{y^{14}}{64} + \frac{y^{16}}{68} + \frac{y^{18}}{72} + \frac{y^{20}}{76} + \frac{y^{22}}{80} + \&c$, or, if we substitute the single letter z in it's terms instead of yy , (which will be found convenient in the following processes,) will be $\frac{1}{36} + \frac{z}{40} + \frac{z^2}{44} + \frac{z^3}{48} + \frac{z^4}{52} + \frac{z^5}{56} + \frac{z^6}{60} + \frac{z^7}{64} + \frac{z^8}{68} + \frac{z^9}{72} + \frac{z^{10}}{76} + \frac{z^{11}}{80} + \&c \text{ ad infinitum}$. We will therefore now endeavour to find the value of this last series:

Art. 31. Now this series $\frac{1}{36} + \frac{z}{40} + \frac{z^2}{44} + \frac{z^3}{48} + \frac{z^4}{52} + \frac{z^5}{56} + \frac{z^6}{60} + \frac{z^7}{64} + \frac{z^8}{68} + \frac{z^9}{72} + \frac{z^{10}}{76} + \frac{z^{11}}{80} + \&c \text{ ad infinitum}$ is equal to the sum of the two following serieses, to wit, $\frac{1}{36} - \frac{z}{40} + \frac{z^2}{44} - \frac{z^3}{48} + \frac{z^4}{52} - \frac{z^5}{56} + \frac{z^6}{60} - \frac{z^7}{64} + \frac{z^8}{68} - \frac{z^9}{72} + \frac{z^{10}}{76} - \frac{z^{11}}{80} + \&c \text{ ad infinitum}$ and $\frac{2z}{40} + \frac{2z^2}{44} + \frac{2z^3}{50} + \frac{2z^4}{54} + \frac{2z^5}{58} + \frac{2z^6}{62} + \frac{2z^7}{66} + \frac{2z^8}{70} + \frac{2z^9}{74} + \frac{2z^{10}}{78} + \frac{2z^{11}}{82} + \&c \text{ ad infinitum}$, and consequently to the series $\frac{1}{36} - \frac{z}{40} + \frac{z^2}{44} - \frac{z^3}{48} + \frac{z^4}{52} - \frac{z^5}{56} + \frac{z^6}{60} - \frac{z^7}{64} + \frac{z^8}{68} - \frac{z^9}{72} + \frac{z^{10}}{76} - \frac{z^{11}}{80} + \&c \text{ ad infinitum}$, together with $2z \times$ the series $\frac{1}{40} + \frac{z}{48} + \frac{z^2}{56} + \frac{z^3}{64} + \frac{z^4}{72} + \frac{z^5}{80} + \&c \text{ ad infinitum}$; of which two serieses the former, to wit, the series $\frac{1}{36} - \frac{z}{40} + \frac{z^2}{44} - \frac{z^3}{48} + \frac{z^4}{52} - \frac{z^5}{56} + \frac{z^6}{60} - \frac{z^7}{64} + \frac{z^8}{68} - \frac{z^9}{72} + \frac{z^{10}}{76} - \frac{z^{11}}{80} + \&c \text{ ad infinitum}$, may be summed by means of the foregoing serieses, and the latter, to wit, the series $\frac{1}{40} + \frac{z}{48} + \frac{z^2}{56} + \frac{z^3}{64} + \frac{z^4}{72} + \frac{z^5}{80} + \&c \text{ ad infinitum}$, converges by the powers of zx , or of y^4 , or of x^4 , and

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consequently much swifter than the series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \frac{x^8}{41} + \frac{x^9}{42} + \frac{x^{10}}{43} + \frac{x^{11}}{44} + \&c$ *ad infinitum*, of which it is the object of all these processes to find the value. Our next operation therefore must be to find the value of the infinite series $\frac{1}{36} - \frac{x}{40} + \frac{x^2}{44} - \frac{x^3}{48} + \frac{x^4}{52} - \frac{x^5}{56} + \frac{x^6}{60} - \frac{x^7}{64} + \frac{x^8}{68} - \frac{x^9}{72} + \frac{x^{10}}{76} - \frac{x^{11}}{80} + \&c$ by means of the foregoing differential series. This may be done in the manner following:

The Summation of the Infinite Series $\frac{1}{36} - \frac{x}{40} + \frac{x^2}{44} - \frac{x^3}{48} + \frac{x^4}{52} - \frac{x^5}{56} + \frac{x^6}{60} - \frac{x^7}{64} + \&c$ by means of the Differential Series above-mentioned.

Art. 32. The differential series to which this series is equal is the series $a - \frac{bx}{1+x} - \frac{D^1 x^2}{1+x)^2} - \frac{D^2 x^3}{1+x)^3} - \frac{D^3 x^4}{1+x)^4} - \frac{D^4 x^5}{1+x)^5} - \frac{D^5 x^6}{1+x)^6} - \frac{D^6 x^7}{1+x)^7} - \frac{D^7 x^8}{1+x)^8} - \frac{D^8 x^9}{1+x)^9} - \frac{D^9 x^{10}}{1+x)^{10}} - \frac{D^{10} x^{11}}{1+x)^{11}} - \&c$ *ad infinitum*, in which a is $= \frac{1}{36}$, and b is $= \frac{1}{40}$, and $c, d, e, f, g, h, i, k, l$, and m , are equal to $\frac{1}{44}, \frac{1}{48}, \frac{1}{52}, \frac{1}{56}, \frac{1}{60}, \frac{1}{64}, \frac{1}{68}, \frac{1}{72}, \frac{1}{76}$, and $\frac{1}{80}$, respectively, and $D^1, D^2, D^3, D^4, D^5, D^6, D^7, D^8, D^9, D^{10}, D^{11}, D^{12}, D^{13}, D^{14}, D^{15}, D^{16}, D^{17}, D^{18}, D^{19}, D^{20}$, are the first differences of the several successive orders of differences of the co-efficients $b, c, d, e, f, g, h, i, k, l$, and m , &c, or $\frac{1}{40}, \frac{1}{44}, \frac{1}{48}, \frac{1}{52}, \frac{1}{56}, \frac{1}{60}, \frac{1}{64}, \frac{1}{68}, \frac{1}{72}, \frac{1}{76}$, and $\frac{1}{80}$, &c.

Here therefore we shall have

$$a = \frac{1}{36} = 0.027,777,777,777,777,777,$$

$$\text{and } b = \frac{1}{40} = 0.025,000,000,000,000,000,$$

$$\text{and } c = \frac{1}{44} = 0.022,727,272,727,272,727,$$

$$\text{and } d = \frac{1}{48} = 0.020,833,333,333,333,333,$$

$$\text{and } e = \frac{1}{52} = 0.019,230,769,230,769,230,$$

$$\text{and } f = \frac{1}{56} = 0.017,857,142,857,142,857,$$

and

$$\text{and } g = \frac{1}{60} = 0.016,666,666,666,666,666,$$

$$\text{and } h = \frac{1}{64} = 0.015,625,000,000,000,000,$$

$$\text{and } i = \frac{1}{68} = 0.014,705,882,352,941,176,$$

$$\text{and } k = \frac{1}{72} = 0.013,888,888,888,888,888,$$

$$\text{and } l = \frac{1}{76} = 0.013,157,894,736,842,105,$$

$$\text{and } m = \frac{1}{80} = 0.012,500,000,000,000,000.$$

$$\begin{aligned} \text{Therefore } b - c \text{ will be} &= \left\{ \begin{array}{l} 0.025,000,000,000,000,000 \\ - 0.022,727,272,727,272,727 \end{array} \right\} \\ &= 0.002,272,727,272,727,273; \end{aligned}$$

$$\begin{aligned} \text{and } c - d \text{ will be} &= \left\{ \begin{array}{l} 0.022,727,272,727,272,727 \\ - 0.020,833,333,333,333,333 \end{array} \right\} \\ &= 0.001,893,939,393,939,394; \end{aligned}$$

$$\begin{aligned} \text{and } d - e \text{ will be} &= \left\{ \begin{array}{l} 0.020,833,333,333,333,333 \\ - 0.019,230,769,230,769,230 \end{array} \right\} \\ &= 0.001,602,564,102,564,103; \end{aligned}$$

$$\begin{aligned} \text{and } e - f \text{ will be} &= \left\{ \begin{array}{l} 0.019,230,769,230,769,230 \\ - 0.017,857,142,857,142,857 \end{array} \right\} \\ &= 0.001,373,626,373,626,373; \end{aligned}$$

$$\begin{aligned} \text{and } f - g \text{ will be} &= \left\{ \begin{array}{l} 0.017,857,142,857,142,857 \\ - 0.016,666,666,666,666,666 \end{array} \right\} \\ &= 0.001,190,476,190,476,191; \end{aligned}$$

$$\begin{aligned} \text{and } g - h \text{ will be} &= \left\{ \begin{array}{l} 0.016,666,666,666,666,666 \\ - 0.015,625,000,000,000,000 \end{array} \right\} \\ &= 0.001,041,666,666,666,666; \end{aligned}$$

$$\begin{aligned} \text{and } h - i \text{ will be} &= \left\{ \begin{array}{l} 0.015,625,000,000,000,000 \\ - 0.014,705,882,352,941,176 \end{array} \right\} \\ &= 0.000,919,117,647,058,824; \end{aligned}$$

$$\begin{aligned} \text{and } i - k \text{ will be} &= \left\{ \begin{array}{l} 0.014,705,882,352,941,176 \\ - 0.013,888,888,888,888,888 \end{array} \right\} \\ &= 0.000,816,993,464,052,288; \end{aligned}$$

$$\begin{aligned} \text{and } k - l \text{ will be} &= \left\{ \begin{array}{l} 0.013,888,888,888,888,888 \\ - 0.013,157,894,736,842,105 \end{array} \right\} \\ &= 0.000,730,994,152,046,783; \end{aligned}$$

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and

$$\text{and } l - m \text{ will be } = \left\{ \begin{array}{l} 0.013, 157, 894, 736, 842, 105 \\ - 0.012, 500, 000, 000, 0.000000 \\ = 0.000, 657, 894, 736, 842, 105; \end{array} \right\}$$

that is, the first differences of the co-efficients $b, c, d, e, f, g, h, i, k, l$, and m ,

or $\frac{1}{40}, \frac{1}{44}, \frac{1}{48}, \frac{1}{52}, \frac{1}{56}, \frac{1}{60}, \frac{1}{64}, \frac{1}{68}, \frac{1}{72}, \frac{1}{76},$ and $\frac{1}{80}$, will be

$$\begin{aligned} &0.002, 272, 727, 272, 727, 273, \\ &0.001, 893, 939, 593, 939, 394, \\ &0.001, 002, 564, 102, 564, 103, \\ &0.001, 373, 624, 373, 626, 373, \\ &0.001, 140, 476, 140, 476, 191, \\ &0.000, 041, 666, 666, 666, 666, \\ &0.000, 919, 117, 547, 058, 824, \\ &0.000 816, 993, 464, 052, 288, \\ &0.000, 730, 994, 152, 046, 783, \\ &\text{and } 0.000, 657, 894, 736, 842, 105. \end{aligned}$$

Therefore the second differences of these co-efficients will be

$$\begin{aligned} &0.000, 378, 787, 878, 787, 879, \\ &0.000, 291, 375, 291, 375, 291, \\ &0.000, 228, 937, 728, 937, 730, \\ &0.000, 183, 150, 183, 150, 182, \\ &0.000, 148, 809, 523, 809, 525, \\ &0.000, 122, 549, 019, 607, 842, \\ &0.000, 102, 124, 183, 006, 536, \\ &0.000, 085, 999, 312, 005, 505, \\ &\text{and } 0.000, 073, 099, 415, 204, 678; \end{aligned}$$

And their third differences will be

$$\begin{aligned} &0.000, 087, 412, 587, 412, 588, \\ &0.000, 062, 437, 562, 437, 561, \\ &0.000, 045, 787, 545, 787, 548, \\ &0.000, 034, 340, 659, 340, 657, \\ &0.000, 026, 260, 504, 201, 683, \\ &0.000, 020, 424, 836, 601, 306, \\ &0.000, 016, 124, 871, 001, 031, \\ &\text{and } 0.000, 012, 899, 896, 800, 827; \end{aligned}$$

And

And their fourth differences will be

0.000,024,975,024,975,027,
 0.000,016,650,016,650,013,
 0.000,011,446,886,446,891,
 0.000,008,080,155,138,974,
 0.000,005,835,667,600,377,
 0.000,004,299,965,600,275,
 and 0.000,003,224,974,200,204 ;

And their fifth differences will be

0.000,008,325,008,325,014,
 0.000,005,203,130,203,122,
 0.000,003,366,731,307,917,
 0.000,002,244,487,538,597,
 0.000,001,535,702,000,102,
 and 0.000,001,074,991,400,071 ;

And their sixth differences will be

0.000,003,121,878,121,892,
 0.000,001,836,398,895,205,
 0.000,001,122,243,769,320,
 0.000,000,708,785,538,495,
 and 0.000,000,460,710,600,031 ;

And their seventh differences will be

0.000,001,285,479,226,687,
 0.000,000,714,155,125,885,
 0.000,000,413,458,230,825,
 and 0.000,000,248,074,938,464 ;

And their eighth differences will be

0.000,000,571,324,100,802,
 0.000,000,300,696,895,060,
 and 0.000,000,165,383,292,361 ;

And their ninth differences will be

0.000,000,270,627,205,742,
 and 0.000,000,135,313,602,699 ;

And their tenth difference will be

0.000,000,135,313,603,043.

Therefore

Therefore D^1 , or the first difference of the said co-efficients b, c, d, e, f, g, h , i, k, l , and m , or $\frac{1}{40}, \frac{1}{44}, \frac{1}{48}, \frac{1}{52}, \frac{1}{56}, \frac{1}{60}, \frac{1}{64}, \frac{1}{68}, \frac{1}{72}, \frac{1}{76}$, and $\frac{1}{80}$, of the first order, will be $= 0.002,272,727,272,727,273$; and D'' , or the first difference of the same co-efficients, of the second order, will be $= 0.000,378,787,878,787,789$; and D''' , or their first difference, of the third order, will be $= 0.000,087,412,587,412,588$; and D^{IV} , or their first difference, of the fourth order, will be $= 0.000,024,975,024,975,027$; and D^V , or their first difference, of the fifth order, will be $= 0.000,008,325,008,325,014$; and D^{VI} , or their first difference, of the sixth order, will be $= 0.000,003,121,878,121,892$; and D^{VII} , or their first difference, of the seventh order, will be $= 0.000,001,285,479,226,687$; and D^{VIII} , or their first difference, of the eighth order, will be $= 0.000,000,571,324,100,802$; and D^X , or their first difference, of the ninth order, will be $= 0.000,000,270,627,205,742$; and D^4 , or their difference, of the tenth order, will be $= 0.000,000,135,313,603,043$.

Therefore the differential series $a = \frac{bx}{1+x} - \frac{D^1 x^2}{(1+x)^2} - \frac{D^2 x^3}{(1+x)^3} - \frac{D^3 x^4}{(1+x)^4} - \frac{D^4 x^5}{(1+x)^5} - \frac{D^5 x^6}{(1+x)^6} - \frac{D^6 x^7}{(1+x)^7} - \frac{D^7 x^8}{(1+x)^8} - \frac{D^8 x^9}{(1+x)^9} - \frac{D^9 x^{10}}{(1+x)^{10}} - \frac{D^{10} x^{11}}{(1+x)^{11}} -$
 &c *ad infinitum* (to which the series $a = bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + iz^8 - kz^9 + lz^{10} - mz^{11} +$ &c *ad infinitum* is equal) will be $=$

0.027,777,777,777,777,777, &c

$$\begin{aligned}
 &- 0.025,000,000,000,000,000 \times \frac{x}{1+x} \\
 &- 0.002,272,727,272,727,273 \times \frac{x^2}{(1+x)^2} \\
 &- 0.000,378,787,878,787,789 \times \frac{x^3}{(1+x)^3} \\
 &- 0.000,087,412,587,412,588 \times \frac{x^4}{(1+x)^4} \\
 &- 0.000,024,975,024,975,027 \times \frac{x^5}{(1+x)^5} \\
 &- 0.000,008,325,008,325,014 \times \frac{x^6}{(1+x)^6} \\
 &- 0.000,003,121,878,121,892 \times \frac{x^7}{(1+x)^7} \\
 &- 0.000,001,285,479,226,687 \times \frac{x^8}{(1+x)^8} \\
 &- 0.000,000,571,324,100,802 \times \frac{x^9}{(1+x)^9}
 \end{aligned}$$

$$\begin{aligned}
& - 0.000,000,270,627,205,742 \times \frac{z^{10}}{1+z}^{10} \\
& - 0.000,000,135,313,603,043 \times \frac{z^{11}}{1+z}^{11} \\
& - \&c \text{ ad infinitum.}
\end{aligned}$$

But, because z is $= y$, and y is $= x$, z will be $= x^4$, or $\left(\frac{9}{10}\right)^4$, or $\frac{9^4}{10^4}$, or $\frac{6561}{10,000}$. And consequently $1+z$ will be $= 1 + \frac{6561}{10,000} (= \frac{10,000}{10,000} + \frac{6561}{10,000})$
 $= \frac{10,000 + 6561}{10,000} = \frac{16,561}{10,000}$, and $\frac{z}{1+z}$ will be $= \frac{\frac{6561}{10,000}}{\frac{16,561}{10,000}} = \frac{6561}{16,561}$.

Therefore the foregoing differential series $a - \frac{bx}{1+x} - \frac{b^2x^2}{(1+x)^2} - \frac{b^3x^3}{(1+x)^3} - \frac{b^4x^4}{(1+x)^4} - \frac{b^5x^5}{(1+x)^5} - \frac{b^6x^6}{(1+x)^6} - \frac{b^7x^7}{(1+x)^7} - \frac{b^8x^8}{(1+x)^8} - \frac{b^9x^9}{(1+x)^9} - \frac{b^{10}x^{10}}{(1+x)^{10}} - \frac{b^{11}x^{11}}{(1+x)^{11}} - \frac{b^{12}x^{12}}{(1+x)^{12}} - \&c \text{ ad infinitum}$ will be $=$

$$\begin{aligned}
& 0.027,777,777,777,777 \\
& - 0.025,000,000,000,000,000 \times \frac{6561}{16,561} \\
& - 0.002,272,727,272,727,273 \times \frac{6561^2}{16,561^2} \\
& - 0.000,378,787,878,787,879 \times \frac{6561^3}{16,561^3} \\
& - 0.000,087,412,587,412,588 \times \frac{6561^4}{16,561^4} \\
& - 0.000,024,975,024,975,027 \times \frac{6561^5}{16,561^5} \\
& - 0.000,008,325,008,325,014 \times \frac{6561^6}{16,561^6} \\
& - 0.000,003,121,878,121,892 \times \frac{6561^7}{16,561^7} \\
& - 0.000,001,285,479,226,687 \times \frac{6561^8}{16,561^8} \\
& - 0.000,000,571,324,100,802 \times \frac{6561^9}{16,561^9} \\
& - \&c \text{ ad infinitum.}
\end{aligned}$$

$$= 0.000,000,270,627,205,742 \times \frac{6561^{10}}{16,561^{10}}$$

$$= 0.000,000,135,313,603,043 \times \frac{6561^{11}}{16,561^{11}}$$

= &c *ad infinitum*

$$= 0.027,777,777,777,777,777$$

$$= 0.009,904,293,219,008,513$$

$$= 0.000,356,709,178,793,083$$

$$= 0.000,023,553,015,337,856$$

$$= 0.000,002,153,316,647,057$$

$$= 0.000,000,243,738,051,037$$

$$= 0.000,000,032,187,375,014$$

$$= 0.000,000,004,781,898,001$$

$$= 0.000,000,000,780,068,799$$

$$= 0.000,000,000,137,351,646$$

$$= 0.000,000,000,025,775,450$$

$$= 0.000,000,000,005,105,751$$

= &c *ad infinitum*

$$= 0.027,777,777,777,777,777$$

$$= 0.010,286,990,385,412,207$$

= 0.017,490,787,392,365,570; of which number, (if no mistake has been made in the calculation,) the first eleven figures, reckoned from the place of units, to wit, the figures 0.017,490,787,39, will be exact.

Therefore the series $\frac{1}{36} - \frac{x}{40} + \frac{x^2}{44} - \frac{x^3}{48} + \frac{x^4}{52} - \frac{x^5}{56} + \frac{x^6}{60} - \frac{x^7}{64} + \frac{x^8}{68} - \frac{x^9}{72} + \frac{x^{10}}{76} - \frac{x^{11}}{80} + \&c$ *ad infinitum*, (which is equal to the said differential series,) will also be equal to 0.017,490,787,392,365,570; of which number (if no mistake has been made in the calculation,) the first eleven figures 0.017,490,787,39, will be exact.

Q. E. D.

End of the Summation of the Infinite Series $\frac{1}{36} - \frac{x}{40} + \frac{x^2}{44} - \frac{x^3}{48} + \frac{x^4}{52} - \frac{x^5}{56} + \frac{x^6}{60} - \frac{x^7}{64} + \&c$ *by means of the Differential Series above-mentioned.*

Art. 33. Having thus found the value of the series $\frac{1}{36} - \frac{x}{40} + \frac{x^3}{44} - \frac{x^5}{48} + \frac{x^7}{52} - \frac{x^9}{56} + \frac{x^{11}}{60} - \frac{x^{13}}{64} + \frac{x^{15}}{68} - \frac{x^{17}}{72} + \frac{x^{19}}{76} - \frac{x^{21}}{80} + \frac{x^{23}}{84} - \frac{x^{25}}{88} + \frac{x^{27}}{92} - \frac{x^{29}}{96} + \frac{x^{31}}{100} - \frac{x^{33}}{104} + \frac{x^{35}}{108} - \frac{x^{37}}{112} + \frac{x^{39}}{116} - \frac{x^{41}}{120} + \frac{x^{43}}{124} - \frac{x^{45}}{128} + \frac{x^{47}}{132} - \frac{x^{49}}{136} + \frac{x^{51}}{140} - \frac{x^{53}}{144} + \frac{x^{55}}{148} - \frac{x^{57}}{152} + \frac{x^{59}}{156} - \frac{x^{61}}{160} + \frac{x^{63}}{164} - \frac{x^{65}}{168} + \frac{x^{67}}{172} - \frac{x^{69}}{176} + \frac{x^{71}}{180} - \frac{x^{73}}{184} + \frac{x^{75}}{188} - \frac{x^{77}}{192} + \frac{x^{79}}{196} - \frac{x^{81}}{200} + \frac{x^{83}}{204} - \frac{x^{85}}{208} + \frac{x^{87}}{212} - \frac{x^{89}}{216} + \frac{x^{91}}{220} - \frac{x^{93}}{224} + \frac{x^{95}}{228} - \frac{x^{97}}{232} + \frac{x^{99}}{236} - \frac{x^{101}}{240} + \frac{x^{103}}{244} - \frac{x^{105}}{248} + \frac{x^{107}}{252} - \frac{x^{109}}{256} + \frac{x^{111}}{260} - \frac{x^{113}}{264} + \frac{x^{115}}{268} - \frac{x^{117}}{272} + \frac{x^{119}}{276} - \frac{x^{121}}{280} + \frac{x^{123}}{284} - \frac{x^{125}}{288} + \frac{x^{127}}{292} - \frac{x^{129}}{296} + \frac{x^{131}}{300} - \frac{x^{133}}{304} + \frac{x^{135}}{308} - \frac{x^{137}}{312} + \frac{x^{139}}{316} - \frac{x^{141}}{320} + \frac{x^{143}}{324} - \frac{x^{145}}{328} + \frac{x^{147}}{332} - \frac{x^{149}}{336} + \frac{x^{151}}{340} - \frac{x^{153}}{344} + \frac{x^{155}}{348} - \frac{x^{157}}{352} + \frac{x^{159}}{356} - \frac{x^{161}}{360} + \frac{x^{163}}{364} - \frac{x^{165}}{368} + \frac{x^{167}}{372} - \frac{x^{169}}{376} + \frac{x^{171}}{380} - \frac{x^{173}}{384} + \frac{x^{175}}{388} - \frac{x^{177}}{392} + \frac{x^{179}}{396} - \frac{x^{181}}{400} + \frac{x^{183}}{404} - \frac{x^{185}}{408} + \frac{x^{187}}{412} - \frac{x^{189}}{416} + \frac{x^{191}}{420} - \frac{x^{193}}{424} + \frac{x^{195}}{428} - \frac{x^{197}}{432} + \frac{x^{199}}{436} - \frac{x^{201}}{440} + \frac{x^{203}}{444} - \frac{x^{205}}{448} + \frac{x^{207}}{452} - \frac{x^{209}}{456} + \frac{x^{211}}{460} - \frac{x^{213}}{464} + \frac{x^{215}}{468} - \frac{x^{217}}{472} + \frac{x^{219}}{476} - \frac{x^{221}}{480} + \frac{x^{223}}{484} - \frac{x^{225}}{488} + \frac{x^{227}}{492} - \frac{x^{229}}{496} + \frac{x^{231}}{500} - \frac{x^{233}}{504} + \frac{x^{235}}{508} - \frac{x^{237}}{512} + \frac{x^{239}}{516} - \frac{x^{241}}{520} + \frac{x^{243}}{524} - \frac{x^{245}}{528} + \frac{x^{247}}{532} - \frac{x^{249}}{536} + \frac{x^{251}}{540} - \frac{x^{253}}{544} + \frac{x^{255}}{548} - \frac{x^{257}}{552} + \frac{x^{259}}{556} - \frac{x^{261}}{560} + \frac{x^{263}}{564} - \frac{x^{265}}{568} + \frac{x^{267}}{572} - \frac{x^{269}}{576} + \frac{x^{271}}{580} - \frac{x^{273}}{584} + \frac{x^{275}}{588} - \frac{x^{277}}{592} + \frac{x^{279}}{596} - \frac{x^{281}}{600} + \frac{x^{283}}{604} - \frac{x^{285}}{608} + \frac{x^{287}}{612} - \frac{x^{289}}{616} + \frac{x^{291}}{620} - \frac{x^{293}}{624} + \frac{x^{295}}{628} - \frac{x^{297}}{632} + \frac{x^{299}}{636} - \frac{x^{301}}{640} + \frac{x^{303}}{644} - \frac{x^{305}}{648} + \frac{x^{307}}{652} - \frac{x^{309}}{656} + \frac{x^{311}}{660} - \frac{x^{313}}{664} + \frac{x^{315}}{668} - \frac{x^{317}}{672} + \frac{x^{319}}{676} - \frac{x^{321}}{680} + \frac{x^{323}}{684} - \frac{x^{325}}{688} + \frac{x^{327}}{692} - \frac{x^{329}}{696} + \frac{x^{331}}{700} - \frac{x^{333}}{704} + \frac{x^{335}}{708} - \frac{x^{337}}{712} + \frac{x^{339}}{716} - \frac{x^{341}}{720} + \frac{x^{343}}{724} - \frac{x^{345}}{728} + \frac{x^{347}}{732} - \frac{x^{349}}{736} + \frac{x^{351}}{740} - \frac{x^{353}}{744} + \frac{x^{355}}{748} - \frac{x^{357}}{752} + \frac{x^{359}}{756} - \frac{x^{361}}{760} + \frac{x^{363}}{764} - \frac{x^{365}}{768} + \frac{x^{367}}{772} - \frac{x^{369}}{776} + \frac{x^{371}}{780} - \frac{x^{373}}{784} + \frac{x^{375}}{788} - \frac{x^{377}}{792} + \frac{x^{379}}{796} - \frac{x^{381}}{800} + \frac{x^{383}}{804} - \frac{x^{385}}{808} + \frac{x^{387}}{812} - \frac{x^{389}}{816} + \frac{x^{391}}{820} - \frac{x^{393}}{824} + \frac{x^{395}}{828} - \frac{x^{397}}{832} + \frac{x^{399}}{836} - \frac{x^{401}}{840} + \frac{x^{403}}{844} - \frac{x^{405}}{848} + \frac{x^{407}}{852} - \frac{x^{409}}{856} + \frac{x^{411}}{860} - \frac{x^{413}}{864} + \frac{x^{415}}{868} - \frac{x^{417}}{872} + \frac{x^{419}}{876} - \frac{x^{421}}{880} + \frac{x^{423}}{884} - \frac{x^{425}}{888} + \frac{x^{427}}{892} - \frac{x^{429}}{896} + \frac{x^{431}}{900} - \frac{x^{433}}{904} + \frac{x^{435}}{908} - \frac{x^{437}}{912} + \frac{x^{439}}{916} - \frac{x^{441}}{920} + \frac{x^{443}}{924} - \frac{x^{445}}{928} + \frac{x^{447}}{932} - \frac{x^{449}}{936} + \frac{x^{451}}{940} - \frac{x^{453}}{944} + \frac{x^{455}}{948} - \frac{x^{457}}{952} + \frac{x^{459}}{956} - \frac{x^{461}}{960} + \frac{x^{463}}{964} - \frac{x^{465}}{968} + \frac{x^{467}}{972} - \frac{x^{469}}{976} + \frac{x^{471}}{980} - \frac{x^{473}}{984} + \frac{x^{475}}{988} - \frac{x^{477}}{992} + \frac{x^{479}}{996} - \frac{x^{481}}{1000} + \frac{x^{483}}{1004} - \frac{x^{485}}{1008} + \frac{x^{487}}{1012} - \frac{x^{489}}{1016} + \frac{x^{491}}{1020} - \frac{x^{493}}{1024} + \frac{x^{495}}{1028} - \frac{x^{497}}{1032} + \frac{x^{499}}{1036} - \frac{x^{501}}{1040} + \frac{x^{503}}{1044} - \frac{x^{505}}{1048} + \frac{x^{507}}{1052} - \frac{x^{509}}{1056} + \frac{x^{511}}{1060} - \frac{x^{513}}{1064} + \frac{x^{515}}{1068} - \frac{x^{517}}{1072} + \frac{x^{519}}{1076} - \frac{x^{521}}{1080} + \frac{x^{523}}{1084} - \frac{x^{525}}{1088} + \frac{x^{527}}{1092} - \frac{x^{529}}{1096} + \frac{x^{531}}{1100} - \frac{x^{533}}{1104} + \frac{x^{535}}{1108} - \frac{x^{537}}{1112} + \frac{x^{539}}{1116} - \frac{x^{541}}{1120} + \frac{x^{543}}{1124} - \frac{x^{545}}{1128} + \frac{x^{547}}{1132} - \frac{x^{549}}{1136} + \frac{x^{551}}{1140} - \frac{x^{553}}{1144} + \frac{x^{555}}{1148} - \frac{x^{557}}{1152} + \frac{x^{559}}{1156} - \frac{x^{561}}{1160} + \frac{x^{563}}{1164} - \frac{x^{565}}{1168} + \frac{x^{567}}{1172} - \frac{x^{569}}{1176} + \frac{x^{571}}{1180} - \frac{x^{573}}{1184} + \frac{x^{575}}{1188} - \frac{x^{577}}{1192} + \frac{x^{579}}{1196} - \frac{x^{581}}{1200} + \frac{x^{583}}{1204} - \frac{x^{585}}{1208} + \frac{x^{587}}{1212} - \frac{x^{589}}{1216} + \frac{x^{591}}{1220} - \frac{x^{593}}{1224} + \frac{x^{595}}{1228} - \frac{x^{597}}{1232} + \frac{x^{599}}{1236} - \frac{x^{601}}{1240} + \frac{x^{603}}{1244} - \frac{x^{605}}{1248} + \frac{x^{607}}{1252} - \frac{x^{609}}{1256} + \frac{x^{611}}{1260} - \frac{x^{613}}{1264} + \frac{x^{615}}{1268} - \frac{x^{617}}{1272} + \frac{x^{619}}{1276} - \frac{x^{621}}{1280} + \frac{x^{623}}{1284} - \frac{x^{625}}{1288} + \frac{x^{627}}{1292} - \frac{x^{629}}{1296} + \frac{x^{631}}{1300} - \frac{x^{633}}{1304} + \frac{x^{635}}{1308} - \frac{x^{637}}{1312} + \frac{x^{639}}{1316} - \frac{x^{641}}{1320} + \frac{x^{643}}{1324} - \frac{x^{645}}{1328} + \frac{x^{647}}{1332} - \frac{x^{649}}{1336} + \frac{x^{651}}{1340} - \frac{x^{653}}{1344} + \frac{x^{655}}{1348} - \frac{x^{657}}{1352} + \frac{x^{659}}{1356} - \frac{x^{661}}{1360} + \frac{x^{663}}{1364} - \frac{x^{665}}{1368} + \frac{x^{667}}{1372} - \frac{x^{669}}{1376} + \frac{x^{671}}{1380} - \frac{x^{673}}{1384} + \frac{x^{675}}{1388} - \frac{x^{677}}{1392} + \frac{x^{679}}{1396} - \frac{x^{681}}{1400} + \frac{x^{683}}{1404} - \frac{x^{685}}{1408} + \frac{x^{687}}{1412} - \frac{x^{689}}{1416} + \frac{x^{691}}{1420} - \frac{x^{693}}{1424} + \frac{x^{695}}{1428} - \frac{x^{697}}{1432} + \frac{x^{699}}{1436} - \frac{x^{701}}{1440} + \frac{x^{703}}{1444} - \frac{x^{705}}{1448} + \frac{x^{707}}{1452} - \frac{x^{709}}{1456} + \frac{x^{711}}{1460} - \frac{x^{713}}{1464} + \frac{x^{715}}{1468} - \frac{x^{717}}{1472} + \frac{x^{719}}{1476} - \frac{x^{721}}{1480} + \frac{x^{723}}{1484} - \frac{x^{725}}{1488} + \frac{x^{727}}{1492} - \frac{x^{729}}{1496} + \frac{x^{731}}{1500} - \frac{x^{733}}{1504} + \frac{x^{735}}{1508} - \frac{x^{737}}{1512} + \frac{x^{739}}{1516} - \frac{x^{741}}{1520} + \frac{x^{743}}{1524} - \frac{x^{745}}{1528} + \frac{x^{747}}{1532} - \frac{x^{749}}{1536} + \frac{x^{751}}{1540} - \frac{x^{753}}{1544} + \frac{x^{755}}{1548} - \frac{x^{757}}{1552} + \frac{x^{759}}{1556} - \frac{x^{761}}{1560} + \frac{x^{763}}{1564} - \frac{x^{765}}{1568} + \frac{x^{767}}{1572} - \frac{x^{769}}{1576} + \frac{x^{771}}{1580} - \frac{x^{773}}{1584} + \frac{x^{775}}{1588} - \frac{x^{777}}{1592} + \frac{x^{779}}{1596} - \frac{x^{781}}{1600} + \frac{x^{783}}{1604} - \frac{x^{785}}{1608} + \frac{x^{787}}{1612} - \frac{x^{789}}{1616} + \frac{x^{791}}{1620} - \frac{x^{793}}{1624} + \frac{x^{795}}{1628} - \frac{x^{797}}{1632} + \frac{x^{799}}{1636} - \frac{x^{801}}{1640} + \frac{x^{803}}{1644} - \frac{x^{805}}{1648} + \frac{x^{807}}{1652} - \frac{x^{809}}{1656} + \frac{x^{811}}{1660} - \frac{x^{813}}{1664} + \frac{x^{815}}{1668} - \frac{x^{817}}{1672} + \frac{x^{819}}{1676} - \frac{x^{821}}{1680} + \frac{x^{823}}{1684} - \frac{x^{825}}{1688} + \frac{x^{827}}{1692} - \frac{x^{829}}{1696} + \frac{x^{831}}{1700} - \frac{x^{833}}{1704} + \frac{x^{835}}{1708} - \frac{x^{837}}{1712} + \frac{x^{839}}{1716} - \frac{x^{841}}{1720} + \frac{x^{843}}{1724} - \frac{x^{845}}{1728} + \frac{x^{847}}{1732} - \frac{x^{849}}{1736} + \frac{x^{851}}{1740} - \frac{x^{853}}{1744} + \frac{x^{855}}{1748} - \frac{x^{857}}{1752} + \frac{x^{859}}{1756} - \frac{x^{861}}{1760} + \frac{x^{863}}{1764} - \frac{x^{865}}{1768} + \frac{x^{867}}{1772} - \frac{x^{869}}{1776} + \frac{x^{871}}{1780} - \frac{x^{873}}{1784} + \frac{x^{875}}{1788} - \frac{x^{877}}{1792} + \frac{x^{879}}{1796} - \frac{x^{881}}{1800} + \frac{x^{883}}{1804} - \frac{x^{885}}{1808} + \frac{x^{887}}{1812} - \frac{x^{889}}{1816} + \frac{x^{891}}{1820} - \frac{x^{893}}{1824} + \frac{x^{895}}{1828} - \frac{x^{897}}{1832} + \frac{x^{899}}{1836} - \frac{x^{901}}{1840} + \frac{x^{903}}{1844} - \frac{x^{905}}{1848} + \frac{x^{907}}{1852} - \frac{x^{909}}{1856} + \frac{x^{911}}{1860} - \frac{x^{913}}{1864} + \frac{x^{915}}{1868} - \frac{x^{917}}{1872} + \frac{x^{919}}{1876} - \frac{x^{921}}{1880} + \frac{x^{923}}{1884} - \frac{x^{925}}{1888} + \frac{x^{927}}{1892} - \frac{x^{929}}{1896} + \frac{x^{931}}{1900} - \frac{x^{933}}{1904} + \frac{x^{935}}{1908} - \frac{x^{937}}{1912} + \frac{x^{939}}{1916} - \frac{x^{941}}{1920} + \frac{x^{943}}{1924} - \frac{x^{945}}{1928} + \frac{x^{947}}{1932} - \frac{x^{949}}{1936} + \frac{x^{951}}{1940} - \frac{x^{953}}{1944} + \frac{x^{955}}{1948} - \frac{x^{957}}{1952} + \frac{x^{959}}{1956} - \frac{x^{961}}{1960} + \frac{x^{963}}{1964} - \frac{x^{965}}{1968} + \frac{x^{967}}{1972} - \frac{x^{969}}{1976} + \frac{x^{971}}{1980} - \frac{x^{973}}{1984} + \frac{x^{975}}{1988} - \frac{x^{977}}{1992} + \frac{x^{979}}{1996} - \frac{x^{981}}{2000} + \frac{x^{983}}{2004} - \frac{x^{985}}{2008} + \frac{x^{987}}{2012} - \frac{x^{989}}{2016} + \frac{x^{991}}{2020} - \frac{x^{993}}{2024} + \frac{x^{995}}{2028} - \frac{x^{997}}{2032} + \frac{x^{999}}{2036} - \frac{x^{1001}}{2040} + \frac{x^{1003}}{2044} - \frac{x^{1005}}{2048} + \frac{x^{1007}}{2052} - \frac{x^{1009}}{2056} + \frac{x^{1011}}{2060} - \frac{x^{1013}}{2064} + \frac{x^{1015}}{2068} - \frac{x^{1017}}{2072} + \frac{x^{1019}}{2076} - \frac{x^{1021}}{2080} + \frac{x^{1023}}{2084} - \frac{x^{1025}}{2088} + \frac{x^{1027}}{2092} - \frac{x^{1029}}{2096} + \frac{x^{1031}}{2100} - \frac{x^{1033}}{2104} + \frac{x^{1035}}{2108} - \frac{x^{1037}}{2112} + \frac{x^{1039}}{2116} - \frac{x^{1041}}{2120} + \frac{x^{1043}}{2124} - \frac{x^{1045}}{2128} + \frac{x^{1047}}{2132} - \frac{x^{1049}}{2136} + \frac{x^{1051}}{2140} - \frac{x^{1053}}{2144} + \frac{x^{1055}}{2148} - \frac{x^{1057}}{2152} + \frac{x^{1059}}{2156} - \frac{x^{1061}}{2160} + \frac{x^{1063}}{2164} - \frac{x^{1065}}{2168} + \frac{x^{1067}}{2172} - \frac{x^{1069}}{2176} + \frac{x^{1071}}{2180} - \frac{x^{1073}}{2184} + \frac{x^{1075}}{2188} - \frac{x^{1077}}{2192} + \frac{x^{1079}}{2196} - \frac{x^{1081}}{2200} + \frac{x^{1083}}{2204} - \frac{x^{1085}}{2208} + \frac{x^{1087}}{2212} - \frac{x^{1089}}{2216} + \frac{x^{1091}}{2220} - \frac{x^{1093}}{2224} + \frac{x^{1095}}{2228} - \frac{x^{1097}}{2232} + \frac{x^{1099}}{2236} - \frac{x^{1101}}{2240} + \frac{x^{1103}}{2244} - \frac{x^{1105}}{2248} + \frac{x^{1107}}{2252} - \frac{x^{1109}}{2256} + \frac{x^{1111}}{2260} - \frac{x^{1113}}{2264} + \frac{x^{1115}}{2268} - \frac{x^{1117}}{2272} + \frac{x^{1119}}{2276} - \frac{x^{1121}}{2280} + \frac{x^{1123}}{2284} - \frac{x^{1125}}{2288} + \frac{x^{1127}}{2292} - \frac{x^{1129}}{2296} + \frac{x^{1131}}{2300} - \frac{x^{1133}}{2304} + \frac{x^{1135}}{2308} - \frac{x^{1137}}{2312} + \frac{x^{1139}}{2316} - \frac{x^{1141}}{2320} + \frac{x^{1143}}{2324} - \frac{x^{1145}}{2328} + \frac{x^{1147}}{2332} - \frac{x^{1149}}{2336} + \frac{x^{1151}}{2340} - \frac{x^{1153}}{2344} + \frac{x^{1155}}{2348} - \frac{x^{1157}}{2352} + \frac{x^{1159}}{2356} - \frac{x^{1161}}{2360} + \frac{x^{1163}}{2364} - \frac{x^{1165}}{2368} + \frac{x^{1167}}{2372} - \frac{x^{1169}}{2376} + \frac{x^{1171}}{2380} - \frac{x^{1173}}{2384} + \frac{x^{1175}}{2388} - \frac{x^{1177}}{2392} + \frac{x^{1179}}{2396} - \frac{x^{1181}}{2400} + \frac{x^{1183}}{2404} - \frac{x^{1185}}{2408} + \frac{x^{1187}}{2412} - \frac{x^{1189}}{2416} + \frac{x^{1191}}{2420} - \frac{x^{1193}}{2424} + \frac{x^{1195}}{2428} - \frac{x^{1197}}{2432} + \frac{x^{1199}}{2436} - \frac{x^{1201}}{2440} + \frac{x^{1203}}{2444} - \frac{x^{1205}}{2448} + \frac{x^{1207}}{2452} - \frac{x^{1209}}{2456} + \frac{x^{1211}}{2460} - \frac{x^{1213}}{2464} + \frac{x^{1215}}{2468} - \frac{x^{1217}}{2472} + \frac{x^{1219}}{2476} - \frac{x^{1221}}{2480} + \frac{x^{1223}}{2484} - \frac{x^{1225}}{2488} + \frac{x^{1227}}{2492} - \frac{x^{1229}}{2496} + \frac{x^{1231}}{2500} - \frac{x^{1233}}{2504} + \frac{x^{1235}}{2508} - \frac{x^{1237}}{2512} + \frac{x^{1239}}{2516} - \frac{x^{1241}}{2520} + \frac{x^{1243}}{2524} - \frac{x^{1245}}{2528} + \frac{x^{1247}}{2532} - \frac{x^{1249}}{2536} + \frac{x^{1251}}{2540} - \frac{x^{1253}}{2544} + \frac{x^{1255}}{2548} - \frac{x^{1257}}{2552} + \frac{x^{1259}}{2556} - \frac{x^{1261}}{2560} + \frac{x^{1263}}{2564} - \frac{x^{1265}}{2568} + \frac{x^{1267}}{2572} - \frac{x^{1269}}{2576} + \frac{x^{1271}}{2580} - \frac{x^{1273}}{2584} + \frac{x^{1275}}{2588} - \frac{x^{1277}}{2592} + \frac{x^{1279}}{2596} - \frac{x^{1281}}{2600} + \frac{x^{1283}}{2604} - \frac{x^{1285}}{2608} + \frac{x^{1287}}{2612} - \frac{x^{1289}}{2616} + \frac{x^{1291}}{2620} - \frac{x^{1293}}{2624} + \frac{x^{1295}}{2628} - \frac{x^{1297}}{2632} + \frac{x^{1299}}{2636} - \frac{x^{1301}}{2640} + \frac{x^{1303}}{2644} - \frac{x^{1305}}{2648} + \frac{x^{1307}}{2652} - \frac{x^{1309}}{2656} + \frac{x^{1311}}{2660} - \frac{x^{1313}}{2664} + \frac{x^{1315}}{2668} - \frac{x^{1317}}{2672} + \frac{x^{1319}}{2676} - \frac{x^{1321}}{2680} + \frac{x^{1323}}{2684} - \frac{x^{1325}}{2688} + \frac{x^{1327}}{2692} - \frac{x^{1329}}{2696} + \frac{x^{1331}}{2700} - \frac{x^{1333}}{2704} + \frac{x^{1335}}{2708} - \frac{x^{1337}}{2712} + \frac{x^{1339}}{2716} - \frac{x^{1341}}{2720} + \frac{x^{1343}}{2724} - \frac{x^{1345}}{2728} + \frac{x^{1347}}{2732} - \frac{x^{1349}}{2736} + \frac{x^{1351}}{2740} - \frac{x^{1353}}{2744} + \frac{x^{1355}}{2748} - \frac{x^{1357}}{2752} + \frac{x^{1359}}{2756} - \frac{x^{1361}}{2760} + \frac{x^{1363}}{2764} - \frac{x^{1365}}{2768} + \frac{x^{1367}}{2772} - \frac{x^{1369}}{2776} + \frac{x^{1371}}{2780} - \frac{x^{1373}}{2784} + \frac{x^{1375}}{2788} - \frac{x^{1377}}{2792} + \frac{x^{1379}}{2796} - \frac{x^{1381}}{2800} + \frac{x^{1383}}{2804} - \frac{x^{1385}}{2808} + \frac{x^{1387}}{2812} - \frac{x^{1389}}{2816} + \frac{x^{1391}}{2820} - \frac{x^{1393}}{2824} + \frac{x^{1395}}{2828} - \frac{x^{1397}}{2832} + \frac{x^{1399}}{2836} - \frac{x^{1401}}{2840} + \frac{x^{1403}}{2844} - \frac{x^{1405}}{2848} + \frac{x^{1407}}{2852} - \frac{x^{1409}}{2856} + \frac{x^{1411}}{2860} - \frac{x^{1413}}{2864} + \frac{x^{1415}}{2868} - \frac{x^{1417}}{2872} + \frac{x^{1419}}{2876} - \frac{x^{1421}}{2880} + \frac{x^{1423}}{2884} - \frac{x^{1425}}{2888} + \frac{x^{1427}}{2892} - \frac{x^{1429}}{2896} + \frac{x^{1431}}{2900} - \frac{x^{1433}}{2904} + \frac{x^{1435}}{2908} - \frac{x^{1437}}{2912} + \frac{x^{1439}}{2916} - \frac{x^{1441}}{2920} + \frac{x^{1443}}{2924} - \frac{x^{1445}}{2928} + \frac{x^{1447}}{2932} - \frac{x^{1449}}{2936} + \frac{x^{1451}}{2940} - \frac{x^{1453}}{2944} + \frac{x^{1455}}{2948} - \frac{x^{1457}}{2952} + \frac{x^{1459}}{2956} - \frac{x^{1461}}{2960} + \frac{x^{1463}}{2964} - \frac{x^{1465}}{2968} + \frac{x^{1467}}{2972} - \frac{x^{1469}}{2976} + \frac{x^{1471}}{2980} - \frac{x^{1473}}{2984} + \frac{x^{1475}}{2988} - \frac{x^{1477}}{2992} + \frac{x^{1479}}{2996} - \frac{x^{1481}}{3000} + \frac{x^{1483}}{3004} - \frac{x^{1485}}{3008} + \frac{x^{1487}}{3012} - \frac{x^{1489}}{3016} + \frac{x^{1491}}{3020} - \frac{x^{1493}}{3024} + \frac{x^{1495}}{3028} - \frac{x^{1497}}{3032} + \frac{x^{1499}}{3036} - \frac{x^{1501}}{3040} + \frac{x^{1503}}{3044} - \frac{x^{1505}}{3048} + \frac{x^{1507}}{3052} - \frac{x^{1509}}{3056} + \frac{x^{1511}}{3060} - \frac{x^{1513}}{3064} + \frac{x^{1515}}{3068} - \frac{x^{1517}}{3072} + \frac{x^{1519}}{3076} - \frac{x^{1521}}{3080} + \frac{x^{1523}}{3084} - \frac{x^{1525}}{3088} + \frac{x^{1527}}{3092} - \frac{x^{1529}}{3096} + \frac{x^{1531}}{3100} - \frac{x^{1533}}{3104} + \frac{x^{1535}}{3108} - \frac{x^{1537}}{3112} + \frac{x^{1539}}{3116} - \frac{x^{1541}}{3120} + \frac{x^{1543}}{3124} - \frac{x^{1545}}{3128} + \frac{x^{1547}}{3132} - \frac{x^{1549}}{3136} + \frac{x^{1551}}{3140} - \frac{x^{1553}}{3144} + \frac{x^{1555}}{3148} - \frac{x^{1557}}{3152} + \frac{x^{1559}}{3156} - \frac{x^{1561}}{3160} + \frac{x^{1563}}{3164} - \frac{x^{1565}}{3168} + \frac{x^{1567}}{3172} - \frac{x^{1569}}{3176} + \frac{x^{1571}}{3180} - \frac{x^{1573}}{3184} + \frac{x^{1575}}{3188} - \frac{x^{1577}}{3192} + \frac{x^{1579}}{3196} - \frac{x^{1581}}{3200} + \frac{x^{1583}}{3204} - \frac{x^{1585}}{3208} + \frac{x^{1587}}{3212} - \frac{x^{1589}}{3216} + \frac{x^{1591}}{3220} - \frac{x^{1593}}{3224} + \frac{x^{1595}}{3228} - \frac{x^{1597}}{3232} + \frac{x^{1599}}{3236} - \frac{x^{1601}}{3240} + \frac{x^{1603}}{3244} - \frac{x^{1605}}{3248} + \frac{x^{1607}}{3252} - \frac{x^{1609}}{3256} + \frac{x^{1611}}{3260} - \frac{x^{1613}}{3264} + \frac{x^{1615}}{3268} - \frac{x^{1617}}{3272} + \frac{x^{1619}}{3276} - \frac{x^{1621}}{3280} + \frac{x^{1623}}{3284} - \frac{x^{1625}}{3288} + \frac{x^{1627}}{3292} - \frac{x^{1629}}{3296} + \frac{x^{1631}}{3300} - \frac{x^{1633}}{3304} + \frac{x$

$\frac{v^9}{112} + \frac{v^{10}}{120} - \frac{v^{11}}{128} + \&c$ by means of the foregoing differential series. This may be done in the manner following :

*The Summation of the Infinite Series $\frac{1}{40} - \frac{v}{48} + \frac{v^2}{56} - \frac{v^3}{64} + \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88}$
 $- \frac{v^7}{96} + \&c$ by means of the Differential Series above-mentioned.*

Art. 34. The differential series to which the series $\frac{1}{40} - \frac{v}{48} + \frac{v^2}{56} - \frac{v^3}{64}$
 $+ \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88} - \frac{v^7}{96} + \frac{v^8}{104} - \frac{v^9}{112} + \frac{v^{10}}{120} - \frac{v^{11}}{128} + \&c$ *ad infinitum* is
 equal, is the series $a - \frac{bv}{1+v} - \frac{D^1 v^2}{1+v)^2} - \frac{D^2 v^3}{1+v)^3} - \frac{D^3 v^4}{1+v)^4} - \frac{D^4 v^5}{1+v)^5}$
 $- \frac{D^5 v^6}{1+v)^6} - \frac{D^6 v^7}{1+v)^7} - \frac{D^7 v^8}{1+v)^8} - \frac{D^8 v^9}{1+v)^9} - \frac{D^9 v^{10}}{1+v)^{10}} - \frac{D^{10} v^{11}}{1+v)^{11}} - \&c$ *ad in-*
finitum, in which a is $= \frac{1}{40}$, and b is $= \frac{1}{48}$, and $D^1, D^2, D^3, D^4, D^5, D^6, D^7, D^8,$
 $D^9, D^{10}, D^{11}, D^{12}, \&c$ are the first differences of the several successive orders
 of differences of the co-efficients $b, c, d, e, f, g, h, i, k, l, m, \&c$, or $\frac{1}{48},$
 $\frac{1}{56}, \frac{1}{64}, \frac{1}{72}, \frac{1}{80}, \frac{1}{88}, \frac{1}{96}, \frac{1}{104}, \frac{1}{112}, \frac{1}{120}, \frac{1}{128}, \&c$.

Here therefore we shall have

$$\begin{aligned} a &= \frac{1}{40} = 0.025,000,000,000,000,000, \\ \text{and } b &= \frac{1}{48} = 0.020,833,333,333,333,333, \\ \text{and } c &= \frac{1}{56} = 0.017,857,142,857,142,857, \\ \text{and } d &= \frac{1}{64} = 0.015,625,000,000,000,000, \\ \text{and } e &= \frac{1}{72} = 0.013,888,888,888,888,888, \\ \text{and } f &= \frac{1}{80} = 0.012,500,000,000,000,000, \\ \text{and } g &= \frac{1}{88} = 0.011,363,636,363,636,363, \\ \text{and } h &= \frac{1}{96} = 0.010,416,666,666,666,666, \end{aligned}$$

and

$$\text{and } i = \frac{1}{104} = 0.009,615,384,615,384,615,$$

$$\text{and } k = \frac{1}{112} = 0.008,928,571,428,571,428,$$

$$\text{and } l = \frac{1}{120} = 0.008,333,333,333,333,333,$$

$$\text{and } m = \frac{1}{128} = 0.007,812,500,000,000,000.$$

$$\begin{aligned} \text{Therefore } b - c \text{ will be} &= \left\{ \begin{array}{l} 0.020,833,333,333,333,333 \\ - 0.017,857,142,857,142,857 \end{array} \right\} \\ &= 0.002,976,190,476,190,476; \end{aligned}$$

$$\begin{aligned} \text{and } c - d \text{ will be} &= \left\{ \begin{array}{l} 0.017,857,142,857,142,857 \\ - 0.015,625,000,000,000,000 \end{array} \right\} \\ &= 0.002,232,142,857,142,857; \end{aligned}$$

$$\begin{aligned} \text{and } d - e \text{ will be} &= \left\{ \begin{array}{l} 0.015,625,000,000,000,000 \\ - 0.013,888,888,888,888,888 \end{array} \right\} \\ &= 0.001,736,111,111,111,112; \end{aligned}$$

$$\begin{aligned} \text{and } e - f \text{ will be} &= \left\{ \begin{array}{l} 0.013,888,888,888,888,888 \\ - 0.012,500,000,000,000,000 \end{array} \right\} \\ &= 0.001,388,888,888,888,888; \end{aligned}$$

$$\begin{aligned} \text{and } f - g \text{ will be} &= \left\{ \begin{array}{l} 0.012,500,000,000,000,000 \\ - 0.011,363,636,363,636,363 \end{array} \right\} \\ &= 0.001,136,363,636,363,637; \end{aligned}$$

$$\begin{aligned} \text{and } g - b \text{ will be} &= \left\{ \begin{array}{l} 0.011,363,636,363,636,363 \\ - 0.010,416,666,666,666,666 \end{array} \right\} \\ &= 0.000,946,969,696,969,697; \end{aligned}$$

$$\begin{aligned} \text{and } b - i \text{ will be} &= \left\{ \begin{array}{l} 0.010,416,666,666,666,666 \\ - 0.009,615,384,615,384,615 \end{array} \right\} \\ &= 0.000,801,282,051,282,051; \end{aligned}$$

$$\begin{aligned} \text{and } i - k \text{ will be} &= \left\{ \begin{array}{l} 0.009,615,384,615,384,615 \\ - 0.008,928,571,428,571,428 \end{array} \right\} \\ &= 0.000,686,813,186,813,187; \end{aligned}$$

$$\begin{aligned} \text{and } k - l \text{ will be} &= \left\{ \begin{array}{l} 0.008,928,571,428,571,428 \\ - 0.008,333,333,333,333,333 \end{array} \right\} \\ &= 0.000,595,238,095,238,095; \end{aligned}$$

$$\begin{aligned} \text{and } l - m \text{ will be} &= \left\{ \begin{array}{l} 0.008,333,333,333,333,333 \\ - 0.007,812,500,000,000,000 \end{array} \right\} \\ &= 0.000,520,833,333,333,333; \end{aligned}$$

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that

that is, the first differences of the co-efficients $b, c, d, e, f, g, h, i, k, l$, and m ,

or $\frac{1}{48}, \frac{1}{56}, \frac{1}{64}, \frac{1}{72}, \frac{1}{80}, \frac{1}{88}, \frac{1}{96}, \frac{1}{104}, \frac{1}{112}, \frac{1}{120}$, and $\frac{1}{128}$, will be

0.002,976,190,476,190,476,
 0.002,232,142,857,142,857,
 0.001,736,111,111,111,112,
 0.001,383,888,888,888,888,
 0.001,136,363,636,363,637,
 0.000,946,969,696,969,697,
 0.000,801,282,051,282,051,
 0.000,686,813,186,813,187,
 0.000,595,238,095,238,095,
 and 0.000,520,833,333,333,333.

Therefore the second differences of these co-efficients will be

0.000,744,047,619,047,619,
 0.000,496,031,746,031,745,
 0.000,347,222,222,222,224,
 0.000,252,525,252,525,251,
 0.000,189,393,939,393,940,
 0.000,145,687,645,687,646,
 0.000,114,468,864,468,864,
 0.000,091,575,091,575,092,
 and 0.000,074,104,761,904,762;

And their third differences will be

0.000,248,015,873,015,874,
 0.000,148,809,523,809,521,
 0.000,094,696,969,696,973,
 0.000,003,131,313,131,311,
 0.000,043,706,293,706,294,
 0.000,031,218,781,218,782,
 0.000,022,893,772,893,772,
 and 0.000,017,170,329,670,330;

And their fourth differences will be

0.000,099,206,349,206,352,
 0.000,054,112,554,112,548,

0.000,

$0.000,031,565,656,565,662,$
 $0.000,019,425,019,425,017,$
 $0.000,012,487,512,487,512,$
 $0.000,008,325,008,325,010,$
 and $0.000,005,723,443,223,442 ;$

And their fifth differences will be

$0.000,045,093,795,093,805,$
 $0.000,022,546,897,546,886,$
 $0.000,012,140,637,140,645,$
 $0.000,006,937,506,937,505,$
 $0.000,004,162,504,162,502,$
 and $0.000,002,601,565,101,568 ;$

And their sixth differences will be

$0.000,022,546,897,546,919,$
 $0.000,010,406,260,406,241,$
 $0.000,005,203,130,203,140,$
 $0.000,002,775,002,775,003,$
 and $0.000,001,560,939,060,934 ;$

And their seventh differences will be

$0.000,012,140,637,140,678,$
 $0.000,005,203,130,203,101,$
 $0.000,002,428,127,428,137,$
 and $0.000,001,214,063,714,069 ;$

And their eighth differences will be

$0.000,006,937,506,937,577,$
 $0.000,002,775,002,774,964,$
 and $0.000,001,214,063,714,068 ;$

And their ninth differences will be

$0.000,004,162,504,162,613,$
 and $0.000,001,560,939,060,896 ;$

And their tenth difference will be

$0.000,002,601,565,101,717.$

Therefore

Therefore D' , or the first difference of the said co-efficients $b, c, d, e, f, g, h, i, k, l, m$, &c, or $\frac{1}{43}, \frac{1}{56}, \frac{1}{64}, \frac{1}{72}, \frac{1}{80}, \frac{1}{88}, \frac{1}{96}, \frac{1}{104}, \frac{1}{112}, \frac{1}{120}, \frac{1}{128}$, &c, of the first order, will be $= 0.002,976,190,476,190,476$; and D'' , or the first difference of the same co-efficients, of the second order, will be $= 0.000,744,047,619,047,619$; and D''' , or the first difference of the same co-efficients, of the third order, will be $= 0.000,248,015,873,015,874$; and D^{IV} , or their first difference of the fourth order, will be $= 0.000,099,206,349,206,353$; and D^V , or their first difference of the fifth order, will be $= 0.000,045,093,795,093,805$; and D^VI , or their first difference of the sixth order, will be $= 0.000,022,546,897,546,919$; and D^{VII} , or their first difference of the seventh order, will be $= 0.000,012,140,637,140,678$; and D^{VIII} , or their first difference of the eighth order, will be $= 0.000,006,937,506,937,577$; and D^IX , or their first difference of the ninth order, will be $= 0.000,004,162,504,162,613$; and D^{X} , or their difference of the tenth order, will be $= 0.000,002,601,565,101,717$.

Therefore the differential series $a = \frac{bv}{1+v} - \frac{D^1 v^2}{1+v)^2} - \frac{D^2 v^3}{1+v)^3} - \frac{D^3 v^4}{1+v)^4} - \frac{D^4 v^5}{1+v)^5} - \frac{D^5 v^6}{1+v)^6} - \frac{D^6 v^7}{1+v)^7} - \frac{D^7 v^8}{1+v)^8} - \frac{D^8 v^9}{1+v)^9} - \frac{D^9 v^{10}}{1+v)^{10}} - \frac{D^{10} v^{11}}{1+v)^{11}} - \text{&c ad infinitum}$ (to which the series $a = bv + cv^2 - dv^3 + ev^4 - fv^5 + gv^6 - hv^7 + iv^8 - kv^9 + lv^{10} - mv^{11} + \text{&c ad infinitum}$, or $\frac{1}{40} - \frac{v}{48} + \frac{v^2}{56} - \frac{v^3}{64} + \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88} - \frac{v^7}{96} + \frac{v^8}{104} - \frac{v^9}{112} + \frac{v^{10}}{120} - \frac{v^{11}}{128} + \text{&c ad infinitum}$, is equal) will be $=$

$$\begin{aligned}
 & 0.025,000,000,000,000,000 \\
 & - 0.020,833,333,333,333,333 \times \frac{v}{1+v} \\
 & - 0.002,976,190,476,190,476 \times \frac{v^2}{(1+v)^2} \\
 & - 0.000,744,047,619,047,619 \times \frac{v^3}{(1+v)^3} \\
 & - 0.000,248,015,873,015,874 \times \frac{v^4}{(1+v)^4} \\
 & - 0.000,099,206,349,206,353 \times \frac{v^5}{(1+v)^5} \\
 & - 0.000,045,093,795,093,805 \times \frac{v^6}{(1+v)^6} \\
 & - 0.000,022,546,897,546,919 \times \frac{v^7}{(1+v)^7} \\
 & - 0.000,012,140,637,140,678 \times \frac{v^8}{(1+v)^8}
 \end{aligned}$$

$$\begin{aligned}
& - 0.000,006,937,506,937,577 \times \frac{v^9}{1+v} \\
& - 0.000,004,162,504,162,613 \times \frac{v^{10}}{1+v} \\
& - 0.000,002,601,565,101,717 \times \frac{v^{11}}{1+v} \\
& - \&c \text{ ad infinitum.}
\end{aligned}$$

But, because v is $= \sqrt[9]{\frac{9}{10}}$, $v^4 = \sqrt[9]{\frac{81}{1000}}$, $v^8 = \sqrt[9]{\frac{6561}{1000000}}$, $v^9 = \frac{9}{10}$, we shall have $1+v = 1 + \frac{43,046,721}{100,000,000} = \frac{100,000,000}{100,000,000} + \frac{43,046,721}{100,000,000}$

$= \frac{143,046,721}{100,000,000}$, and consequently $\frac{v}{1+v} = \frac{\frac{43,046,721}{100,000,000}}{\frac{143,046,721}{100,000,000}} = \frac{43,046,721}{143,046,721}$.

Therefore the foregoing differential series $a - \frac{bv}{1+v} - \frac{b^2v^2}{1+v^2} - \frac{b^3v^3}{1+v^3} - \frac{b^4v^4}{1+v^4} - \frac{b^5v^5}{1+v^5} - \frac{b^6v^6}{1+v^6} - \frac{b^7v^7}{1+v^7} - \frac{b^8v^8}{1+v^8} - \frac{b^9v^9}{1+v^9} - \frac{b^{10}v^{10}}{1+v^{10}} - \frac{b^{11}v^{11}}{1+v^{11}} - \&c \text{ ad infinitum}$ (to which the series $a - bv + cv^2 - dv^3 + ev^4 - fv^5 + gv^6 - hv^7 + iv^8 - kv^9 + lv^{10} - mv^{11} + \&c \text{ ad infinitum}$, or $\frac{x}{40} - \frac{v}{48} + \frac{v^2}{56} - \frac{v^3}{64} + \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88} - \frac{v^7}{96} + \frac{v^8}{104} - \frac{v^9}{112} + \frac{v^{10}}{120} - \frac{v^{11}}{128} + \&c \text{ ad infinitum}$, is equal,) will be =

$$\begin{aligned}
& 0.025,000,000,000,000,000 \\
& - 0.020,833,333,333,333,333 \times \frac{43,046,721}{143,046,721} \\
& - 0.002,976,190,476,190,476 \times \frac{43,046,721^2}{143,046,721^2} \\
& - 0.000,744,047,619,047,619 \times \frac{43,046,721^3}{143,046,721^3} \\
& - 0.000,248,015,873,015,874 \times \frac{43,046,721^4}{143,046,721^4} \\
& - 0.000,099,206,349,206,353 \times \frac{43,046,721^5}{143,046,721^5} \\
& - 0.000,045,093,795,093,805 \times \frac{43,046,721^6}{143,046,721^6} \\
& - 0.000,022,546,897,546,919 \times \frac{43,046,721^7}{143,046,721^7}
\end{aligned}$$

$$\begin{aligned}
& - 0.000,012,140,637,140,678 \times \frac{43,046,721^8}{143,046,721^8} \\
& - 0.000,006,937,506,937,577 \times \frac{43,046,721^9}{143,046,721^9} \\
& - 0.000,004,162,504,162,613 \times \frac{43,046,721^{10}}{143,046,721^{10}} \\
& - 0.000,002,601,565,101,717 \times \frac{43,046,721^{11}}{143,046,721^{11}} \\
& - \&c \text{ ad infinitum}
\end{aligned}$$

$$\begin{aligned}
& = 0.025,000,000,000,000,000 \\
& - 0.006,269,327,120,752,386 \\
& - 0.000,269,516,314,608,009 \\
& - 0.000,020,276,231,287,886 \\
& - 0.000,002,033,893,227,518 \\
& - 0.000,000,244,821,925,861 \\
& - 0.000,000,033,488,045,226 \\
& - 0.000,000,005,038,740,243 \\
& - 0.000,000,000,816,467,357 \\
& - 0.000,000,000,140,398,654 \\
& - 0.000,000,000,025,349,906 \\
& - 0.000,000,000,004,767,805 \\
& - \&c \text{ ad infinitum} \\
& = 0.025,000,000,000,000,000 \\
& - 0.006,561,437,895,570,851 \\
& - \&c \text{ ad infinitum}
\end{aligned}$$

$= 0.018,438,562,104,429,149$; of which number, (if no mistake has been made in the calculation,) the first eleven figures, reckoned from the place of units, to wit, the figures $0.018,438,562,10$, must be exact.

Therefore the series $\frac{1}{40} - \frac{v}{48} + \frac{v^2}{56} - \frac{v^3}{64} + \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88} - \frac{v^7}{96} + \frac{v^8}{104} - \frac{v^9}{112} + \frac{v^{10}}{120} - \frac{v^{11}}{128} + \&c \text{ ad infinitum}$ (which is equal to the said differential series,) will also be equal to $0.018,438,562,104,429,149$. Q. E. D.

End of the Summation of the Infinite Series $\frac{1}{40} - \frac{v}{48} + \frac{v^2}{56} - \frac{v^3}{64} + \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88} - \frac{v^7}{96} + \&c$ by means of the above-mentioned Differential Series.

Art. 35.

Art. 35. Having thus found the value of the series $\frac{1}{40} - \frac{v}{48} + \frac{v^2}{50} - \frac{v^3}{64} + \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88} - \frac{v^7}{96} + \frac{v^8}{104} - \frac{v^9}{112} + \frac{v^{10}}{120} - \frac{v^{11}}{128} + \&c$, we must now endeavour to find the value of the series $\frac{1}{48} + \frac{v^2}{64} + \frac{v^4}{80} + \frac{v^6}{96} + \frac{v^8}{112} + \frac{v^{10}}{128} + \&c$ *ad infinitum*.

Now this series converges with a sufficient degree of swiftness to make it capable of being summed in the common way, or by merely computing a moderate number of it's terms and adding them together. For, since v is $= x^4$, vv will be $= x^8$, which we have found above (in art. 18) to be $= 0.185,302,018,885,184,100$; that is, the quantity vv , (by the powers of which the said series converges,) will be $= 0.185,302,018,885,184,100$; which is less than 0.20 , or than $\frac{20}{100}$, or $\frac{2}{10}$, or $\frac{1}{5}$. Therefore every term in the said series must be less than the 5th part of the term immediately preceeding it, and than the 25th part of the last term but one, and than the 125th part of the last term but two; and consequently every three new terms of the series will give us the value of the whole series exact to at least two new places of decimal figures, and therefore eighteen terms of it will give us the value of the whole exact to at least twelve places of figures. We will therefore now proceed to compute the first twenty terms of the said series.

A Computation of the Value of the Sum of the first twenty Terms of the Series

$$\frac{1}{48} + \frac{v^2}{64} + \frac{v^4}{80} + \frac{v^6}{96} + \frac{v^8}{112} + \frac{v^{10}}{128} + \&c \text{ ad infinitum.}$$

Art. 36. Since vv is $= 0.185,302,018,885,184,100$, we shall have

$$\begin{aligned} v^4 & (= 0.185,302,018,885,184,100^2) = 0.034,336,838,202,925,120; \\ \text{and } v^6 & (= v^4 \times v^2 = 0.034,336,838,202,925,120 \times 0.185,302,018,885,184,100) = 0.006,362,685,441,135,942; \\ \text{and } v^8 & (= v^6 \times v^2 = 0.006,362,685,441,135,942 \times 0.185,302,018,885,184,100) = 0.001,179,018,457,773,858; \\ \text{and } v^{10} & (= v^8 \times v^2 = 0.001,179,018,457,773,858 \times 0.185,302,018,885,184,100) = 0.000,218,474,500,528,392; \\ \text{and } v^{12} & (= v^{10} \times v^2 = 0.000,218,474,500,528,392 \times 0.185,302,018,885,184,100) = 0.000,040,483,766,022,843; \end{aligned}$$

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and

$$\text{and } v^{14} (= v^{13} \times v^1 = 0.000,040,483,766,022,843 \times 0.185,302,018,885, \\ 184,100) = 0.000,007,501,723,576,108;$$

$$\text{and } v^{15} (= v^{14} \times v^1 = 0.000,007,501,723,576,108 \times 0.185,302,018,885, \\ 184,100) = 0.000,001,390,084,523,771;$$

$$\text{and } v^{16} (= v^{15} \times v^1 = 0.000,001,390,084,523,771 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,257,585,460,675;$$

$$\text{and } v^{17} (= v^{16} \times v^1 = 0.000,000,257,585,460,675 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,047,731,107,380;$$

$$\text{and } v^{18} (= v^{17} \times v^1 = 0.000,000,047,731,107,380 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,008,844,670,561;$$

$$\text{and } v^{19} (= v^{18} \times v^1 = 0.000,000,008,844,670,561 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,001,638,935,311;$$

$$\text{and } v^{20} (= v^{19} \times v^1 = 0.000,000,001,638,935,311 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,000,303,698,021;$$

$$\text{and } v^{21} (= v^{20} \times v^1 = 0.000,000,000,303,698,021 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,000,056,275,856;$$

$$\text{and } v^{22} (= v^{21} \times v^1 = 0.000,000,000,056,275,856 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,000,010,428,029;$$

$$\text{and } v^{23} (= v^{22} \times v^1 = 0.000,000,000,010,428,029 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,000,001,932,334;$$

$$\text{and } v^{24} (= v^{23} \times v^1 = 0.000,000,000,001,932,334 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,000,000,358,065;$$

$$\text{and } v^{25} (= v^{24} \times v^1 = 0.000,000,000,000,358,065 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,000,000,066,350;$$

$$\text{and } v^{26} (= v^{25} \times v^1 = 0.000,000,000,000,066,350 \times 0.185,302,018,885, \\ 184,100) = 0.000,000,000,000,012,294.$$

$$\text{Therefore } \frac{v^1}{64} \text{ will be } (= \frac{0.185,302,018,885,184,100}{64}) = \\ 0.002,895,344,045,081,001;$$

$$\text{and } \frac{v^4}{80} \text{ will be } (= \frac{0.004,156,838,202,925,120}{80}) = 0.000,429,210,477,536,564;$$

$$\text{and } \frac{v^6}{96} \text{ will be } (= \frac{0.006,361,685,441,135,942}{96}) = 0.000,066,277,973,345,166;$$

$$\text{and } \frac{v^8}{112} \text{ will be } (= \frac{0.001,179,018,457,73,858}{112}) = 0.000,010,526,950,515,838;$$

and

$$\text{and } \frac{v^{10}}{128} \text{ will be } (= \frac{0.000,218,474,500,518,392}{128}) = 0.000,001,706,832,035,378;$$

$$\text{and } \frac{v^{12}}{144} \text{ will be } (= \frac{0.000,040,481,766,022,843}{144}) = 0.000,000,281,137,264,047;$$

$$\text{and } \frac{v^{14}}{160} \text{ will be } (= \frac{0.000,007,501,723,556,108}{160}) = 0.000,000,046,885,772,350;$$

$$\text{and } \frac{v^{16}}{176} \text{ will be } (= \frac{0.000,001,390,084,523,771}{176}) = 0.000,000,007,898,207,521;$$

$$\text{and } \frac{v^{18}}{192} \text{ will be } (= \frac{0.000,000,257,585,468,675}{192}) = 0.000,000,001,341,590,982;$$

$$\text{and } \frac{v^{20}}{208} \text{ will be } (= \frac{0.000,000,047,731,107,380}{208}) = 0.000,000,000,229,476,477;$$

$$\text{and } \frac{v^{22}}{224} \text{ will be } (= \frac{0.000,000,008,844,670,561}{224}) = 0.000,000,000,039,485,136;$$

$$\text{and } \frac{v^{24}}{240} \text{ will be } (= \frac{0.000,000,001,638,935,311}{240}) = 0.000,000,000,006,828,897;$$

$$\text{and } \frac{v^{26}}{256} \text{ will be } (= \frac{0.000,000,000,301,608,021}{256}) = 0.000,000,000,001,186,320;$$

$$\text{and } \frac{v^{28}}{272} \text{ will be } (= \frac{0.000,000,000,056,275,856}{272}) = 0.000,000,000,000,206,896;$$

$$\text{and } \frac{v^{30}}{288} \text{ will be } (= \frac{0.000,000,000,010,428,029}{288}) = 0.000,000,000,000,036,208;$$

$$\text{and } \frac{v^{32}}{304} \text{ will be } (= \frac{0.000,000,000,001,932,334}{304}) = 0.000,000,000,000,006,356;$$

$$\text{and } \frac{v^{34}}{320} \text{ will be } (= \frac{0.000,000,000,000,358,055}{320}) = 0.000,000,000,000,001,118;$$

$$\text{and } \frac{v^{36}}{336} \text{ will be } (= \frac{0.000,000,000,000,066,350}{336}) = 0.000,000,000,000,000,197;$$

$$\text{and } \frac{v^{38}}{352} \text{ will be } (= \frac{0.000,000,000,000,012,204}{352}) = 0.000,000,000,000,000,034.$$

$$\text{Therefore the series } \frac{1}{48} + \frac{v^2}{64} + \frac{v^4}{80} + \frac{v^6}{96} + \frac{v^8}{112} + \frac{v^{10}}{128} + \frac{v^{12}}{144} + \frac{v^{14}}{160} + \frac{v^{16}}{176} + \frac{v^{18}}{192} + \frac{v^{20}}{208} + \frac{v^{22}}{224} + \frac{v^{24}}{240} + \frac{v^{26}}{256} + \frac{v^{28}}{272} + \frac{v^{30}}{288} + \frac{v^{32}}{304} + \frac{v^{34}}{320} + \frac{v^{36}}{336} + \frac{v^{38}}{352} + \&c \text{ ad infinitum will be } =$$

$$\begin{aligned} & 0.020,833,333,333,333,333, \\ & + 0.002,895,344,045,081,001, \\ & + 0.000,429,210,477,536,564, \\ & + 0.000,066,277,973,345,166, \\ & + 0.000,010,526,950,515,838, \\ & + 0.000,001,706,832,035,378, \\ & + 0.000,000,281,137,264,047, \\ & + 0.000,000,046,885,772,350, \end{aligned}$$

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$$\begin{aligned}
&+ 0.000,000,007,898,207,521, \\
&+ 0.000,000,001,341,590,982, \\
&+ 0.000,000,000,229,476,477, \\
&+ 0.000,000,000,039,485,136, \\
&+ 0.000,000,000,006,828,897, \\
&+ 0.000,000,000,001,186,320, \\
&+ 0.000,000,000,000,206,896, \\
&+ 0.000,000,000,000,036,208, \\
&+ 0.000,000,000,000,006,356, \\
&+ 0.000,000,000,000,001,118, \\
&+ 0.000,000,000,000,000,197, \\
&+ 0.000,000,000,000,000,034, \\
&+ \&c \text{ ad infinitum}
\end{aligned}$$

= 0.024,236,737,151,909,819; of which number, (if no mistakes have been made in the calculations,) the first sixteen figures, (reckoning from the place of units,) to wit, the figures 0.024,236,737,151,909,8, will be exact.

Q. E. I.

End of the Computation of the Value of the Sum of the first twenty Terms of the Series

$$\frac{1}{48} + \frac{v^2}{64} + \frac{v^4}{80} + \frac{v^6}{96} + \frac{v^8}{112} + \frac{v^{10}}{128} + \&c \text{ ad infinitum.}$$

The Application of all the foregoing Computations to the Computation of the Value of the Series A + R, or $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ ad infinitum.

Art. 37. Having thus found the series $\frac{1}{48} + \frac{v^2}{64} + \frac{v^4}{80} + \frac{v^6}{96} + \frac{v^8}{112} + \frac{v^{10}}{128} + \frac{v^{12}}{144} + \&c$, continued to twenty terms, to be = 0.024,236,737,151,909,819, we must multiply this value into $2v$, in order to obtain the value of the series $\frac{2v}{48} + \frac{2v^3}{64} + \frac{2v^5}{80} + \frac{2v^7}{96} + \frac{2v^9}{112} + \frac{2v^{11}}{128} + \&c$ mentioned in art. 33.

Now, since v is ($= \lambda^1$) = 0.430,467,210.000,000,000, we shall have $2v$ ($= 2 \times 0.430,467,210.000,000,000$) = 0.860,934,420,000,000,000, and consequently $2v \times 0.024,236,737,151,909,819$ ($= 0.860,934,420,000,000,000, \times 0.024,236,737,151,909,819$) = 0.020,816,241,242,571,931. Therefore the series $\frac{2v}{48} + \frac{2v^3}{64} + \frac{2v^5}{80} + \frac{2v^7}{96} + \frac{2v^9}{112} + \frac{2v^{11}}{128} + \&c$ will be = 0.020,866,241,242,571,931.

Q. E. I.

Art. 38. But the series $\frac{2v}{48} + \frac{2v^2}{64} + \frac{2v^3}{80} + \frac{2v^4}{96} + \frac{2v^5}{112} + \frac{2v^6}{128} + \&c$ together with the series $\frac{1}{40} - \frac{v}{48} + \frac{v^2}{56} - \frac{v^3}{64} + \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88} - \frac{v^7}{96} + \frac{v^8}{104} - \frac{v^9}{112} + \frac{v^{10}}{120} - \frac{v^{11}}{128} + \&c$ is equal to the series $\frac{1}{40} + \frac{v}{48} + \frac{v^2}{56} + \frac{v^3}{64} + \frac{v^4}{72} + \frac{v^5}{80} + \frac{v^6}{88} + \frac{v^7}{96} + \frac{v^8}{104} + \frac{v^9}{112} + \frac{v^{10}}{120} + \frac{v^{11}}{128} + \&c$.

Therefore $0.020,866,241,242,571,931$, together with the series $\frac{1}{40} - \frac{v}{48} + \frac{v^2}{56} - \frac{v^3}{64} + \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88} - \frac{v^7}{96} + \frac{v^8}{104} - \frac{v^9}{112} + \frac{v^{10}}{120} - \frac{v^{11}}{128} + \&c$ *ad infinitum* will be equal to the series $\frac{1}{40} + \frac{v}{48} + \frac{v^2}{56} + \frac{v^3}{64} + \frac{v^4}{72} + \frac{v^5}{80} + \frac{v^6}{88} + \frac{v^7}{96} + \frac{v^8}{104} + \frac{v^9}{112} + \frac{v^{10}}{120} + \frac{v^{11}}{128} + \&c$ *ad infinitum*.

But it has been shewn above, in art. 34, that the series $\frac{1}{40} - \frac{v}{48} + \frac{v^2}{56} - \frac{v^3}{64} + \frac{v^4}{72} - \frac{v^5}{80} + \frac{v^6}{88} - \frac{v^7}{96} + \frac{v^8}{104} - \frac{v^9}{112} + \frac{v^{10}}{120} - \frac{v^{11}}{128} + \&c$ *ad infinitum* is $= 0.018,438,562,104,429,149$.

Therefore the series $\frac{1}{40} + \frac{v}{48} + \frac{v^2}{56} + \frac{v^3}{64} + \frac{v^4}{72} + \frac{v^5}{80} + \frac{v^6}{88} + \frac{v^7}{96} + \frac{v^8}{104} + \frac{v^9}{112} + \frac{v^{10}}{120} + \frac{v^{11}}{128} + \&c$ *ad infinitum* will be $(= 0.020,866,241,242,571,931 + 0.018,438,562,104,429,149) = 0.039,304,803,347,001,080$; and consequently the series $\frac{1}{40} + \frac{v}{48} + \frac{v^2}{56} + \frac{v^3}{64} + \frac{v^4}{72} + \frac{v^5}{80} + \frac{v^6}{88} + \frac{v^7}{96} + \frac{v^8}{104} + \frac{v^9}{112} + \frac{v^{10}}{120} + \frac{v^{11}}{128} + \&c$ (which is equal to the series $\frac{1}{40} + \frac{v}{48} + \frac{v^2}{56} + \frac{v^3}{64} + \frac{v^4}{72} + \frac{v^5}{80} + \frac{v^6}{88} + \frac{v^7}{96} + \frac{v^8}{104} + \frac{v^9}{112} + \frac{v^{10}}{120} + \frac{v^{11}}{128} + \&c$, and from which the latter series was derived by the mere substitution of the single letter v in it's terms instead of zx .) will also be $= 0.039,304,803,347,001,080$.

Art. 39. Having thus found the value of the series $\frac{1}{40} + \frac{v}{48} + \frac{v^2}{56} + \frac{v^3}{64} + \frac{v^4}{72} + \frac{v^5}{80} + \frac{v^6}{88} + \frac{v^7}{96} + \frac{v^8}{104} + \frac{v^9}{112} + \frac{v^{10}}{120} + \frac{v^{11}}{128} + \&c$ to be $0.039,304,803,347,001,080$, we must multiply the said value by zx , in order to obtain the value of the series $\frac{zx}{40} + \frac{zx^2}{48} + \frac{zx^3}{56} + \frac{zx^4}{64} + \frac{zx^5}{72} + \frac{zx^6}{80} + \&c$.

Now z is $(= xy = x^2) = 0.6561$; and therefore zx is $(= 2 \times 0.6561) = 1.3122$.

1.3122. Therefore $23 \times 0.039,304,803,347,001,080$ will be $(= 1.3122 \times 0.039,304,803,347,001,080) = 0.051,575,762,951,934,817$. Consequently the series $\frac{2\pi}{40} + \frac{2\pi^3}{48} + \frac{2\pi^5}{56} + \frac{2\pi^7}{64} + \frac{2\pi^9}{72} + \frac{2\pi^{11}}{80} + \&c$ will be $= 0.051,575,762,951,934,817$.

But the series $\frac{2\pi}{40} + \frac{2\pi^3}{48} + \frac{2\pi^5}{56} + \frac{2\pi^7}{64} + \frac{2\pi^9}{72} + \frac{2\pi^{11}}{80} + \&c$ *ad infinitum* together with the series $\frac{1}{36} - \frac{\pi}{40} + \frac{\pi^3}{44} - \frac{\pi^5}{48} + \frac{\pi^7}{52} - \frac{\pi^9}{56} + \frac{\pi^{11}}{60} - \frac{\pi^{13}}{64} + \frac{\pi^{15}}{68} - \frac{\pi^{17}}{72} + \frac{\pi^{19}}{76} - \frac{\pi^{21}}{80} + \&c$ *ad infinitum* are equal to the series $\frac{1}{36} + \frac{\pi}{40} + \frac{\pi^3}{44} + \frac{\pi^5}{48} + \frac{\pi^7}{52} + \frac{\pi^9}{56} + \frac{\pi^{11}}{60} + \frac{\pi^{13}}{64} + \frac{\pi^{15}}{68} + \frac{\pi^{17}}{72} + \frac{\pi^{19}}{76} + \frac{\pi^{21}}{80} + \&c$ *ad infinitum*; and it has been shewn in art. 32, that the series $\frac{1}{36} - \frac{\pi}{40} + \frac{\pi^3}{44} - \frac{\pi^5}{48} + \frac{\pi^7}{52} - \frac{\pi^9}{56} + \frac{\pi^{11}}{60} - \frac{\pi^{13}}{64} + \frac{\pi^{15}}{68} - \frac{\pi^{17}}{72} + \frac{\pi^{19}}{76} - \frac{\pi^{21}}{80} + \&c$ *ad infinitum* is $= 0.017,490,787,392,365,570$. Therefore the series $\frac{1}{36} + \frac{\pi}{40} + \frac{\pi^3}{44} + \frac{\pi^5}{48} + \frac{\pi^7}{52} + \frac{\pi^9}{56} + \frac{\pi^{11}}{60} + \frac{\pi^{13}}{64} + \frac{\pi^{15}}{68} + \frac{\pi^{17}}{72} + \frac{\pi^{19}}{76} + \frac{\pi^{21}}{80} + \&c$ *ad infinitum* will be $= 0.051,575,762,951,934,817 + 0.017,490,787,392,365,570 = 0.069,066,550,344,300,387$.

Art. 40. But, by art. 30, the series $\frac{1}{36} + \frac{\pi}{40} + \frac{\pi^3}{44} + \frac{\pi^5}{48} + \frac{\pi^7}{52} + \frac{\pi^9}{56} + \frac{\pi^{11}}{60} + \frac{\pi^{13}}{64} + \frac{\pi^{15}}{68} + \frac{\pi^{17}}{72} + \frac{\pi^{19}}{76} + \frac{\pi^{21}}{80} + \&c$ *ad infinitum* is equal to the series $\frac{1}{36} + \frac{2\pi}{40} + \frac{2^3\pi^3}{44} + \frac{2^5\pi^5}{48} + \frac{2^7\pi^7}{52} + \frac{2^9\pi^9}{56} + \frac{2^{11}\pi^{11}}{60} + \frac{2^{13}\pi^{13}}{64} + \frac{2^{15}\pi^{15}}{68} + \frac{2^{17}\pi^{17}}{72} + \frac{2^{19}\pi^{19}}{76} + \frac{2^{21}\pi^{21}}{80} + \&c$ *ad infinitum*. Therefore this last series is also $= 0.069,066,550,344,300,387$. And consequently the series $\frac{2\pi}{36} + \frac{2^3\pi^3}{40} + \frac{2^5\pi^5}{44} + \frac{2^7\pi^7}{48} + \frac{2^9\pi^9}{52} + \frac{2^{11}\pi^{11}}{56} + \&c$ *ad infinitum* (which arises from the multiplication of the said series into 2π) will be $= 2\pi \times 0.069,066,550,344,300,387 (= 2 \times \pi \times 0.069,066,550,344,300,387 = 2 \times 0.81 \times 0.069,066,550,344,300,387 = 1.62 \times 0.069,066,550,344,300,387) = 0.111,887,811,557,766,626$.

Art. 41. But the series $\frac{2\pi}{36} + \frac{2^3\pi^3}{40} + \frac{2^5\pi^5}{44} + \frac{2^7\pi^7}{48} + \frac{2^9\pi^9}{52} + \frac{2^{11}\pi^{11}}{56} + \&c$ *ad infinitum* together with the series $\frac{1}{34} - \frac{\pi}{36} + \frac{2\pi}{38} - \frac{\pi^3}{40} + \frac{2^3\pi^3}{42} - \frac{\pi^5}{44} + \frac{2^5\pi^5}{46} - \frac{2^7\pi^7}{48} + \&c$ *ad infinitum*

$\frac{y^2}{48} + \frac{y^4}{50} - \frac{y^6}{52} + \frac{y^{10}}{54} - \frac{y^{11}}{56} + \&c$ *ad infinitum*, is equal to the series $\frac{1}{34} + \frac{y}{36} + \frac{y^2}{38} + \frac{y^3}{40} + \frac{y^4}{42} + \frac{y^5}{44} + \frac{y^6}{46} + \frac{y^7}{48} + \frac{y^8}{50} + \frac{y^9}{52} + \frac{y^{10}}{54} + \frac{y^{11}}{56} + \&c$ *ad infinitum*; and it has been shewn in art. 29, that the series $\frac{1}{34} - \frac{y}{36} + \frac{y^2}{38} - \frac{y^3}{40} + \frac{y^4}{42} - \frac{y^5}{44} + \frac{y^6}{46} - \frac{y^7}{48} + \frac{y^8}{50} - \frac{y^9}{52} + \frac{y^{10}}{54} - \frac{y^{11}}{56} + \&c$ *ad infinitum* is $= 0.016,674,011,463,729,394$. Therefore the series $\frac{1}{34} + \frac{y}{36} + \frac{y^2}{38} + \frac{y^3}{40} + \frac{y^4}{42} + \frac{y^5}{44} + \frac{y^6}{46} + \frac{y^7}{48} + \frac{y^8}{50} + \frac{y^9}{52} + \frac{y^{10}}{54} + \frac{y^{11}}{56} + \&c$ *ad infinitum* will be $= 0.111,887,811,557,766,626 + 0.016,674,011,463,729,394 = 0.128,561,823,021,496,020$.

Art. 42. But, because y is $= xx$, the series $\frac{x}{34} + \frac{y}{36} + \frac{y^2}{38} + \frac{y^3}{40} + \frac{y^4}{42} + \frac{y^5}{44} + \frac{y^6}{46} + \frac{y^7}{48} + \frac{y^8}{50} + \frac{y^9}{52} + \frac{y^{10}}{54} + \frac{y^{11}}{56} + \&c$ *ad infinitum* is equal to the series $\frac{1}{34} + \frac{xx}{36} + \frac{x^4}{38} + \frac{x^6}{40} + \frac{x^8}{42} + \frac{x^{10}}{44} + \frac{x^{12}}{46} + \frac{x^{14}}{48} + \frac{x^{16}}{50} + \frac{x^{18}}{52} + \frac{x^{20}}{54} + \frac{x^{22}}{56} + \&c$. Therefore the said series $\frac{1}{34} + \frac{xx}{36} + \frac{x^4}{38} + \frac{x^6}{40} + \frac{x^8}{42} + \frac{x^{10}}{44} + \frac{x^{12}}{46} + \frac{x^{14}}{48} + \frac{x^{16}}{50} + \frac{x^{18}}{52} + \frac{x^{20}}{54} + \frac{x^{22}}{56} + \&c$ *ad infinitum* will be $= 0.128,561,823,021,496,020$. Therefore the series $\frac{2x}{34} + \frac{2x^3}{36} + \frac{2x^5}{38} + \frac{2x^7}{40} + \frac{2x^9}{42} + \frac{2x^{11}}{44} + \&c$ *ad infinitum* (which arises from the multiplication of the said series into $2x$), will be $= 2x \times 0.128,561,823,021,496,020 (= 2 \times 0.9 \times 0.128,561,823,021,496,020 = 1.8 \times 0.128,561,823,021,496,020) = 0.231,411,281,438,652,836$.

Art. 43. But the series $\frac{2x}{34} + \frac{x^2}{36} + \frac{2x^3}{38} + \frac{2x^5}{40} + \frac{2x^7}{42} + \frac{2x^9}{44} + \&c$ *ad infinitum*, being added to the series $\frac{1}{33} - \frac{x}{34} + \frac{x^2}{35} - \frac{x^3}{36} + \frac{x^4}{37} - \frac{x^5}{38} + \frac{x^6}{39} - \frac{x^7}{40} + \frac{x^8}{41} - \frac{x^9}{42} + \frac{x^{10}}{43} - \frac{x^{11}}{44} + \&c$ *ad infinitum*, is equal to the series $\frac{1}{33} - \frac{x^2}{40} + \frac{x^3}{41} - \frac{x^6}{42} + \frac{x^{10}}{43} - \frac{x^{11}}{44} + \&c$ *ad infinitum*; and it is shewn in art. 24, that the series $\frac{1}{33} - \frac{x}{34} + \frac{x^2}{35} - \frac{x^3}{36} + \frac{x^4}{37} - \frac{x^5}{38} + \frac{x^6}{39} - \frac{x^7}{40} + \frac{x^8}{41} - \frac{x^9}{42} + \frac{x^{10}}{43} - \frac{x^{11}}{44} + \&c$ *ad infinitum* is

is = 0.016,177,427,297,506,659. Therefore the said series $\frac{x}{33} + \frac{x^2}{34} + \frac{x^3}{35} + \frac{x^4}{36} + \frac{x^5}{37} + \frac{x^6}{38} + \frac{x^7}{39} + \frac{x^8}{40} + \frac{x^9}{41} + \frac{x^{10}}{42} + \frac{x^{11}}{43} + \frac{x^{12}}{44} + \&c$ *ad infinitum* will be = 0.231,411,281,438,692,836 + 0.016,177,427,297,506,659 = 0.247,588,708,736,199,495. And consequently the series $\frac{x^{13}}{33} + \frac{x^{14}}{34} + \frac{x^{15}}{35} + \frac{x^{16}}{36} + \frac{x^{17}}{37} + \frac{x^{18}}{38} + \frac{x^{19}}{39} + \frac{x^{20}}{40} + \&c$ *ad infinitum* (which arises from the multiplication of the last series into x^{13}), will be = $x^{13} \times 0.247,588,708,736,199,495 = 0.9^{13} \times 0.247,588,708,736,199,495 = 0.034,336,838,202,925,124 \times 0.247,588,708,736,199,495 = 0.008,501,413,432,746,036$; that is, the whole remainder of the infinite series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum* after the first thirty-two terms, will be = 0.008,501,413,432,746,036; or, according to the notation of art. 5, the quantity R will be = 0.008,501,413,432,746,036.

ART. 44. But the quantity A, or the sum of the first thirty-two terms of the said infinite series, has been shewn above, in art. 19, 10 be equal to 2.549,926,467,204,303,322. Therefore the quantity A + R, or the whole of the said infinite series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum*, will be = 2.549,926,467,204,303,322 + 0.008,501,413,432,746,036 = 2.558,427,880,637,049,358. And consequently $A + R \times x$, or the whole original series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$ *ad infinitum*, will be = 2.558,427,880,637,049,358 $\times x$ (= 2.558,427,880,637,049,358 $\times 0.9$) = 2.302,585,092,573,344,422. Q. E. D.

This number is exact in the first ten figures 2.302,585,092, the more-accurate value of the said original series (which is Napier's Logarithm of the ratio of 10 to 1,) being 2.302,585,092,994,045,684,017, &c.

A more summary Statement of the Processes of the foregoing Computation.

ART. 45. The last series obtained in the foregoing computation is the series $\frac{x}{48} + \frac{x^2}{64} + \frac{x^4}{80} + \frac{x^6}{96} + \frac{x^8}{112} + \frac{x^{10}}{128} + \&c$ *ad infinitum*, in which xv is = x^{16}

$= 0.185,302,018,885,184,100$, which is less than 0.2 , or than $\frac{1}{5}$. This series converges swiftly enough to be summed in the common way by computing some of its first terms and adding them together. And twenty of these terms are found to be $= 0.024,236,737,151,909,819$; of which number, if no mistake has been made in the calculation, the first sixteen figures, $0.024,236,737,151,909,8$, must be exact.

Therefore the series $\frac{2^0}{48} + \frac{2^1}{64} + \frac{2^2}{80} + \frac{2^3}{96} + \frac{2^4}{112} + \frac{2^5}{128} + \&c$ (which is $= 2^0 \times$ the series $\frac{1}{48} + \frac{2^0}{64} + \frac{2^1}{80} + \frac{2^2}{96} + \frac{2^3}{112} + \frac{2^4}{128} + \&c$) will be $= 2^0 \times 0.024,236,737,151,909,819$ ($= 2 \times 2^1 \times 0.024,236,737,151,909,819 = 2 \times 0.430,467,210 \times 0.024,236,737,151,909,819 = 0.860,934,420 \times 0.024,236,737,151,909,819$) $= 0.020,866,241,242,571,931$.

The series $\frac{2^0}{48} + \frac{2^1}{64} + \frac{2^2}{80} + \frac{2^3}{96} + \frac{2^4}{112} + \frac{2^5}{128} + \&c$ and the series $\frac{1}{40} - \frac{2^0}{48} + \frac{2^1}{56} - \frac{2^2}{64} + \frac{2^3}{72} - \frac{2^4}{80} + \frac{2^5}{88} - \frac{2^6}{96} + \frac{2^7}{104} - \frac{2^8}{112} + \frac{2^9}{120} - \frac{2^{10}}{128} + \&c$ are together equal to the series $\frac{1}{40} + \frac{2^1}{48} + \frac{2^2}{56} + \frac{2^3}{64} + \frac{2^4}{72} + \frac{2^5}{80} + \frac{2^6}{88} + \frac{2^7}{96} + \frac{2^8}{104} + \frac{2^9}{112} + \frac{2^{10}}{120} + \frac{2^{11}}{128} + \&c$. And the series $\frac{1}{40} - \frac{2^0}{48} + \frac{2^1}{56} - \frac{2^2}{64} + \frac{2^3}{72} - \frac{2^4}{80} + \frac{2^5}{88} - \frac{2^6}{96} + \frac{2^7}{104} - \frac{2^8}{112} + \frac{2^9}{120} - \frac{2^{10}}{128} + \&c$ is $= 0.018,438,562,104,429,149$; of which number, if no mistake has been made in the calculation, the first eleven figures $0.018,438,562,10$ must be exact.

Therefore the series $\frac{1}{40} + \frac{2^0}{48} + \frac{2^1}{56} + \frac{2^2}{64} + \frac{2^3}{72} + \frac{2^4}{80} + \frac{2^5}{88} + \frac{2^6}{96} + \frac{2^7}{104} + \frac{2^8}{112} + \frac{2^9}{120} + \frac{2^{10}}{128} + \&c$ will be $(= 0.020,866,241,242,571,931 + 0.018,438,562,104,429,149) = 0.039,304,803,347,001,080$; of which number the first eleven figures, $0.039,304,803,34$, must be exact.

Therefore the series $\frac{1}{40} + \frac{2^1}{48} + \frac{2^2}{56} + \frac{2^3}{64} + \frac{2^4}{72} + \frac{2^5}{80} + \frac{2^6}{88} + \frac{2^7}{96} + \&c$ (which is equal to the series $\frac{1}{40} + \frac{2^0}{48} + \frac{2^1}{56} + \frac{2^2}{64} + \frac{2^3}{72} + \frac{2^4}{80} + \frac{2^5}{88} + \frac{2^6}{96} + \&c$) will also be $= 0.039,304,803,347,001,080$; of which number the first eleven figures, $0.039,304,803,34$, must be exact.

Therefore the series $\frac{2^0}{40} + \frac{2^1}{48} + \frac{2^2}{56} + \frac{2^3}{64} + \frac{2^4}{72} + \frac{2^5}{80} + \frac{2^6}{88} + \frac{2^7}{96} + \&c$ (which is $= 2 \times$ the series $\frac{1}{40} + \frac{2^0}{48} + \frac{2^1}{56} + \frac{2^2}{64} + \frac{2^3}{72} + \frac{2^4}{80} + \frac{2^5}{88} + \frac{2^6}{96} + \&c$)

+ $\frac{2^{14}}{96} + \&c$) will be $= 2z \times 0.039,304,803,347,001,080 (= 2 \times yy \times 0.039,304,803,347,001,080 = 2 \times x^4 \times 0.039,304,803,347,001,080 = 2 \times 0.6561 \times 0.039,304,803,347,001,080 = 1.3122 \times 0.039,304,803,347,001,080) = 0.051,575,762,951,934,817$; of which number the first eleven figures 0.051,575,762,951 must be exact.

But the series $\frac{3z}{40} + \frac{2z^2}{48} + \frac{2z^3}{56} + \frac{2z^4}{64} + \frac{2z^5}{72} + \frac{2z^6}{80} + \&c$, being added to the series $\frac{1}{36} - \frac{z}{40} + \frac{z^2}{44} - \frac{z^3}{48} + \frac{z^4}{52} - \frac{z^5}{56} + \frac{z^6}{60} - \frac{z^7}{64} + \frac{z^8}{68} - \frac{z^9}{72} + \frac{z^{10}}{76} - \frac{z^{11}}{80} + \&c$, is equal to the series $\frac{1}{36} + \frac{z}{40} + \frac{z^2}{44} + \frac{z^3}{48} + \frac{z^4}{52} + \frac{z^5}{56} + \frac{z^6}{60} + \frac{z^7}{64} + \frac{z^8}{68} + \frac{z^9}{72} + \frac{z^{10}}{76} + \frac{z^{11}}{80} + \&c$. And the series $\frac{1}{36} - \frac{z}{40} + \frac{z^2}{44} - \frac{z^3}{48} + \frac{z^4}{52} - \frac{z^5}{56} + \frac{z^6}{60} - \frac{z^7}{64} + \frac{z^8}{68} - \frac{z^9}{72} + \frac{z^{10}}{76} - \frac{z^{11}}{80} + \&c$ has been found to be $= 0.017,490,787,392,365,570$; of which number the first eleven figures 0.017,490,787,392, will be exact.

Therefore the series $\frac{1}{36} + \frac{z}{40} + \frac{z^2}{44} + \frac{z^3}{48} + \frac{z^4}{52} + \frac{z^5}{56} + \frac{z^6}{60} + \frac{z^7}{64} + \frac{z^8}{68} + \frac{z^9}{72} + \frac{z^{10}}{76} + \frac{z^{11}}{80} + \&c$ will be $(= 0.051,575,762,951,934,817 + 0.017,490,787,392,365,570) = 0.069,066,550,344,300,387$; of which number the first eleven figures 0.069,066,550,344 will be exact.

But z is yy , and consequently the series $\frac{1}{36} + \frac{z}{40} + \frac{z^2}{44} + \frac{z^3}{48} + \frac{z^4}{52} + \frac{z^5}{56} + \frac{z^6}{60} + \frac{z^7}{64} + \&c$ is equal to the series $\frac{1}{36} + \frac{yy}{40} + \frac{y^2}{44} + \frac{y^3}{48} + \frac{y^4}{52} + \frac{y^5}{56} + \frac{y^6}{60} + \frac{y^7}{64} + \&c$. Therefore the series $\frac{1}{36} + \frac{yy}{40} + \frac{y^2}{44} + \frac{y^3}{48} + \frac{y^4}{52} + \frac{y^5}{56} + \frac{y^6}{60} + \frac{y^7}{64} + \&c$ is $= 0.069,066,550,344,300,387$; of which number the eleven first figures 0.069,066,550,344 will be exact. And consequently the series $\frac{2y}{36} + \frac{2y^2}{40} + \frac{2y^3}{44} + \frac{2y^4}{48} + \frac{2y^5}{52} + \frac{2y^6}{56} + \frac{2y^7}{60} + \frac{2y^8}{64} + \&c$, (which arises from the multiplication of the last series into $2y$), will be $= 2y \times 0.069,066,550,344,300,387 (= 2 \times xx \times 0.069,066,550,344,300,387 = 2 \times 0.81 \times 0.069,066,550,344,300,387 = 1.62 \times 0.069,066,550,344,300,387) = 0.111,887,811,557,766,626$; of which the first eleven figures 0.111,887,811,557 will be exact.

But the series $\frac{2y}{36} + \frac{2y^2}{40} + \frac{2y^3}{44} + \frac{2y^4}{48} + \frac{2y^5}{52} + \frac{2y^6}{56} + \&c$, being added to the series $\frac{1}{34} - \frac{y}{36} + \frac{y^2}{38} - \frac{y^3}{40} + \frac{y^4}{42} - \frac{y^5}{44} + \frac{y^6}{46} - \frac{y^7}{48} + \&c$, is equal

to

to the series $\frac{x}{34} + \frac{x^2}{36} + \frac{x^3}{38} + \frac{x^4}{40} + \frac{x^5}{42} + \frac{x^6}{44} + \frac{x^7}{46} + \frac{x^8}{48} + \&c$; and the series $\frac{x}{34} - \frac{x^2}{36} + \frac{x^3}{38} - \frac{x^4}{40} + \frac{x^5}{42} - \frac{x^6}{44} + \frac{x^7}{46} - \frac{x^8}{48} + \&c$, has been found to be $= 0.016,674,011,463,729,394$; of which number the first eleven figures 0.016,674,011,463 will be exact.

Therefore the series $\frac{x}{34} + \frac{x^2}{36} + \frac{x^3}{38} + \frac{x^4}{40} + \frac{x^5}{42} + \frac{x^6}{44} + \frac{x^7}{46} + \frac{x^8}{48} + \&c$ *ad infinitum* will be $= 0.111,887,811,557,766,626 + 0.016,674,011,463,729,394 = 0.128,561,823,021,496,020$; of which number the first eleven figures 0.128,561,823,021 will be exact.

But, because $y = xx$, the series $\frac{1}{34} + \frac{x}{36} + \frac{x^2}{38} + \frac{x^3}{40} + \frac{x^4}{42} + \frac{x^5}{44} + \frac{x^6}{46} + \frac{x^7}{48} + \&c$ will be equal to the series $\frac{1}{34} + \frac{xx}{36} + \frac{x^2}{38} + \frac{x^3}{40} + \frac{x^4}{42} + \frac{x^5}{44} + \frac{x^6}{46} + \frac{x^7}{48} + \&c$. Therefore the series $\frac{1}{34} + \frac{x}{36} + \frac{x^2}{38} + \frac{x^3}{40} + \frac{x^4}{42} + \frac{x^5}{44} + \frac{x^6}{46} + \frac{x^7}{48} + \&c$ will also be $= 0.128,561,823,021,496,020$; of which number the first eleven figures 0.128,561,823,021 will be exact.

Therefore the series $\frac{2x}{34} + \frac{2x^2}{36} + \frac{2x^3}{38} + \frac{2x^4}{40} + \frac{2x^5}{42} + \frac{2x^6}{44} + \frac{2x^7}{46} + \frac{2x^8}{48} + \&c$ (which arises from the multiplication of the last series into $2x$), will be $= 2x \times 0.128,561,823,021,496,020 = 2 \times 0.9 \times 0.128,561,823,021,496,020 = 1.8 \times 0.128,561,823,021,496,020 = 0.231,411,281,438,692,836$; of which number the first eleven figures 0.231,411,281,438 will be exact.

But the series $\frac{2x}{34} + \frac{2x^2}{36} + \frac{2x^3}{38} + \frac{2x^4}{40} + \frac{2x^5}{42} + \frac{2x^6}{44} + \&c$ *ad infinitum*, being added to the series $\frac{x}{33} - \frac{x^2}{34} + \frac{x^3}{35} - \frac{x^4}{36} + \frac{x^5}{37} - \frac{x^6}{38} + \frac{x^7}{39} - \frac{x^8}{40} + \frac{x^9}{41} - \frac{x^{10}}{42} + \frac{x^{11}}{43} - \frac{x^{12}}{44} + \&c$ *ad infinitum* is equal to the series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \frac{x^8}{41} + \frac{x^9}{42} + \frac{x^{10}}{43} + \frac{x^{11}}{44} + \&c$ *ad infinitum*. And the series $\frac{1}{33} - \frac{x}{34} + \frac{x^2}{35} - \frac{x^3}{36} + \frac{x^4}{37} - \frac{x^5}{38} + \frac{x^6}{39} - \frac{x^7}{40} + \frac{x^8}{41} - \frac{x^9}{42} + \frac{x^{10}}{43} - \frac{x^{11}}{44} + \&c$ *ad infinitum* has been found to be $= 0.016,177,427,297,506,659$; of which number the first eleven figures 0.016,177,427,297 are exact. Therefore the series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \frac{x^8}{41} + \frac{x^9}{42} + \frac{x^{10}}{43} + \frac{x^{11}}{44} + \&c$ *ad infinitum* will be $= 0.016,177,427,297,506,659 + 0.231,411,281,438,692,836 = 0.247,588,708,736,209,495$; of which number the first eleven figures 0.247,588,708,736 will be exact.

$\frac{x^6}{39} + \frac{x^7}{40} + \frac{x^8}{41} + \frac{x^9}{42} + \frac{x^{10}}{43} + \frac{x^{11}}{44} + \&c$ *ad infinitum* will be $= 0.231,411,281,438,692,836 + 0.016,177,427,297,506,659 = 0.247,588,708,736,199,495$; of which number the first eleven figures, $0.247,588,708,736$, (if no mistakes have been made in the calculations,) will be exact.

Therefore the series $\frac{x^{11}}{33} + \frac{x^{12}}{34} + \frac{x^{13}}{35} + \frac{x^{14}}{36} + \frac{x^{15}}{37} + \frac{x^{16}}{38} + \frac{x^{17}}{39} + \frac{x^{18}}{40} + \&c$ *ad infinitum*, (which is equal to the last series multiplied into x^{11}), will be $= 0.247,588,708,736,199,495 \times x^{11} (= 0.247,588,708,736,199,495 \times 0.034,336,838,202,925,120) = 0.008,501,413,432,746,036$; that is, the whole series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum*, except the first thirty-two terms, will be $= 0.008,501,413,432,746,036$.

But the sum of the first thirty-two terms of the said series is $= 2.549,926,467,204,303,322$.

Therefore the whole of the said series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum* will be $= 2.549,926,467,204,303,322 + 0.008,501,413,432,746,036 = 2.558,427,880,637,049,358$. And consequently the whole original series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$ *ad infinitum*, (which is equal to the last series multiplied into x), will be $= 2.558,427,880,637,049,358 \times x (= 2.558,427,880,637,049,358 \times 0.9) = 2.302,585,092,573,344,422$. Q. E. I.

Another Statement of the Processes of the foregoing Computation, expressed in the Notation used in art. 14.

Art. 46. If we express these several processes in the notation used in art. 14, they will be as follows:

The series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \&c$ *ad infinitum*, or $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ *ad infinitum*, will be

$= A + R =$ the sum of the first thirty-two terms of the said series, together with the remainder of it, or the series $\frac{x^{11}}{33} + \frac{x^{12}}{34} + \frac{x^{13}}{35} + \frac{x^{14}}{36} + \frac{x^{15}}{37} + \frac{x^{16}}{38} + \frac{x^{17}}{39} + \frac{x^{18}}{40} + \&c$ *ad infinitum*

=

$$\begin{aligned}
&= 2.549,926,467,204,303,322 + x^{13} \times \text{the series } \frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \\
&\quad \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c \text{ ad infinitum} \\
&= 2.549,926,467,204,303,322 + 0.034,336,838,202,925,120 \times \text{the series} \\
&\quad \frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c \text{ ad infinitum} \\
&= 2.549,926,467,204,303,322 + 0.034,336,838,202,925,120 \times \text{the series} \\
&\quad \frac{1}{33} - \frac{x}{34} + \frac{x^2}{35} - \frac{x^3}{36} + \frac{x^4}{37} - \frac{x^5}{38} + \frac{x^6}{39} - \frac{x^7}{40} + \&c \text{ ad infinitum} \\
&+ 0.034,336,838,202,925,120 \times \text{the series } \frac{2x}{34} + \frac{2x^3}{36} + \frac{2x^5}{38} + \frac{2x^7}{40} + \\
&\quad \&c \text{ ad infinitum} \\
&= (\text{by art. 8.}) 2.549,926,467,204,303,322 + 0.034,336,838,202,925,120 \times \\
&\quad (\alpha, \text{ or } 0.016,177,427,297,506,659 \\
&+ 0.034,336,838,202,925,120 \times 2x \times \text{the series } \frac{1}{34} + \frac{xx}{36} + \frac{x^4}{38} + \frac{x^6}{40} + \\
&\quad \&c \text{ ad infinitum} \quad * \\
&= A + x^{13} \times \alpha + x^{13} \times 2x \times \text{the series } \frac{1}{34} + \frac{xx}{36} + \frac{x^4}{38} + \frac{x^6}{40} + \frac{x^8}{42} + \\
&\quad \frac{x^{10}}{44} + \frac{x^{12}}{46} + \frac{x^{14}}{48} + \&c \text{ ad infinitum} \\
&= A + x^{13} \times \alpha + x^{13} \times 2x \times \text{the series } \frac{1}{34} - \frac{x^2}{36} + \frac{x^4}{38} - \frac{x^6}{40} + \frac{x^8}{42} - \frac{x^{10}}{44} \\
&\quad + \frac{x^{12}}{46} - \frac{x^{14}}{48} + \&c \text{ ad infinitum} + x^{13} \times 2x \times \text{the series } \frac{2x^2}{36} + \frac{2x^6}{40} + \frac{2x^{10}}{44} \\
&\quad + \frac{2x^{14}}{48} + \&c \text{ ad infinitum} \\
&= A + x^{13} \times \alpha + x^{13} \times 2x \times \text{the series } \frac{1}{34} - \frac{x^2}{36} + \frac{x^4}{38} - \frac{x^6}{40} + \frac{x^8}{42} - \\
&\quad \frac{x^{10}}{44} + \frac{x^{12}}{46} - \frac{x^{14}}{48} + \&c \text{ ad infinitum} + x^{13} \times 2x \times \text{the series } \frac{2x^2}{36} + \frac{2x^6}{40} \\
&\quad + \frac{2x^{10}}{44} + \frac{2x^{14}}{48} + \&c \text{ ad infinitum} \\
&= A + x^{13} \times \alpha + x^{13} \times 2x \times \&C + x^{13} \times 2x \times \text{the series } \frac{2x^2}{36} + \frac{2x^6}{40} + \frac{2x^{10}}{44} \\
&\quad + \frac{2x^{14}}{48} + \&c \text{ ad infinitum} \\
&= A + x^{13} \times \alpha + x^{13} \times 2x \times \&C + x^{13} \times 2x \times 2y \times \text{the series } \frac{1}{36} + \frac{xy}{40} \\
&\quad + \frac{x^2}{44} + \frac{x^6}{48} + \frac{x^8}{52} + \frac{x^{10}}{56} + \frac{x^{14}}{60} + \frac{x^{18}}{64} + \&c \text{ ad infinitum} \\
&=
\end{aligned}$$

$$\begin{aligned}
&= A + x^{12} \times \alpha + x^{12} \times 2x \times \zeta + x^{12} \times 2x \times 2xx \times \text{the series } \frac{1}{36} + \frac{27}{40} + \frac{2^4}{44} + \frac{2^6}{48} + \frac{2^8}{52} + \frac{2^{10}}{56} + \frac{2^{12}}{60} + \frac{2^{14}}{64} + \&c \text{ ad infinitum} \\
&= A + x^{12} \times \alpha + x^{12} \times 2x \times \zeta + x^{12} \times 2x \times 2xx \times \text{the series } \frac{1}{36} - \frac{27}{40} + \frac{2^4}{44} - \frac{2^6}{48} + \frac{2^8}{52} - \frac{2^{10}}{56} + \frac{2^{12}}{60} - \frac{2^{14}}{64} + \&c \text{ ad infinitum} + x^{12} \times 2x \times 2xx \times \text{the series } \frac{27}{40} + \frac{27^2}{48} + \frac{27^{10}}{56} + \frac{27^{14}}{64} + \&c \text{ ad infinitum} \\
&= A + x^{12} \times \alpha + x^{12} \times 2x \times \zeta + x^{12} \times 2x \times 2xx \times \text{the series } \frac{1}{36} - \frac{\pi}{40} + \frac{\pi^2}{44} - \frac{\pi^2}{48} + \frac{\pi^4}{52} - \frac{\pi^4}{56} + \frac{\pi^6}{60} - \frac{\pi^6}{64} + \&c \text{ ad infinitum} + x^{12} \times 2x \times 2xx \times \text{the series } \frac{2\pi}{40} + \frac{2\pi^2}{48} + \frac{2\pi^3}{56} + \frac{2\pi^7}{64} + \&c \text{ ad infinitum} \\
&= A + x^{12} \times \alpha + x^{12} \times 2x \times \zeta + x^{12} \times 2x \times 2xx \times \gamma + x^{12} \times 2x \times 2xx \times \text{the series } \frac{2\pi}{40} + \frac{2\pi^2}{48} + \frac{2\pi^3}{56} + \frac{2\pi^7}{64} + \frac{2\pi^9}{72} + \frac{2\pi^{11}}{80} + \frac{2\pi^{13}}{88} + \frac{2\pi^{15}}{96} + \&c \text{ ad infinitum} \\
&= A + x^{12} \times \alpha + x^{12} \times 2x \times \zeta + x^{12} \times 2x \times 2xx \times \gamma + x^{12} \times 2x \times 2xx \times 2z \times \text{the series } \frac{1}{40} + \frac{\pi^2}{48} + \frac{\pi^4}{56} + \frac{\pi^6}{64} + \frac{\pi^8}{72} + \frac{\pi^{10}}{80} + \frac{\pi^{12}}{88} + \frac{\pi^{14}}{96} + \&c \text{ ad infinitum} \\
&= A + x^{12} \times \alpha + x^{12} \times 2x \times \zeta + x^{12} \times 2x \times 2xx \times \gamma + x^{12} \times 2x \times 2xx \times 2x^4 \times \text{the series } \frac{1}{40} + \frac{2\pi^2}{48} + \frac{\pi^4}{56} + \frac{\pi^6}{64} + \frac{\pi^8}{72} + \frac{\pi^{10}}{80} + \frac{\pi^{12}}{88} + \frac{\pi^{14}}{96} + \&c \text{ ad infinitum} \\
&= A + x^{12} \times \alpha + x^{12} \times 2x \times \zeta + x^{12} \times 2x \times 2xx \times \gamma + x^{12} \times 2x \times 2xx \times 2x^4 \times \text{the series } \frac{1}{40} + \frac{2\pi^2}{48} + \frac{2\pi^2}{56} + \frac{2\pi^3}{64} + \frac{2\pi^4}{72} + \frac{2\pi^5}{80} + \frac{2\pi^6}{88} + \frac{2\pi^7}{96} + \&c \text{ ad infinitum} \\
&= A + x^{12} \times \alpha + x^{12} \times 2x \times \zeta + x^{12} \times 2x \times 2xx \times \gamma + x^{12} \times 2x \times 2xx \times 2x^4 \times \text{the series } \frac{1}{40} - \frac{2\pi^2}{48} + \frac{2\pi^2}{56} - \frac{2\pi^3}{64} + \frac{2\pi^4}{72} - \frac{2\pi^5}{80} + \frac{2\pi^6}{88} - \frac{2\pi^7}{96} + \&c \text{ ad infinitum} + x^{12} \times 2x \times 2xx \times 2x^4 \times \text{the series } \frac{2\pi^2}{48} + \frac{2\pi^3}{64} + \frac{2\pi^3}{80} + \frac{2\pi^7}{96} + \&c \text{ ad infinitum} \\
&= A + x^{12} \times \alpha + x^{12} \times 2x \times \zeta + x^{12} \times 2x \times 2xx \times \gamma + x^{12} \times 2x \times 2xx \times 2x^4 \times \delta + x^{12} \times 2x \times 2xx \times 2x^4 \times 2v \times \text{the series } \frac{1}{48} + \frac{2\pi^2}{64} + \frac{2\pi^2}{80} + \frac{2\pi^6}{96} + \&c \text{ ad infinitum}
\end{aligned}$$

=

$$= A + x^{13} \times \alpha + x^{13} \times 2x \times 6 + x^{13} \times 2x \times 2xx \times \gamma + x^{13} \times 2x \times 2xx \times 2x^4 \times \delta + x^{13} \times 2x \times 2xx \times 2x^4 \times 2x^8 \times \text{the series } \frac{1}{48} + \frac{v^1}{64}$$

$$+ \frac{v^4}{80} + \frac{v^6}{96} + \frac{v^8}{104} + \frac{v^{10}}{120} + \frac{v^{12}}{136} + \frac{v^{14}}{152} + \frac{v^{16}}{168} + \frac{v^{18}}{184} + \frac{v^{20}}{200} + \frac{v^{22}}{216} + \&c$$

$$= A + x^{13} \times \alpha + x^{13} \times 2x \times 6 + x^{13} \times 2x \times 2xx \times \gamma + x^{13} \times 2x \times 2xx \times 2x^4 \times \delta + x^{13} \times 2x \times 2xx \times 2x^4 \times 2x^8 \times M$$

$$= A + x^{13} \times \alpha + x^{13} \times 2x \times 6 + x^{13} \times 4x^1 \times \gamma + x^{13} \times 8x^7 \times \delta + x^{13} \times 16x^{13} \times M$$

$$= A + x^{13} \times \text{the quinquinomial quantity } \alpha + 2x \times 6 + 4x^1 \times \gamma + 8x^7 \times \delta + 16x^{13} \times M$$

$$= A + x^{13} \times \text{the quinquinomial quantity}$$

$$\begin{aligned} & 0.016,177,427,297,506,659, \\ & + 2x \times 0.016,674,011,463,729,394, \\ & + 4x^1 \times 0.017,490,787,392,365,570, \\ & + 8x^7 \times 0.018,438,562,104,429,149, \\ & + 16x^{13} \times 0.024,236,737,151,909,819 \end{aligned}$$

$$= A + x^{13} \times \text{the quinquinomial quantity}$$

$$\begin{aligned} & 0.016,177,427,297,506,659, \\ & + 2 \times 0.9 \times 0.016,674,011,463,729,394, \\ & + 4 \times 0.729 \times 0.017,490,787,392,365,570, \\ & + 8 \times 0.478,296,9 \times 0.018,438,562,104,429,149, \\ & + 16 \times 0.205,891,132,094,649 \times 0.024,236,737,151,909,819, \end{aligned}$$

$$= A + x^{13} \times \text{the quinquinomial quantity}$$

$$\begin{aligned} & 0.016,177,427,297,506,659, \\ & + 1.8 \times 0.016,674,011,463,729,394, \\ & + 2.916 \times 0.017,490,787,392,365,570, \\ & + 3.826,375,2 \times 0.018,438,562,104,429,149, \\ & + 3.294,258,113,514,384 \times 0.024,236,737,151,909,819, \end{aligned}$$

$$= A + x^{13} \times \text{the quinquinomial quantity}$$

$$\begin{aligned} & 0.016,177,427,297,506,659, \\ & + 0.030,013,220,634,712,909, \\ & + 0.051,003,136,036,138,002, \\ & + 0.070,552,856,760,047,505, \\ & + 0.079,842.068,007,794,424, \end{aligned}$$

$$= A + x^{13} \times 0.247,588,708,736,199,499,$$

=

$$= A + 0.034,336,838,202,925,120 \times 0.247,588,708,736,199,499$$

$$= A + 0.008,501,413,432,746,036$$

$$= 2.549,926,467,204,303,322 + 0.008,501,413,432,746,036$$

$$= 2.558,427,880,637,049,358.$$

Therefore the original series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$ *ad infinitum*, will be $(= x \times 2.558,427,880,637,049,358 = 0.9 \times 2.558,427,880,637,049,358) = 2.302,585,092,573,344,422.$ Q. E. I.

End of the Computation of the Value of the Infinite Series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \&c$ ad infinitum, when x is equal to the Fraction $\frac{9}{10}$, by the foregoing Method of Approximation invented by Mr. Hellins.

Art. 47. I have now gone through the computation of the value of this infinite series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \&c$ *ad infinitum*, when x is equal to $\frac{9}{10}$, by Mr. Hellins's ingenious method of approximation, in as full and plain a manner as I could. But I must confess I have found the labour of performing it much greater than I could have imagined; it having been the employment of myself at first, and afterwards of a very skillfull arithmetician, whose assistance I found it necessary to call-in, for more than three weeks. And therefore, though this method is very ingenious and accurate, it seems much to be wished that the great mathematicians of the age should apply themselves to the subject, and favour us with the discovery of some less laborious method of obtaining, to a great degree of exactness, the values of these slowly-converging serieses of the foregoing form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ *ad infinitum*, in which all the terms are connected with the first term by the sign +, or are added to it.

Art. 48. In the mean time, or until some tolerably easy method of obtaining a very exact value of a slowly-converging series of the said form shall be discovered, the following imperfect and conjectural method of obtaining a value of such a series to a moderate degree of exactness, (as, for example, to four or five places of decimal figures,) will be found to be of considerable use in practice.

A Con-

A Conjectural Method of obtaining the Value of a slowly-converging Series of the foregoing Form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ ad infinitum, (in which all the Terms are connected together by the Sign +, or added to each other,) to a moderate Degree of Exactness.

Art. 49. In these slowly converging serieses the several numbers $a, b, c, d, e, f, g, b, \&c$ almost always decrease more slowly than the terms of a geometrical progression that begins with the same two first terms a and b ; so that the third, fourth, fifth, and other following terms of the series $a + b + c + d + e + f + g + b, \&c$ will be respectively greater than the third, fourth, fifth, and other corresponding terms of the geometrical progression $a + b + \frac{b^2}{a} + \frac{b^3}{a^2} + \frac{b^4}{a^3} + \frac{b^5}{a^4} + \frac{b^6}{a^5} + \frac{b^7}{a^6} + \&c$. And consequently the series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ ad infinitum will be greater than the geometrical series $a + bx + \frac{b^2x^2}{a} + \frac{b^3x^3}{a^2} + \frac{b^4x^4}{a^3} + \frac{b^5x^5}{a^4} + \frac{b^6x^6}{a^5} + \frac{b^7x^7}{a^6} + \&c$ ad infinitum. Now the value of this geometrical series continued-on ad infinitum is equal to the fraction $\frac{aa}{a - bx}$. Therefore the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ will be greater than the fraction $\frac{aa}{a - bx}$; which will therefore be the lesser limit of the magnitude of the value of the said series.

Further, the said series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ ad infinitum, is evidently less than the geometrical series $a + ax + ax^2 + ax^3 + ax^4 + ax^5 + ax^6 + ax^7 + \&c$ ad infinitum, because the numbers $a, b, c, d, e, f, g, b, \&c$ are supposed to be a decreasing progression. But the said geometrical progression $a + ax + ax^2 + ax^3 + ax^4 + ax^5 + ax^6 + ax^7 + \&c$ ad infinitum is equal to the fraction $\frac{aa}{a - ax}$, or $\frac{a}{1 - x}$. Therefore the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ will be less than the said fraction $\frac{a}{1 - x}$; which will therefore be the greater limit of the magnitude of the said series. If therefore we suppose the said series to be equal to any quantity of an intermediate magnitude between the two fractions $\frac{aa}{a - bx}$ and $\frac{aa}{a - ax}$, such quantity will be a near value of the said series. But I have found,

in the case of the foregoing flow series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$ *ad infinitum*, when x is equal to $\frac{9}{10}$, that the true value of the said series approaches much nearer to it's lesser limit, or the fraction $\frac{aa}{a-bx}$,

than to it's greater limit, or the fraction $\frac{a}{1-x}$; and that, if we subtract the said lower limit $\frac{aa}{a-bx}$ from the higher limit $\frac{a}{1-x}$, and divide the remainder, or difference, into 8 equal parts, and add one of these parts to the said lower limit, $\frac{aa}{a-bx}$, the sum thence arising will be very nearly equal to the true value of the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ *ad*

infinitum. For the fraction $\frac{a}{1-x}$ is $(= \frac{\frac{1}{33}}{1 - \frac{9}{10}} = \frac{\frac{1}{33}}{\frac{1}{10}} = \frac{1}{33} \times \frac{10}{1} = \frac{10}{33}) =$

0.303,030, &c, which is considerably greater than 0.247,588, &c, which we have seen in the foregoing articles to be the true value of the proposed series

$\frac{1}{23} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$ *ad infinitum*; and

the fraction $\frac{aa}{a-bx}$ is $(= \frac{\frac{1}{33} \times \frac{1}{34}}{\frac{1}{33} - \frac{x}{34}} = \frac{\frac{1}{33} \times \frac{1}{34}}{\frac{34 - 33x}{33 \times 34}} = \frac{\frac{1}{33}}{\frac{34 - 33x}{34}} = \frac{1}{33} \times \frac{34}{34 - 33x}) =$

$= \frac{\frac{34}{33 \times 34 - 33 \times 33x}}{1122 - 1089x} = \frac{34}{1122 - 1089 \times 0.9} = \frac{34}{1122 - 980.1} = \frac{34}{141.9} = \frac{340}{1419} = 0.239,605, \&c$, which is less than 0.247,588, &c, or the

said true value of the proposed series. But the said true value 0.247,588, &c is much nearer to the lower limit 0.239,605, &c, than to the higher limit, 0.303,030, &c; and, if we subtract the lower limit 0.239,605, &c from the higher limit, 0.303,030, &c, and divide the difference 0.063,425 by 8, and add the quotient of this division, to wit, 0.007,928, to the lower limit, 0.239,605, the sum will be equal to 0.247,533, which is very nearly equal to, but something less than, 0.247,588, or the true value of the said proposed series; or, in other words, the said proposed series will be very nearly equal to, but a little greater than, the first, or least, of seven arithmetical mean proportionals interposed between the values of the said two geometrical serieses $a + bx + \frac{bx^2}{a} + \frac{bx^3}{a^2} + \frac{bx^4}{a^3} + \frac{bx^5}{a^4} + \frac{bx^6}{a^5} + \&c$ *ad infinitum* and $a + ax + ax^2 + ax^3 + ax^4 + ax^5 + \&c$ *ad infinitum*.

Q. E. I.

Art. 50.

Art. 50. If we suppose the series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$ *ad infinitum* to be equal to the value just now found for it by this conjectural method of approximation, to wit, the number 0.247,533, the series $\frac{x^{10}}{33} + \frac{x^{11}}{34} + \frac{x^{12}}{35} + \frac{x^{13}}{36} + \frac{x^{14}}{37} + \frac{x^{15}}{38} + \frac{x^{16}}{39} + \frac{x^{17}}{40} + \&c$ *ad infinitum* will be $(= x^{10} \times 0.247,533 = 0.034,336,838,202,925,120 \times 0.247,533) = 0.008,499,500,570,884,163$; which, being added to 2549,926,467 224,303, 322, or the sum of the first thirty-two terms of the series $1 + \frac{x}{9} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + \frac{x^8}{9} + \frac{x^9}{10} + \&c$ *ad infinitum*, will give us the number 2.558,425,967,775,187,985 for the value of the whole of the said series. And consequently the original series $x' + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \&c$ *ad infinitum* will be $(= x \times 2.558,425,967,775,187,985 = 0.9 \times 2.558,425,967,775,187,985) = 2.302,583,370,997,669,186$; which agrees with the true value of the said series, to wit, 2 302,585,092, 994,045,684,017, &c, in the first six figures 2.302,58, and falls short of the said true value by a less quantity than 0.000,002, or the 1,155,292nd part of the said true value; which, for such a conjectural method of proceeding, may be considered as a surprising degree of exactness. Q. E. T.

Art. 51. As this method of obtaining a near value of the slowly-converging series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$ *ad infinitum*, though very successful, is only conjectural, and therefore would leave us in doubt as to the degree of exactness with which it approaches to the true value of the said series, if we did not know already, from other grounds, that the said true value is 0.247,538, &c, it will, I think, be useful to have recourse on these occasions to more than one method of obtaining a near value of the same series, to the end that the near agreement which the several values of the proposed series, obtained by these different methods, may be found to have with each other, may serve to increase our confidence that each of the values of the said series so obtained, approaches tolerably near to its true value. And therefore I will now proceed to describe a second method of finding a near value of a slowly-converging series of the foregoing form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ *ad infinitum* that is but little different from the foregoing one, and that will enable us to approach as nearly, or almost as nearly, to the true value of the said series, as we did by the foregoing method, but not with quite so little calculation. This method is as follows:

Another Conjectural Method of obtaining the Value of a slowly-converging Infinite Series of the foregoing Form $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7$ &c ad infinitum to a moderate Degree of Exactness.

Art. 52. Let this series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ ad infinitum be divided into the two following serieses, $a + cx^2 + ex^4 + gx^6 + \&c$ ad infinitum and $bx + dx^3 + fx^5 + hx^7 + \&c$ ad infinitum; of which we will call the former y , and the latter z . And let the values of each of these two serieses y and z be sought for in the following manner, (which is nearly the same with that in which we before sought for the value of the whole series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ ad infinitum, or $y + z$), and then added to each other. And the sum thence arising will be a near value of $y + z$, or the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ ad infinitum. The values of the two serieses y and z may be found in the following manner:

Art. 53. The series y , or $a + cx^2 + ex^4 + gx^6 + \&c$ ad infinitum is evidently less than the geometrical series $a + ax^2 + ax^4 + ax^6 + \&c$ ad infinitum, but greater than the geometrical series $a + cx^2 + \frac{c^2x^4}{a} + \frac{c^3x^6}{a^2} + \&c$ ad infinitum; and therefore it will be less than the fraction $\frac{aa}{a - ax^2}$, or $\frac{a}{1 - x^2}$, which is equal to the former of the said geometrical serieses, and greater than the fraction $\frac{aa}{a - cx^2}$, which is equal to the latter of them. And, further, it will be much nearer to the latter series, or it's lower limit, than to the former, or it's higher limit; just as the series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ ad infinitum was much nearer to the geometrical series $a + bx + \frac{bx^2}{a} + \frac{b^2x^3}{a^2} + \frac{b^3x^4}{a^3} + \&c$ ad infinitum, or it's lower limit, than to the geometrical series $a + ax + ax^2 + ax^3 + ax^4 + \&c$ ad infinitum, or it's higher limit. And, as the series $a + cx^2 + ex^4 + gx^6 + \&c$ converges by the powers of the square of x , whereas the series $a + bx + cx^2 + dx^3 + ex^4 + \&c$ converges only by the powers of x itself, and the numeral co-efficients a, c, e, g , &c of the terms of the former series likewise decrease somewhat faster, or less slowly, than the numeral co-efficients a, b, c, d , &c of the corresponding terms of the latter series, it seems reasonable to suppose that the former series will approach something nearer to it's lower limit, or the geometrical series $a + cx^2 + \frac{c^2x^4}{a}$.

+ $\frac{c^2x^4}{a^2}$ + &c, than the latter series $a + bx + cx^2 + dx^3 + ex^4 + \&c$, approaches to it's lower limit, or the geometrical series $a + bx + \frac{b^2x^2}{a} + \frac{b^3x^3}{a^2} + \frac{b^4x^4}{a^3} + \&c$. We will therefore compute the values of the two geometrical series $a + ax^2 + ax^4 + ax^6 + ax^8 + \&c$ *ad infinitum* and $a + cx^2 + \frac{c^2x^4}{a} + \frac{c^3x^6}{a^2} + \frac{c^4x^8}{a^3} + \&c$ *ad infinitum*, or the values of the two fractions $\frac{a}{1 - x^2}$ and $\frac{aa}{a - cx^2}$, (which are equal to the said serieses,) and will then subtract the value of the fraction $\frac{aa}{a - cx^2}$ from the value of the fraction $\frac{a}{1 - x^2}$, and divide the remainder into nine equal parts, (instead of eight equal parts, as in the former method,) and add one of the said parts to the value of the fraction $\frac{aa}{a - cx^2}$. And the sum thence arising will be a near value of the series y , or $a + cx^2 + ex^4 + gx^6 + \&c$ *ad infinitum*. Q. E. I.

Art. 54. And, in like manner, the series z , or $bx + dx^3 + fx^5 + hx^7 + \&c$ *ad infinitum* will be less than the geometrical series $bx + bx^3 + bx^5 + bx^7 + \&c$ *ad infinitum*, but greater than the geometrical series $bx + dx^3 + \frac{d^2x^5}{b} + \frac{d^3x^7}{b^2} + \&c$ *ad infinitum*; and therefore it will be less than the fraction $\frac{b^2x^2}{bx - dx^3}$, or $\frac{bx}{1 - x^2}$, which is equal to the former of the said geometrical serieses, but greater than the fraction $\frac{b^2x^2}{bx - dx^3}$, or $\frac{b^2x}{b - dx^2}$, which is equal to the latter of them.

And, further, the said series z will be much nearer to the latter series, or it's lower limit, than to the former series, or it's higher limit; just as the series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ was much nearer to the geometrical series $a + bx + \frac{b^2x^2}{a} + \frac{b^3x^3}{a^2} + \frac{b^4x^4}{a^3} + \frac{b^5x^5}{a^4} + \&c$ *ad infinitum*, or it's lower limit, than to the geometrical series $a + ax + ax^2 + ax^3 + ax^4 + ax^5 + \&c$ *ad infinitum*, or it's higher limit. And, as the series $bx + dx^3 + fx^5 + bx^7 + \&c$ converges by the powers of the square of x , whereas the series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ converges only by the powers of x itself, and the numeral co-efficients b, d, f, b , &c of the terms of the former series likewise decrease somewhat faster, or less slowly, than the numeral co-efficients a, b, c, d , &c of the corresponding terms of the latter series, it seems reasonable to suppose that the former series will approach something nearer to it's lower limit, or the geometrical series $bx + dx^3 + \frac{d^2x^5}{b} + \frac{d^3x^7}{b^2} + \&c$ *ad infinitum*.

$\frac{d^2x^2}{b^2} + \frac{d^4x^4}{b^4} + \&c$ *ad infinitum*, than the latter series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \&c$ approaches to it's lower limit, or the geometrical series $a + bx + \frac{b^2x^2}{a} + \frac{b^3x^3}{a^2} + \frac{b^4x^4}{a^3} + \frac{b^5x^5}{a^4} + \&c$. We will therefore compute the values of the two geometrical series $bx + bx^2 + bx^3 + bx^4 + \&c$ *ad infinitum* and $bx + dx^2 + \frac{d^2x^3}{b} + \frac{d^3x^4}{b^2} + \&c$ *ad infinitum*, or the values of the two fractions $\frac{b^2x^2}{b^2 - bx}$, or $\frac{bx}{1 - \frac{bx}{a}}$, and $\frac{d^2x^3}{bx - dx^2}$, or $\frac{bx}{b - dx}$, (which are equal to the said series,) and will then subtract the value of the fraction $\frac{bx}{b - dx}$ from the value of the fraction $\frac{bx}{1 - \frac{bx}{a}}$, and divide the remainder into nine equal parts, (instead of eight equal parts, as in the former method,) and add one of the said parts to the value of the fraction $\frac{b^2x}{b - \frac{bx}{a}}$. And the sum will be a near value of the series z , or $bx + dx^2 + fx^3 + bx^4 + \&c$ *ad infinitum*. Q. E. I.

Art. 55. When we have thus found near values of the two series y and z , or $a + cx^3 + ex^4 + gx^5 + \&c$ *ad infinitum* and $bx + dx^2 + fx^3 + lx^4 + \&c$ *ad infinitum*, we must add these two values together; and their sum will be a near value of $y + z$, or the proposed series $a + bx + cx^3 + dx^2 + ex^4 + fx^3 + gx^5 + bx^4 + \&c$ *ad infinitum*. Q. E. I.

And, if this near value of the proposed series $a + bx + cx^3 + dx^2 + ex^4 + fx^3 + gx^5 + bx^4 + \&c$ *ad infinitum* differs but little from the near value of it found above by the former conjectural method described above in art. 49, we may conclude with confidence that each of the conjectural values of the said series will be nearly equal to it's true value.

Art. 56. We will now try this method of obtaining a near value of the said series $a + bx + cx^3 + dx^2 + ex^4 + fx^3 + gx^5 + bx^4 + \&c$ *ad infinitum*, in the case of the foregoing series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$ *ad infinitum*, when x is equal to the fraction $\frac{9}{10}$, or 0.9.

Now in this series a is $= \frac{1}{33} = 0.030,303,030$, and b is $= \frac{1}{34}$, and c is $= \frac{1}{35}$, and d is $= \frac{1}{36}$; and consequently lx will be $(= \frac{1}{34} \times 0.9 = \frac{0.9}{34}) = 0.026,470,588$, and cx^3 will be $(= \frac{1}{35} \times 0.81 = \frac{0.81}{35}) = 0.023,142,857$, and dx^2 will be $(= \frac{1}{36} \times 0.81 = \frac{0.81}{36}) = 0.022,500,$

500,000, and b^2x will be ($= b \times bx = b \times 0.026,470,588 = \frac{1}{34} \times 0.026,470,588 = \frac{0.026,470,588}{34}$) = 0.000,778,546, and $1 - xx$ will be ($= 1 - 0.81 = 1.00 - 0.81$) = 0.19, and $a - cx^4$ will be ($= 0.030,303,030 - 0.023,142,857$) = 0.007,160,173, and $b - dxn$ will be ($= \frac{1}{34} - 0.022,500,000 = 0.029,111,764 - 0.022,500,000$) = 0.006,911,764. Therefore $\frac{a}{1 - xx}$ will be ($= \frac{0.030,303,030}{0.19}$) = 0.159,489, and $\frac{ax}{a - cx^4}$ will be ($= \frac{0.030,303,030 \times 0.007,160,173}{0.007,160,173}$) = 0.128,240; that is, the value of the geometrical series $a + ax^3 + ax^4 + ax^5 + ax^6 + ax^7 + ax^{10} + \&c$ ad infinitum, or $\frac{1}{33} + \frac{x^3}{33} + \frac{x^4}{33} + \frac{x^5}{33} + \frac{x^6}{33} + \frac{x^{10}}{33} + \&c$ ad infinitum, will be 0.159,489, and the value of the geometrical series $a + cx^4 + \frac{c^2x^8}{a} + \frac{c^3x^{12}}{a^2} + \frac{c^4x^{16}}{a^3} + \frac{c^5x^{20}}{a^4} + \&c$ ad infinitum, or $\frac{1}{33} + \frac{x^4}{35} + \frac{33x^8}{35^2} + \frac{3^2 \times x^{12}}{35^3} + \frac{3^3 \times x^{16}}{35^4} + \frac{3^4 \times x^{20}}{35^5} + \&c$ ad infinitum, will be 0.128,240.

Now let 0.128,240 be subtracted from 0.159,489; and the remainder will be 0.031,249; of which the 9th part is 0.003,472. We must therefore add this last number 0.003,472 to 0.128,240, or the value of the geometrical series $\frac{1}{33} + \frac{x^3}{35} + \frac{33x^8}{35^2} + \frac{3^2 \times x^{12}}{35^3} + \frac{3^3 \times x^{16}}{35^4} + \frac{3^4 \times x^{20}}{35^5} + \&c$ ad infinitum; and the sum thence arising, to wit, the number 0.131,712, will be a near value of the series y , or $\frac{1}{33} + \frac{x^3}{35} + \frac{x^4}{37} + \frac{x^5}{39} + \frac{x^6}{41} + \frac{x^{10}}{43} + \&c$ ad infinitum.

Q. E. I.

We must next, in like manner, find the values of the two geometrical series. $bx + bx^3 + bx^4 + bx^5 + bx^6 + bx^{11} + \&c$ ad infinitum, or $\frac{x}{34} + \frac{x^3}{34} + \frac{x^4}{34} + \frac{x^5}{34} + \frac{x^6}{34} + \frac{x^{11}}{34} + \&c$ ad infinitum, and $bx - dx^3 + \frac{a^2x^5}{b} + \frac{a^3x^7}{b^2} + \frac{a^4x^9}{b^3} + \frac{a^5x^{11}}{b^4} + \&c$ ad infinitum, or $\frac{x}{34} + \frac{x^3}{36} + \frac{34x^5}{34^2} + \frac{3^2 \times x^7}{34^3} + \frac{3^3 \times x^9}{34^4} + \frac{3^4 \times x^{11}}{34^5} + \&c$ ad infinitum.

Now the former of these series is equal to the fraction $\frac{bx^2}{bx - bx^2}$, or $\frac{bx}{1 - xx}$; and.

and the latter is equal to the fraction $\frac{b^2x}{bx - dx^2}$, or $\frac{b^2x}{b - dx}$. And therefore we must compute the values of the fractions $\frac{bx}{1 - ax}$ and $\frac{b^2x}{b - dx}$, which may be done as follows :

Since bx has been shewn to be $= 0.026,470,588$, and b^2x to be $= 0.000,773,546$, and $1 - ax$ to be $= 0.19$, and $b - dx$ to be $= 0.006,911,764$, we shall have $\frac{bx}{1 - ax} (= \frac{0.026,470,588}{0.19}) = 0.139,318$, and $\frac{b^2x}{b - dx} (= \frac{0.000,773,546}{0.006,911,764}) = 0.112,640$. Q. E. I.

Now let $0.112,640$ be subtracted from $0.139,318$; and the remainder will be $= 0.026,678$; of which the 9th part is $= 0.002,964$. We must therefore add this last number $0.002,964$ to $0.112,640$; and the sum thence arising, to wit, the number $0.115,604$, will be a near value of the series a , or $bx + dx^2 + fx^3 + bx^2 + kx^3 + mx^4 + \&c$ *ad infinitum*, or $\frac{x}{34} + \frac{x^2}{36} + \frac{x^3}{38} + \frac{x^4}{40} + \frac{x^5}{42} + \frac{x^6}{44} + \&c$ *ad infinitum*. Q. E. I.

Therefore $0.131,712 + 0.115,604$, or $0.247,316$, will be a near value of $y + z$, or the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^2 + \&c$ *ad infinitum*, or $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$ *ad infinitum*. Q. E. I.

Art. 57. This number $0.247,316$ is not quite so near the true value of the said proposed series, (which is $0.247,588$;) as the number $0.247,533$, which was obtained above in art. 49, by means of the first conjectural method of summing the said series there described. But it is near enough to the said former near value, $0.247,533$, of the said series, to render it highly probable (if we did not already know what its true value was,) that each of them approaches pretty nearly to the true value of the said series.

A Third Conjectural Method of finding a near Value of a slowly converging Infinite Series of the foregoing Form, $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + \&c$ ad infinitum.

Art. 58. If we could find the proportion of the series y , or $a + cx^3 + ex^4 + gx^5 + ix^6 + lx^{10} + \&c$ ad infinitum, to the series z , or $bx + dx^2 + fx^3 + bx^7 + kx^9 + mx^{11} + \&c$ ad infinitum, to a great degree of exactness, (as, for example, to the degree of exactness that might be expressed by two numbers of ten or twelve decimal figures in each,) without knowing the actual magnitudes of each of the said two serieses, we might, by means of only one application of the differential series $a - \frac{bx}{1+x} - \frac{b^2x^2}{1+x^2} - \frac{b^3x^3}{1+x^3} - \frac{b^{11}x^{11}}{1+x^{11}} -$

$\frac{b^{12}x^{12}}{1+x^{12}} - \&c$, discover the value of $y + z$, or of the proposed series $a + bx$

$+ cx^3 + dx^2 + ex^4 + fx^5 + gx^6 + bx^7 + ix^6 + kx^9 + lx^{10} + mx^{11} + \&c$ ad infinitum, to the same degree of exactness. For, if in that case the two known numbers that expressed the proportion of y to z , or of the series $a + cx^3 + ex^4 + gx^5 + ix^6 + lx^{10} + \&c$ ad infinitum to the series $bx + dx^2 + fx^3 + bx^7 + kx^9 + mx^{11} + \&c$ ad infinitum, were to be called p and q , we should have $p : q :: y : z$, and consequently (*componendo*,) $p + q : p :: y + z : y$, and (*dividendo*,) $p - q : p :: y - z : y$, and (*invertendo*,) $p : p - q :: y : y - z$, and therefore (*ex æquo*,) $p + q : p - q :: y + z : y - z$, and

$y + z = \frac{p+q}{p-q} \times y - z$. But $y - z$ is equal to the series $a - bx + cx^3 - dx^2 + ex^4 - fx^5 + gx^6 - bx^7 + ix^6 - kx^9 + lx^{10} - mx^{11} + \&c$ ad infinitum, the value of which may be found by means of the differential series

$a - \frac{bx}{1+x} - \frac{b^2x^2}{1+x^2} - \frac{b^3x^3}{1+x^3} - \frac{b^{11}x^{11}}{1+x^{11}} - \frac{b^{12}x^{12}}{1+x^{12}} - \&c$ ad infinitum. Let

it be so found to the same degree of exactness as the proportion of y to z is expressed by the known numbers p and q ; and let it be called K . Then will $y + z$, or the proposed series $a + bx + cx^3 + dx^2 + ex^4 + fx^5 + gx^6 + bx^7$

$+ ix^6 + kx^9 + lx^{10} + mx^{11} + \&c$ ad infinitum be $= \frac{p+q}{p-q} \times K$. But I know

of no way of discovering the proportion of y , or the series $a + cx^3 + ex^4 + gx^5 + ix^6 + lx^{10} + \&c$ ad infinitum, to z , or the series $bx + dx^2 + fx^3 + bx^7 + kx^9 + mx^{11} + \&c$ ad infinitum, to so great a degree of exactness, or to more than the degree of exactness that may be expressed by two numbers containing each three, or, at most, four, decimal figures. And therefore this method of obtaining a near value of $y + z$, or the proposed series $a + bx + cx^3 + dx^2$

+ $ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*, notwithstanding the assistance it seems to receive from the foregoing differential series, will not be found preferable to either of the two foregoing methods, which proceed without the use of that series. It may, however, be used to obtain a third near value of $y + z$, or the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*, to a moderate degree of exactness, as, for example, to three or four places of decimal figures; which value may be compared to the former near values of it found by the two foregoing methods, and will, if found to agree pretty nearly with those former near values of the said series, serve greatly to confirm them, and to give us further grounds for thinking that each of the values of the said series obtained by these three different methods of investigation will be nearly equal to its true value.

Art. 59. In this last method of seeking a near value of the proposed series $y + z$, or $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + ix^8 + kx^9 + lx^{10} + mx^{11} + \&c$ *ad infinitum*, our business is to find, to a moderate degree of exactness, the proportion of y to z , or of the series $a + cx^2 + ex^4 + gx^6 + ix^8 + lx^{10} + \&c$ *ad infinitum* to the series $bx + dx^3 + fx^5 + bx^7 + kx^9 + mx^{11} + \&c$ *ad infinitum*. Now I have found that in the case of the series $\frac{x}{33} + \frac{x^2}{34} + \frac{x^3}{35} + \frac{x^4}{36} + \frac{x^5}{37} + \frac{x^6}{38} + \frac{x^7}{39} + \frac{x^8}{40} + \frac{x^9}{41} + \frac{x^{10}}{42} + \frac{x^{11}}{43} + \frac{x^{12}}{44} + \&c$ *ad infinitum*, when x is $= \frac{9}{10}$, or 0.9, the proportion of the whole series $a + cx^2 + ex^4 + gx^6 + ix^8 + lx^{10} + \&c$ *ad infinitum* to the whole series $bx + dx^3 + fx^5 + bx^7 + kx^9 + mx^{11} + \&c$ *ad infinitum* will be pretty nearly equal to that of the quadrinomial quantity $cx^4 + gx^6 + ix^8 + lx^{10}$, or $\frac{x^4}{37} + \frac{x^6}{39} + \frac{x^8}{41} + \frac{x^{10}}{43}$, to the quadrinomial quantity $fx^5 + bx^7 + kx^9 + mx^{11}$, or $\frac{x^5}{38} + \frac{x^7}{40} + \frac{x^9}{42} + \frac{x^{11}}{44}$, or of the sum of the 3d, 4th, 5th, and 6th terms of the series y , or $a + cx^2 + ex^4 + gx^6 + ix^8 + lx^{10} + \&c$ *ad infinitum*, or $\frac{x^2}{33} + \frac{x^4}{35} + \frac{x^6}{37} + \frac{x^8}{39} + \frac{x^{10}}{41} + \frac{x^{12}}{43} + \&c$ *ad infinitum*, to the sum of the 3d, 4th, 5th, and 6th terms of the series z , or $bx + dx^3 + fx^5 + bx^7 + kx^9 + mx^{11} + \&c$ *ad infinitum*, or $\frac{x}{34} + \frac{x^3}{36} + \frac{x^5}{38} + \frac{x^7}{40} + \frac{x^9}{42} + \frac{x^{11}}{44} + \&c$ *ad infinitum*. For $\frac{x^4}{37}$ is $(= \frac{0.6561}{37}) = 0.017,732,432$, and $\frac{x^6}{39}$ is $(= \frac{0.531,441,000}{39}) = 0.013,626,692$, and $\frac{x^8}{41}$ is $(= \frac{0.430,467,210}{41}) = 0.010,499,200$, and $\frac{x^{10}}{43}$ is $(= \frac{0.343,78,440}{43}) = 0.008,108,800$; and consequently the quadrinomial quantity $\frac{x^4}{37} + \frac{x^6}{39} + \frac{x^8}{41}$

+ $\frac{x^{10}}{43}$ will be ($= 0.017,732,432 + 0.013,626,692 + 0.010,499,200 + 0.008,108,800$) $= 0.049,967,124$. And $\frac{x^5}{38}$ is ($= \frac{0.590,490,000}{38} = 0.015,538,210$, and $\frac{x^7}{40}$ is ($= \frac{0.478,296,900}{40} = 0.011,957,422$, and $\frac{x^9}{42}$ is ($= \frac{0.387,427,489}{42} = 0.009,224,297$, and $\frac{x^{11}}{44}$ is ($= \frac{0.313,810,596}{44} = 0.007,132,059$; and consequently the quadrinomial quantity $\frac{x^5}{38} + \frac{x^7}{40} + \frac{x^9}{42} + \frac{x^{11}}{44}$ will be ($= 0.015,538,210 + 0.011,957,422 + 0.009,224,297 + 0.007,132,059$) $= 0.043,851,988$. Therefore the former quadrinomial quantity will be to the latter as the whole number 49,967,124 to the whole number 43,851,988, or, nearly, as the whole number 49,967 to the whole number 43,852. Now these two numbers are to each other very nearly in the same proportion as the whole series y , or $\frac{1}{33} + \frac{x^3}{35} + \frac{x^5}{37} + \frac{x^7}{39} + \frac{x^9}{41} + \frac{x^{11}}{43} + \&c$ *ad infinitum*, is to the whole series z , or $\frac{x}{34} + \frac{x^3}{36} + \frac{x^5}{38} + \frac{x^7}{40} + \frac{x^9}{42} + \frac{x^{11}}{44} + \&c$ *ad infinitum*. For the true value of the series y is 0.131,882, and the true value of the series z is 0.115,705; and 49,967 is to 43,852 as 0.131,882 is to 0.115,742, which exceeds 0.115,705, or the true value of z , by only the small quantity 0.000,037, which is less than the 3127th part of the said true value. These numbers therefore may be substituted instead of p and q in the expression given above for the value of $y + z$, to wit, the expression $\frac{p+q}{p-q} \times y - z$, or $\frac{p+q}{p-q} \times K$; and consequently the said expression will be $= \frac{49,967 + 43,852}{49,967 - 43,852} \times K = \frac{93,819}{6115} \times K$.

But K , or $y - z$, or the series $\frac{1}{33} - \frac{x}{34} + \frac{x^3}{35} - \frac{x^5}{36} + \frac{x^7}{37} - \frac{x^9}{38} + \frac{x^{11}}{39} - \frac{x^{13}}{40} + \frac{x^{15}}{41} - \frac{x^{17}}{42} + \frac{x^{19}}{43} - \frac{x^{21}}{44} + \&c$ *ad infinitum*, has been shewn above, in art. 24, to be $= 0.016,177,427$.

Therefore $y + z$ will be $= \frac{93,819}{6115} \times 0.016,177,427 (= \frac{1,517,750,023,713}{6115}) = 0.248,201$; that is, the proposed series $\frac{1}{33} + \frac{x}{34} + \frac{x^3}{35} + \frac{x^5}{36} + \frac{x^7}{37} + \frac{x^9}{38} + \frac{x^{11}}{39} + \frac{x^{13}}{40} + \frac{x^{15}}{41} + \frac{x^{17}}{42} + \frac{x^{19}}{43} + \frac{x^{21}}{44} + \&c$ *ad infinitum* will be nearly equal to 0.248,201. Q. E. I.

Art. 60. This near value of the proposed series $\frac{x}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \&c$ *ad infinitum* is greater than it's true value 0.247,588 by the quantity 0.000,613, and is therefore less exact than either of the two former near values of the said series, that were obtained by the two former conjectural methods of investigation, to wit, 0.247,533 and 0.247,316. I therefore think those two former methods are preferable to this last method, notwithstanding the use therein made of the differential series in discovering the value of $y - z$, or the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ *ad infinitum*.

Art. 61. But there is another method of applying our knowledge of the true value of $y - z$, or the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ *ad infinitum*, (obtained by means of the differential series $a - \frac{bx}{1+x} - \frac{b^2x^2}{1+x} - \frac{b^3x^3}{1+x} - \frac{b^4x^4}{1+x} - \frac{b^5x^5}{1+x} - \&c$ *ad infinitum*,) to the discovery of a near value of $y + z$, or the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*, which may, perhaps, prove more successful than the last. This method may be described as follows:

Since $y - z$ is equal to the known quantity K , the square of $y - z$, that is, the trinomial quantity $yy - 2yz + zz$, will be equal to KK , or the square of K . Now $yy + 2yz + zz$, or the square of $y + z$, is $= yy - 2yz + zz + 4yz$, or $KK + 4yz$. Therefore, if we could find the value of yz , or the product of the multiplication of the series y , or $a + cx^2 + cx^3 + gx^4 + ix^5 + lx^{10} + \&c$ *ad infinitum*, into the series z , or $bx + dx^2 + fx^3 + bx^4 + lx^5 + mx^{11} + \&c$, we might, by adding four times the value of that product to $yy - 2yz + zz$, or KK , obtain the value of $yy + 2yz + zz$, or the square of $y + z$; after which, by extracting the square-root of the value of $yy + 2yz + zz$ so obtained, we should obtain the value of $y + z$ itself, or the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + ix^8 + lx^{10} + mx^{11} + \&c$ *ad infinitum*. We must therefore now endeavour to find a tolerably near value of yz , or of the product of the multiplication of the series y , or $a + cx^2 + ex^3 + gx^4 + ix^5 + lx^{10} + \&c$ *ad infinitum* into the series $bx + dx^2 + fx^3 + bx^4 + lx^5 + mx^{11} + \&c$ *ad infinitum*.

Art. 62. Now, in order to find a near value of this product, we may proceed as follows:

Let a near value of y , or the series $a + cx^2 + ex^3 + gx^4 + ix^5 + lx^{10} + \&c$ *ad infinitum* be found in the manner described in the second conjectural method above-mentioned, namely, by computing the values of the two geometrical series $a + ax^2 + ax^3 + ax^4 + ax^5 + ax^{10} + \&c$ *ad infinitum* and

$a + cx^3 + \frac{c^3x^4}{a} + \frac{c^3x^8}{a^2} + \frac{c^4x^8}{a^3} + \frac{c^5x^{10}}{a^4} + \&c$ *ad infinitum*, (between which it lies, and which are equal to the two fractions $\frac{a}{1 - xx}$ and $\frac{aa}{a - cx^3}$;) and then subtracting the latter value from the former, and adding a 9th part of their difference to the latter, or $\frac{aa}{a - cx^3}$. And let this near value of y be called p .

Then let a near value of z , or the series $bx + dx^3 + fx^5 + bx^7 + kx^9 + mx^{11} + \&c$ *ad infinitum* be found in the same manner, to wit, by computing the values of the two geometrical series $bx + bx^3 + bx^5 + bx^7 + bx^9 + bx^{11} + \&c$ *ad infinitum*, and $bx + dx^3 + \frac{d^3x^5}{b} + \frac{d^5x^7}{b^2} + \frac{d^7x^9}{b^3} + \frac{d^9x^{11}}{b^4} + \&c$ *ad infinitum*, (between which it lies, and which are equal to the two fractions $\frac{bx}{1 - xx}$ and $\frac{bx}{b - dxx}$;) and then subtracting the latter value from the former, and adding a 9th part of their difference to the latter, or $\frac{bx}{b - dxx}$. And let this near value of z be called q .

Then will p and q be nearly equal to y and z , but probably somewhat less than the truth. At least they will be so in the case of the series $\frac{1}{33} + \frac{x}{34} + \frac{x^3}{35} + \frac{x^5}{36} + \frac{x^7}{37} + \frac{x^9}{38} + \frac{x^{11}}{39} + \frac{x^{13}}{40} + \frac{x^{15}}{41} + \frac{x^{17}}{42} + \frac{x^{19}}{43} + \frac{x^{21}}{44} + \&c$, as we have seen above in art. 56, page 688.

Art. 63. We will therefore find other near values of y and z that shall be a little greater than p and q , by proceeding in the same manner as before, excepting that we will add the 8th part (instead of the 9th) of the difference of the values of the two fractions $\frac{a}{1 - xx}$ and $\frac{aa}{a - cx^3}$ to the value of $\frac{aa}{a - cx^3}$, in order to obtain a near value of y , and the 8th part (instead of the 9th,) of the difference of the values of the two fractions $\frac{bx}{1 - xx}$ and $\frac{bx}{b - dxx}$ to the value of $\frac{bx}{b - dxx}$, in order to obtain a near value of z . And the near value of y thereby obtained we will call P , and the near value of z thereby obtained we will call Q .

And now we will suppose y to be nearly equal to an arithmetical mean proportional between P and p , or to be nearly equal to $\frac{P+p}{2}$, and z to be nearly equal to an arithmetical mean proportional between Q and q , or to be nearly equal

equal to $\frac{Q+q}{2}$. And then, upon this supposition, yz will be nearly $= \frac{P+p}{2}$
 $\times \frac{Q+q}{2} = \frac{P+p}{4} \times \frac{Q+q}{2}$, and $4yz$ will be nearly $= P+p \times \frac{Q+q}{2}$.
 Therefore $yy + 2yz + zz$ (which is $= yy - 2yz + zz + 4yz = KK + 4yz$.)
 will be nearly $= KK + P+p \times \frac{Q+q}{2}$; and consequently $y + z$ will be
 nearly $= \sqrt{KK + P+p \times \frac{Q+q}{2}}$, or the proposed series $a + bx + cx^2$
 $+ dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + ix^8 + kx^9 + lx^{10} + mx^{11} + \&c$ *ad infi-*
nitum, will be nearly equal to the square-root of the compound quantity
 $KK + P+p \times \frac{Q+q}{2}$. Q. E. I.

Art. 64. We will now try this method of proceeding in the case of the fore-
 going series $\frac{1}{33} + \frac{x}{34} + \frac{x^2}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^5}{38} + \frac{x^6}{39} + \frac{x^7}{40} + \frac{x^8}{41} + \frac{x^9}{42}$
 $+ \frac{x^{10}}{43} + \frac{x^{11}}{44} + \&c$ *ad infinitum*, when x is $= \frac{9}{10}$, or 0.9.

We have already seen in art. 56, pages 687, 688, that p , (or the near value
 of y , derived from it's greater and lesser limits, 0.159,489 and 0.128,240, by
 adding 0.003,472, or the 9th part of 0.031,249, or the difference of the said
 limits, to the lesser limit 0.128,240,) is $= 0.131,712$, and that q , (or the near
 value of z , derived from it's greater and lesser limits, 0.139,318 and 0.112,640,
 by adding 0.002,964, or the 9th part of 0.026,678, or the difference of the
 said limits, to the lesser limit, 0.112,640,) is $= 0.115,604$. It remains there-
 fore that we find the other near values of y and z , which are to be derived from
 their said limits by adding to 0.128,240, the lesser limit of y , the 8th part of
 0.031,249, the difference of it's two limits, and by adding to 0.112,640, the
 lesser limit of z , the 8th part of 0.026,678, the difference of it's two limits.
 Now the 8th part of 0.031,249 is $= 0.003,906$, which, being added to
 0.128,240, makes 0.132,146; and the 8th part of 0.026,678 is $= 0.003,334$,
 which, being added to 0.112,640, makes 0.115,974. Therefore P is $=$
 0.132,146, and Q is $= 0.115,974$. Therefore $P+p$ will be $(= 0.132,146$
 $+ 0.131,712) = 0.263,858$, and $Q+q$ will be $(= 0.115,974 + 0.115,604)$
 $= 0.231,578$; and consequently $P+p \times \frac{Q+q}{2}$ will be $(= 0.263,858 \times$
 $0.231,578) = 0.061,103,707,924$.

But K is $= 0.016,177,427$, and consequently KK is $(= 0.016,177,427^2)$
 $= 0.000,261,709,144,340,329$.

Therefore $KK + P+p \times \frac{Q+q}{2}$ will be $= 0.000,261,709,144$, &c. +
 $0.061,103,707,924 = 0.061,365,417,068$, and consequently the square-root
 of $KK + P+p \times \frac{Q+q}{2}$ will be $= \sqrt{0.061,365,417,068} = 0.247,720$;
 and therefore 0.247,720 will be nearly equal to the proposed series $\frac{1}{33} + \frac{x}{34}$
†.

$$+ \frac{x^3}{35} + \frac{x^3}{36} + \frac{x^4}{37} + \frac{x^4}{38} + \frac{x^4}{39} + \frac{x^7}{40} + \frac{x^8}{41} + \frac{x^9}{42} + \frac{x^{10}}{43} + \frac{x^{11}}{44} + \&c$$

ad infinitum. Q₂ E. 1.

Art. 65. This near value of $y + z$, or the proposed series, is a little greater than it's true value 0.247,588, but approaches nearer to it than the near value of it obtained in art. 59, by means of the conjectural numbers 49,967 and

$$43,852, \text{ expressing nearly the proportion of } y \text{ to } z, \text{ or of the series } \frac{x}{33} + \frac{x^3}{35} + \frac{x^4}{37} + \frac{x^6}{39} + \frac{x^8}{41} + \frac{x^{10}}{43} + \&c \text{ ad infinitum to the series } \frac{x}{34} + \frac{x^3}{36} + \frac{x^5}{38} + \frac{x^7}{40} + \frac{x^9}{42} + \frac{x^{11}}{44} + \&c \text{ ad infinitum; which near value was } 0.248,201.$$

For the excess of 0.248,201 above 0.247,588, the true value of the proposed series is 0.000,613; whereas the excess of the last-found near value of the said series, to wit, 0.247,720, above it's true value, 0.247,588, is only 0.000,132, which is less than the fourth part of 0.000,613. Therefore this last manner of applying our knowledge of the true value of $y - z$, or the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$ *ad infinitum*, to the discovery of a near value of $y + z$, or the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*, seems to be preferable to the former method of applying it to the same purpose by means of the conjectural numbers expressing the proportion of y to z . But both these methods require a good deal of calculation, besides the labour (which, as we have seen above, is by no means trifling,) of computing the differential series, by means of which the true value of $y - z$, or the series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$, is obtained; and, when all is done, the near values of the proposed series that are obtained by them, are less exact than the two former near values of it, to wit, 0.247,533 and 0.247,316, which were obtained by the two first conjectural methods of investigation explained above in art. 49, and 52, &c . . . 56. And therefore I think those two first methods are preferable to this last method, which requires the computation of $y - z$ by means of the differential series. They are not, however, so exact as I could wish. But I have not been able, after much labour and repeated endeavours, to find any better method of obtaining a near value of the proposed series $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ *ad infinitum*. And therefore I shall here give over the further prosecution of this inquiry, and put an end to this Discourse.

End of the Account of Mr. Hellins's Method of finding the Value of a slowly-converging Infinite Series of the foregoing Form, $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$ ad infinitum.

ERRATA.

- In page 8, line 10 from the bottom, instead of FEN, read EFN.
- In page 12, line 8 from the bottom, instead of *cut*, read *cuts*.
- In page 38, line 13, after $\equiv$, and before 0.008,959, insert the sign $\equiv$.
- And in the same page 38, line 12 from the bottom, instead of 0.002,02x³, read 0.002,502x⁴.
- And again in the same page 38, line 10 from the bottom, after x³, insert $\epsilon^3 \epsilon$.
- In page 40, line 9 from the bottom, instead of 0.001,217, read 0.001,220.
- In page 43, line 23, before 0.028,799, insert the sign $\equiv$.
- In page 46, line 19, instead of 0.385,667,3, read 0.385,700,8.
- In page 53, line 7, instead of $\frac{dx}{r}$, read $\frac{dx}{2r}$.
- In page 59, line 13, after the capital F, instead of C, read G.
- In page 61, line 16, instead of 65.5, read 69.5.
- And in the same page 61, and same line, instead of 147.375, read 156.375.
- And again in the same page 61, line 17, instead of 0.036,843, read 0.039,094.
- N. B. This value of ϵ is to be used instead of 0.036,843 in the remainder of the solution.
- In page 87, line 6 from the bottom, instead of $a + \epsilon$, read $\overline{a + \epsilon}$.
- In page 96, line 5, instead of 0.081,201, read 0.081,226.
- And in the same page 96, line 7, instead of 0.081,201, read 0.081,226.
- And again in the same page 96, line 8 from the bottom, instead of 0.081,201, read 0.081,226. N. B. This value of ϵ is to be used instead of 0.081,201 in the two subsequent pages.
- In page 98, line 8, instead of 0.060,332, read 0.060,307.
- And in the same page 98, the bottom line, instead of 0.060,332, read 0.060,307.
- N. B. This number is to be used instead of 0.060,332, as often as it occurs afterwards.
- In page 115, line 4 from the bottom, instead of *erat*, read *erit*.

In page 131, line 18, instead of $\gamma\mu\sigma\theta\alpha\iota$, read $\gamma\mu\sigma\theta\alpha\iota$.

In page 139, the Greek corresponding to the Latin sentence, *Hunc igitur in modum Pontus, ac Bosporus, et Hellespontus à me dimensa sunt*, is wanting.

In page 140, line 18 from the bottom, instead of 500, read 20.

In page 151, line 12, instead of g , read in .

In page 162, line 6, instead of $12\frac{9}{10}\frac{1}{10}\frac{1}{10}$, read $11\frac{1}{10}\frac{1}{10}\frac{1}{10}$.

In page 165, line 10, instead of $curfus$, read $curfus$.

In page 171, line 15 from the bottom, instead of *longitudinis*, read *latitudinis*.

In page 177, line 6, instead of $curfus$, read $curfus$.

In page 191, line 4, instead of oe , read io .

In page 338, line 6, instead of $\frac{14 - \sqrt{26}}{3}$, read $\frac{14 - \sqrt{106}}{3}$.

In page 354, line 1, instead of $1,380ax^2$, read $1,380a^{10}x^2$.

In page 355, line 11, instead of $72x^4$, read $72x^{10}$.

And in the same page 355, line 19, instead of $12,435a^3x^3$, read $12,435a^4x^3$.

In page 356, lines 8 and 9, instead of $4,212a^4x^4$, read $4,212a^5x^4$.

In page 357, line 7, instead of $\frac{24 - 6\sqrt{24} + 9}{3}$, read $\frac{24 - 6\sqrt{24} + 9}{2}$.

In page 358, line 7, instead of $\frac{4 \times 3 + 3 \times 2 \times \sqrt{3} \times \sqrt{2} \times 2}{3}$, read $\frac{4 \times 3 + 2 \times 2 \times \sqrt{3} \times \sqrt{2} \times 2}{3}$.

In page 359, line 10, instead of $12,435a^3x^3$, read $12,435a^4x^3$.

And in the same page 359, line 12, instead of $9x^9$, read $9x^{10}$.

In page 363, lines 3 and 6, instead of $12,625a^4$, read $12,625a^{10}$.

In page 364, line 2, instead of $124ax^7$, read $124a^4x^7$.

And in the same page 364, line 2, after $124ax^7$, insert $+ 24ax^7$.

In page 365, line 9 from the bottom, instead of $14,052a^5a^4$, read $14,052a^9x^4$.

In page 370, line 18, instead of $9,162a^4x^4$, read $9,162a^5x^4$.

And in the same page 370, and line 18, instead of $9,162a^4$, read $9,162a^5$.

And again in the same page 370, line 19, in two places, instead of $9,162a^4$, read $9,162a^5$.

In page 373, line 4, instead of $298,112,058,489,6$, read $297,736,917,476,4$.

And in the same page 373, line 9, instead of $3,022,363,764,741,5$, read $3,021,988,633,738,3$.

And again in the same page 373, line 10, instead of $12,591,587,735,258,5$, read $12,591,962,866,261,7$.

In

In page 375, line 3 from the bottom, instead of 3,021.988,633,728,3, read 3,021.988,633,738,3.

And in the same page 375, bottom line, instead of 12,591.962,866,271,7, read 12,591.962,866,261,7.

In page 376, line 4, instead of 12,591.962,866,271,7, read 12,591.962,866,261,7.

And in the same page 376, line 6, instead of 12,591.962,866,271,7, read 12,591.962,866,261,7.

And again in the same page 376, line 14 from the bottom, instead of 0.160,500,953 $\times x^4$, read 0.160,500,953 $\times a^4$.

In page 378, line 2, instead of 0.632,95 $+ a \times z$, read 0.632,95 $\times a \times z$.

In page 381, line 20, instead of 8,478 a^1x^3 , read 8,748 a^2x^3 .

In page 383, line 10 from the bottom, instead of 90 ax^2x , read 90 x^2x .

In page 384, line 10, instead of 2,156 a^1a^6 , read 2,156 a^2x^6 .

In page 387, line 13 from the bottom, instead of 7760.6835 $\times a^3$, read 7760.6835 $\times a^{10}$.

In page 391, line 7, instead of 15,289.547,896,654 $\times z$, read 15,289.547,896,564 $\times z$.

In page 402, line 3, instead of 516 ax^2 , read 516 a^2x^2 .

In page 411, line 1, instead of 12,435 a^2x^3 , read 12,435 a^2x^2 .

In page 442, line 15 from the bottom, instead of cd , read cd .

In page 489, line 13 from the bottom, instead of *l'hypothèse*, read *l'hypothèse*.

In page 490, line 2 from the bottom, instead of *d'avantage*, read *d'avantage*.

In page 499, line 8 from the bottom, instead of UD, read ND.

In page 550, line 5, instead of *ipso* A, read *ipso* A.

And in the same page 550, line 8, instead of *ascito* A, read *ascito* A.

In page 567, line 15, instead of *terti*, read *terti*.

In page 571, line 13 from the bottom, instead of $\frac{1}{27}$, read $\frac{1}{37}$.

In page 593, line 7, instead of $\frac{D^2x^3}{1+x^2}$, read $\frac{D^2x^3}{1+x^2}$.

In page 595, line 11, instead of $-b$, read $+b$.

In page 597, line 3, instead of $-$, after the capital letter P, put a comma.

In page 600, line 3, in two places, instead of V, read W.

In page 607, line 7, instead of D_4'' , read D_6'' .

In page 619, line 7, instead of fx , read fx^2 .

In page 626, line 9, instead of d''' , read d'''' .

In page 655, line 8 from the bottom, instead of $\frac{6561}{16561}$, read $\frac{6561}{16561}$.

In page 669, line 10 from the bottom, instead of $\frac{x^2}{80}$, read $\frac{x^{10}}{80}$.

In page 687, line 4 from the bottom, instead of $-dx^2$, read $+dx^2$.

In page 693, line 8, instead of $\frac{d^2x^2}{dx^2}$, read $\frac{d^2x^{11}}{dx^2}$.

F I N I S.

DIRECTIONS TO THE BINDER.

Let one Plate of Mr. Pascal's Triangle Arithmetique be placed on the left-hand side, next to page 502, and so as to open upon the reader when he is reading page 503; and the other copy of the same Plate be placed on the right-hand side between pages 532 and 533, so as to open upon the reader when he is reading page 532.

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